INVARIANT DOLBEAULT COHOMOLOGY FOR HOMOGENEOUS ALMOST COMPLEX MANIFOLDS

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1. Introduction

In [CW21] the authors define Dolbeault cohomology for almost complex manifolds, extending the definition from the integrable setting, and showing there is a spectral sequence converging to de Rham cohomology whose first page is the Dolbeault cohomology. The special case of compact Lie groups, and nilmanifolds, were also treated in [CW21], where analogous results are shown for the subcomplex of invariant forms, for any choice of invariant almost complex structure. Additionally, it is shown that each page of the spectral sequence of the invariant forms injects into the spectral sequence of the global forms. This provides an efficient means to compute some piece of the global theory, at least for the case of Lie groups and nilmanifolds.

There are numerous examples of homogenous complex and almost complex manifolds that fall outside the above categories, for which one wishes to compute Dolbeault cohomology and more generally the Frölicher spectral sequence. Among these on the complex side are several non-Kähler examples (so that degeneration at the first page is not guaranteed) as well as several interesting non-integrable examples including homogenous nearly Kähler manifolds, such as S^6, $\mathbb{CP}^3$, and the manifold $F(1,2)$ of complete flags in $\mathbb{C}^3$.

In this note we give a definition of invariant Dolbeault cohomology for homogeneous almost complex manifolds, suitable for both the integrable and non-integrable cases, generalizing the definition given in [CW21, Definition 5.7] for Lie groups. We conclude with some explicit computations, focusing primarily on homogeneous nearly Kähler six–manifolds.

2. Preliminaries

We recall a description of invariant differential forms on homogenous spaces. Let G be a compact connected Lie group, with Lie algebra $\mathfrak{g}$, and let H be a closed connected subgroup, with Lie algebra $\mathfrak{h}$. We consider the right action of H on G, with resulting orbit space G/H of left cosets $\{gH\}$ of H in G. Consider the exterior algebra $\wedge^*(\mathfrak{g}^*)$ on the dual $\mathfrak{g}^*$, which is a differential graded algebra with differential d given by the negative dual of the Lie bracket of $\mathfrak{g}$, extended as a derivation. Namely, for $\omega \in \wedge^k(\mathfrak{g}^*)$,

$$d\omega(x_1,\dots,x_{k+1}) = \sum_{i<j}(-1)^{i+j+1}\omega([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots x_{k+1})$$

Then by a classical result of Chevalley–Eilenberg, $(\wedge^*(\mathfrak{g}^*),d)$ is isomorphic as a differential graded algebra to the left G-invariant forms on G [CE48, Theorem 9.1].

Definition 2.1. (Relative cochains) For a Lie algebra $\mathfrak{g}$ and subalgebra $\mathfrak{h}$, let $C(\mathfrak{g},\mathfrak{h})$ denote the subspace of $\wedge^*(\mathfrak{g}^*)$ consisting of cochains *relative*[*] *to* $\mathfrak{h}$, i.e. those $\omega \in \wedge^*(\mathfrak{g}^*)$ such that

$$\iota_v \omega = 0 \quad \text{and} \quad \iota_v (d\omega) = 0, \qquad \text{for all } v \in \mathfrak{h}.$$

The motivation for the above definition of relative cochains $C(\mathfrak{g},\mathfrak{h})$ is that there is a one-to-one correspondence between relative cochains ω and left G-invariant differential forms $\Omega^*(G/H)$ on G/H [CE48, Theorem 13.1]. The correspondence is given by assigning to any left G-invariant $\eta \in \Omega^*(G/H)$ the value at the identity $e \in G$ of the pullback of η along $\pi : G \to G/H$. The two conditions in Definition 2.1 give a characterization of when a left G-invariant form on G is the pullback of a left G-invariant form on G/H (c.f. [CE48], (13.6)).

Note that $C(\mathfrak{g},\mathfrak{h})$ is a differential graded subalgebra of $\wedge^*(\mathfrak{g}^*)$, with an injection $j : C(\mathfrak{g},\mathfrak{h}) \to \Omega^*(G/H)$.

Theorem 2.2. ([CE48], Theorem 22.1) Let G be a compact connected Lie group, with Lie algebra $\mathfrak{g}$, and let H be a closed connected subgroup, with Lie algebra $\mathfrak{h}$. The inclusion $j : (C(\mathfrak{g},\mathfrak{h}), d) \to (\Omega^*(G/H), d)$ induces a ring isomorphism on cohomology.

We sketch the proof here, as some of the arguments are needed again below. We choose a left invariant measure on G such that G has total measure equal to 1. We define the averaging operator I by

$$I(\omega) = \int_G L_g^* \omega \, dg,$$

giving a left G-invariant form $I(\omega)$, for all $\omega \in \Omega^*(G)$, so that $I : \Omega^*(G) \to \wedge^*(\mathfrak{g}^*)$ is well defined. The operator I satisfies $dI = Id$, and that $I(\omega) = \omega$, if ω is left G-invariant.

For any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there is an induced map

$$I : \Omega^*(G/H) \to C(\mathfrak{g},\mathfrak{h}), \qquad \omega \mapsto I(\pi^*\omega).$$

To prove this is well defined it suffices, by the remarks following Definition 2.1, to show $\iota_v(I\pi^*\omega) = 0$ and $\iota_v d(I\pi^*\omega) = 0$ for all $\omega \in \Omega^*(G/H)$. Let $v \in \mathfrak{h}$, and compute

$$\begin{aligned}\iota_v(I\pi^*\omega)(x_1,\dots,x_k) &= \int_G L_g^*\pi^*\omega(v,x_1,\dots,x_k)\,dg \\ &= \int_G \omega(\pi_*(L_g)_*v, \pi_*(L_g)_*x_1,\dots,\pi_*(L_g)_*x_k)\,dg = 0,\end{aligned}$$

where the last equality holds since the left G-invariance of $\pi : G \to G/H$ implies that $\operatorname{Ker}\pi_*$ is the left orbit of $\mathfrak{h}$. Similarly, $\iota_v d(I\pi^*\omega) = 0$ for $v \in \mathfrak{h}$. Here we also have $dI = Id$ and $I\omega = \omega$ for all $\omega \in C(\mathfrak{g},\mathfrak{h})$.

Now, to show $j : C(\mathfrak{g},\mathfrak{h}) \to \Omega^*(G/H)$ is injective on cohomology, it suffices to note that if $\omega \in C(\mathfrak{g},\mathfrak{h})$ is closed and $j(\omega) = d\eta$ for some $\eta \in \Omega^*(G/H)$, then ω is exact in $C(\mathfrak{g},\mathfrak{h})$ as well, since

$$dI(\eta) = I(d\eta) = I(\omega) = \omega.$$

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[*]In [CE48] these are referred to as cochains that are *orthogonal to* $\mathfrak{h}$. We use the term *relative*, as the notion is metric independent.

Finally, to show that $j : C(\mathfrak{g},\mathfrak{h}) \to \Omega^*(G/H)$ is surjective on cohomology, we use the fact that a closed differential form is exact if it integrates to zero on all cycles. (This a non-trivial point, that closed forms have a pre-dual "testing ground" for being exact; there is no known analogue for $\bar{\partial}$-closed forms.) Suppose $\omega \in \Omega^*(G/H)$ is closed; then by Fubini we have for all cycles $c \in G/H$,

$$\int_c I(\omega) = \int_G \int_{L_g(c)} \omega \, dg = \int_G \int_c \omega \, dg = \int_c \omega,$$

where the second equality holds since $L_g(c)$ is homologous to c for G connected, via $L_{g_t} c$ for some path g_t from g to e, and the last equality holds since G has measure 1. This shows $\int_c (I(\omega) - \omega) = 0$ for all cycles c, so that so that $I(\omega) - \omega$ is exact.

2.1. **Homogeneous almost complex structures.** An almost complex structure J on G/H is said to be left G-invariant if J commutes with the left action of G on G/H. We have the following description of left G-invariant almost complex structures J on G/H.

Proposition 2.3. [KN63, Proposition 6.3] Let J be a linear endomorphism of $\mathfrak{g}$ such that

(1) $J(\mathfrak{h}) \subset \mathfrak{h}$,
(2) $J^2 = -\mathrm{Id} \mod \mathfrak{h}$,
(3) $J[v,w] = [v, Jw] \mod \mathfrak{h}$ for $v \in \mathfrak{h}, w \in \mathfrak{g}$.

Then the left G-extension of J to G descends to a left G-invariant almost complex structure on G/H. Conversely, any left G-invariant almost complex structure J on G/H arises in this way (but not uniquely).

In particular, we do not assume that $J^2 = -1$ on $\mathfrak{g}$, though this will be the case in several examples below.

Now suppose G/H has a left G-invariant almost complex structure. As an almost complex manifold, the complex valued differential forms $\Omega^*(G/H;\mathbb{C})$ inherit a bigrading

$$\Omega^k(G/H;\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(G/H;\mathbb{C}),$$

with projection operators $\pi^{p,q} : \Omega^k(G/H;\mathbb{C}) \to \Omega^{p,q}(G/H;\mathbb{C})$. Let

$$C(\mathfrak{g},\mathfrak{h};\mathbb{C}) := C(\mathfrak{g},\mathfrak{h}) \otimes \mathbb{C}$$

be the complex-valued relative cochains, as in Definition 2.1, which is a subcomplex of $\wedge^*_{\mathbb{C}}(\mathfrak{g}^*) := \wedge^*(\mathfrak{g}^* \otimes \mathbb{C})$. There is a natural sequence of inclusions

$$C(\mathfrak{g},\mathfrak{h};\mathbb{C}) \to \Omega^*(G/H,\mathbb{C}) = \bigoplus_{p,q} \Omega^{p,q}(G/H;\mathbb{C})$$

The next lemma shows that the projection of any complex relative cochain $\omega \in C(\mathfrak{g},\mathfrak{h};\mathbb{C})$ onto any subspace $\Omega^{p,q}(G/H;\mathbb{C})$ is again a complex relative cochain.

Lemma 2.4. Let G/H be a homogeneous almost complex manifold with left invariant almost complex structure J. Then $\omega \in C(\mathfrak{g},\mathfrak{h};\mathbb{C})$ is a complex relative cochain if and only if every (p,q)-component $\pi^{p,q}\omega$ of ω is a complex relative cochain.

Proof. We give two proofs. Since J is left G-invariant on G/H, the pointwise operators $\pi^{p,q}$ commute with the G-action on $\Omega^*(G/H;\mathbb{C})$, so $\pi^{p,q}\omega$ is also left G-invariant on G/H, and therefore $\pi^{p,q}\omega$ is in the image of a relative cochain.

For a second proof, we use the conditions of Proposition 2.3 to check that $\pi^{p,q}\omega$ is a relative cochain whenever ω is a relative $(k+1)$-cochain . To simplify notation, regard $\pi^{p,q}\omega$ as a form on G via pullback along $\pi : G \to G/H$, which is still left G-invariant, and completely determined by its value at the identity $e \in G$.

First, we have $i_v \pi^{p,q}\omega = 0$ when $v \in \mathfrak{h}$, since

$$(i_v \pi^{p,q}\omega)(v_1, \dots, v_k) = (\pi^{p,q}\omega)(v, v_1, \dots, v_k) = \omega((\pi_{p,q})(v, v_1, \dots, v_k)) = 0.$$

The last equality holds since ω is a relative cochain, $v \in \mathfrak{h}$, $J(\mathfrak{h}) \subset \mathfrak{h}$ by condition (1) of Proposition 2.3, and

$$\pi_{1,0}v = \frac{1}{2}(Id - iJ)v \in \mathfrak{h} \otimes \mathbb{C} \qquad \text{and} \qquad \pi_{0,1}v = \frac{1}{2}(Id + iJ)v \in \mathfrak{h} \otimes \mathbb{C}.$$

Next we show that if ω is a complex-valued relative cochain, then $i_v d(\pi^{p,q}\omega) = 0$ for all $v \in \mathfrak{h}$. We first consider the case where ω is a 1-form. By condition (3) of Proposition 2.3 we have

$$\omega(J[v, u]) = \omega([v, Ju]) \qquad \text{for all } v \in \mathfrak{h}.$$

This implies $\omega(\pi_{1,0}[v, u]) = \omega([v, \pi_{1,0}u])$ for all $v \in \mathfrak{h}$ when ω is a relative 1-form. Therefore, using Cartan's formula,

$$\begin{aligned} i_v(d\pi^{1,0}\omega)(u) = (d\pi^{1,0}\omega)(v, u) &= -(\pi^{1,0}\omega)([v, u]) + v(\pi^{1,0}\omega(u)) - u(\pi^{1,0}\omega(v)) \\ &= -\omega(\pi_{1,0}[v, u]) + v(\omega(\pi_{1,0}u)) - u(\omega(\pi_{1,0}(v))) \end{aligned}$$

The last term is zero, and can be replaced by $(\pi_{1,0}u)(\omega(v))$ which is also zero, and we get

$$-\omega([v, \pi_{1,0}u]) + v(\omega(\pi_{1,0}u)) - (\pi_{1,0}u)(\omega(v)) = (i_v d\omega)(\pi_{1,0}u) = 0.$$

Similarly, $\pi^{0,1}\omega$ is a relative form whenever ω is a relative form.

For the general case, let ω be a relative k-form, let $v_1 \in \mathfrak{h}$, and we show

$$i_{v_1}(d\,(\pi^{p,q}\omega)) = (i_{v_1}(d\omega)) \circ \pi_{p,q},$$

so the left hand side is zero. Then

$$\begin{aligned} i_{v_1}(d\,(\pi^{p,q}\omega))(v_2, \dots, v_{k+1}) &= d(\pi^{p,q}\omega)(v_1, v_2, \dots, v_{k+1}) \\ &= (\pi^{p,q}\omega)\Bigg(\sum_{j=2}^{k+1}(-1)^{j+1}\left([v_1, v_j], v_2, \dots, \hat{v}_j, \dots, v_{k+1}\right) \\ &\quad + \sum_{1<i<j\le k+1}(-1)^{i+j}([v_i, v_j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{k+1})\Bigg) \\ &\quad + v_1(\omega(\pi_{p,q}(v_2, \dots, v_k))) + \sum_{j=2}^{k+1}(-1)^{j+1} v_j(\omega(\pi_{p,q}(v_1, \dots, \hat{v}_j, \dots, v_{k+1}))). \end{aligned}$$

(Since we are dealing with invariant forms, some summands here and below trivially vanish, but we leave them in for clarity.) The summands involving $[v_i, v_j]$ vanish upon evaluation by $\pi^{p,q}\omega$, since $v_1 \in \mathfrak{h}$, and the last term above is zero, and can be replaced by

$$A = \sum_{j=2}^{k+1}(-1)^{j+1}\,(\pi_{1,0}v_j)(\omega(v_1, \pi_{p-1,q}(v_2, \ldots, \hat{v}_j, \ldots, v_{k+1}))$$
$$+ \sum_{j=2}^{k+1}(-1)^{j+1}\,(\pi_{0,1}v_j)(\omega(v_1, \pi_{p,q-1}(v_2, \ldots, \hat{v}_j, \ldots, v_{k+1}))$$

which is also zero. Writing $\pi_{p,q} = \pi_{1,0} \wedge \pi_{p-1,q} + \pi_{0,1} \wedge \pi_{p,q-1}$ and using

$$\begin{aligned}\omega(\pi_{p,q}([v_1, u], -)) &= \omega(\pi_{1,0}[v_1, u], \pi_{p-1,q}(-)) + \omega(\pi_{0,1}[v_1, u], \pi_{p,q-1}(-)) \\ &= \omega([v_1, \pi_{1,0}u], \pi_{p-1,q}(-)) + \omega([v_1, \pi_{0,1}u], \pi_{p,q-1}(-)),\end{aligned}$$

by condition (3) of Proposition 2.3 as before, we have

$$i_v(d\,(\pi^{p,q}\omega))(v_1, \ldots, v_k) = \sum_{j=2}^{k+1}(-1)^{j+1}\omega\,([v_1, \pi_{1,0}v_j], \pi_{p-1,q}(v_2, \ldots, \hat{v}_j, \ldots, v_{k+1}))$$
$$+\sum_{j=2}^{k+1}(-1)^{j+1}\omega\,([v_1, \pi_{0,1}v_j], \pi_{p,q-1}(v_2, \ldots, \hat{v}_j, \ldots, v_{k+1}))+v_1(\omega(\pi_{p,q}(v_2, \ldots, v_{k+1})))+A$$
$$= (d\omega)(v_1, \pi_{p,q}(v_2, \ldots, v_{k+1})) = i_{v_1}(d\omega)(\pi_{p,q}(v_2, \ldots, v_{k+1})). \quad \square$$

Corollary 2.5. For any homogeneous manifold G/H with left invariant almost complex structure J, the complex-valued relative cochains have a bigrading,

$$C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{C}) = \bigoplus_{p+q=k} C^{p,q}(\mathfrak{g}, \mathfrak{h}; \mathbb{C}), \qquad C^{p,q}(\mathfrak{g}, \mathfrak{h}; \mathbb{C}) := C^{p+q}(\mathfrak{g}, \mathfrak{h}) \cap \Omega^{p,q}(G/H; \mathbb{C}),$$

such that the inclusion map

$$C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{C}) \to \Omega^*(G/H; \mathbb{C})$$

is a map of bigraded algebras.

Proof. By Lemma 2.4 every relative cochain $\omega \in \Omega^*(G/H; \mathbb{C})$ is equal to the sum of its (p, q)-components with respect to the bigrading of $\Omega^*(G/H; \mathbb{C})$. $\square$

Remark 2.6. Note that for a typical homogeneous almost complex manifold G/H with left G-invariant J, the space $\wedge^*_{\mathbb{C}}(\mathfrak{g}^*)$ does not have a natural bigrading. Nevertheless, in several examples below, we begin with a left G-invariant J on G that induces a left invariant J on G/H. In this case there is a natural bigrading on $\wedge^*_{\mathbb{C}}(\mathfrak{g}^*)$, and the inclusion of $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{C}) \to \wedge^*_{\mathbb{C}}(\mathfrak{g}^*)$ is a map of bigraded algebras and complexes.

2.2. **Hodge filtration and Frölicher sequence.** For any almost complex manifold (M, J) the exterior differential decomposes as $d = \bar{\mu} + \bar{\partial} + \partial + \mu$, where the bidegrees of each component are given by

$$|\bar{\mu}| = (-1, 2),\ |\bar{\partial}| = (0, 1),\ |\partial| = (1, 0),\ \text{and}\ \ |\mu| = (2, -1).$$

The terms $\bar{\mu}$ and μ are linear over functions and vanish if and only if the structure J on G/H is integrable, c.f. [CW21, Lemma 2.1].

For any such complex $(\mathcal{A}, d)$ with (p,q)-bigrading, there is an associated Hodge-type filtration, defined in [CW21, Definition 3.2] by

$$F^p\mathcal{A}^n := \mathrm{Ker}\,(\bar{\mu}) \cap \mathcal{A}^{p,n-p} \oplus \bigoplus_{i>p} \mathcal{A}^{i,n-i},$$

and resulting Frölicher sequence $E_r(M,J)$ of bigraded algebras for $r \geq 0$. In particular, this applies to any homogeneous space G/H with left invariant almost complex structure J, yielding a spectral sequence $E_r(G/H, J)$. We similarly denote by $(E_r^I(\mathfrak{g}, J), d_r)$ the Frölicher sequence obtained by the filtered differential graded algebra $(\wedge^*(\mathfrak{g}^*), d, F)$ on the left invariant forms $(\wedge^*_{\mathbb{C}}(\mathfrak{g}^*))$ of G with left invariant J.

Definition 2.7. The *invariant Frölicher spectral sequence* $\{{}^L E_r^{*,*}(\mathfrak{g}, \mathfrak{h}; J), d_r\}_{r\geq 0}$ of the homogeneous space G/H with left G-invariant J is the spectral sequence

$${}^L E_r^{*,*}(\mathfrak{g}, \mathfrak{h}) := E_r^{*,*}(C(\mathfrak{g}, \mathfrak{h}; \mathbb{C}), F),$$

associated to the filtered differential graded algebra $(C(\mathfrak{g}, \mathfrak{h}), d, F)$. For each $r \geq 0$, $({}^L E_r^{*,*}(\mathfrak{g}, \mathfrak{h}; J), d_r)$ is a commutative bigraded algebra.

By [CE48, Theorem 22.1], this spectral sequence converges to the complex de Rham cohomology of G/H,

$$({}^L E_r(\mathfrak{g}, \mathfrak{h}; J), d_r) \implies H^*_{\mathrm{dR}}(G/H, \mathbb{C}) = H^*_{\mathrm{dR}}(G/H, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Next we give some explicit formulas for the pages $(E_r^I(\mathfrak{g}, \mathfrak{h}; J), d_r)$. In [CW21] explicit formulas are given for the pages $E_r(M,J)$ and differentials d_r for any almost complex manifold, in term of the components of $d = \bar{\mu} + \bar{\partial} + \partial + \mu$. By Corollary 2.5, $C(\mathfrak{g}, \mathfrak{h}; \mathbb{C})$ is a bigraded subalgebra of $\Omega^*(G/H; \mathbb{C})$, and since the Hodge filtration and associated spectral sequences are functorial constructions of the bigrading, the same formulas give expressions for the differentials d_r of the pages $\{E_r^I(\mathfrak{g}, \mathfrak{h}; J), d_r\}$, for all $r \geq 0$. We make the first few of these explicit here, and refer the reader to the Appendix of [CW21] for the higher pages.

For $r = 0$, we have the $\bar{\mu}$-cohomology of relative cochains $C := C(\mathfrak{g}, \mathfrak{h}; \mathbb{C})$,

$${}^L E_0^{p,q}(\mathfrak{g}, \mathfrak{h}; J) = H_{\bar{\mu}}^{p,q}(C) := \frac{\mathrm{Ker}\,(\bar{\mu} : C^{p,q} \longrightarrow C^{p-1,q+2})}{\mathrm{Im}(\bar{\mu} : C^{p+1,q-2} \longrightarrow C^{p,q})},$$

with differential $d_0 = \bar{\partial}$.

Next, for $r = 1$, we make the following:

Definition 2.8. The *invariant Dolbeault cohomology* is given by

$$H^{p,q}_{\mathrm{Dol}}(G/H; J) := {}^L E_1^{p,q}(\mathfrak{g}, \mathfrak{h}; J).$$

Succinctly, this may be written as the quotient

$${}^L E_1^{p,q}(\mathfrak{g}, \mathfrak{h}; J) \cong \frac{\{x \in C^{p,q}; \bar{\mu}\omega = 0, \bar{\partial}\omega = \bar{\mu}\omega_1\}}{\{x = \bar{\mu}\eta_1 + \bar{\partial}\eta_2; \bar{\mu}\eta_2 = 0\}}$$

with induced differential d_1 on ${}^L E_1(\mathfrak{g}, \mathfrak{h}; J)$ given by

$$d_1[\omega] = [\partial\omega - \bar{\partial}\omega_1],$$

where ω_1 is any choice such that $\bar{\partial}\omega = \bar{\mu}\omega_1$.

The second page is given by the quotient

$$ {}^{L}E_2^{p,q}(\mathfrak{g},\mathfrak{h};J) \cong \frac{\{\omega \in C^{p,q}; \bar{\mu}\omega = 0, \bar{\partial}\omega = \bar{\mu}\omega_1, \partial\omega = \bar{\partial}\omega_1 + \bar{\mu}\omega_3\}}{\{\omega = \bar{\mu}\eta_1 + \bar{\partial}\eta_2 + \partial\eta_3; \bar{\mu}\eta_2 + \bar{\partial}\eta_3 = 0, \bar{\mu}\eta_3 = 0\}}. $$

There is a commutative pullback diagram of complexes

$$ \begin{tikzcd} C(\mathfrak{g},\mathfrak{h};\mathbb{C}) \arrow[r] \arrow[d] & \Omega^*(G/H;\mathbb{C}) \arrow[d, "\pi^*"] \\ \wedge_{\mathbb{C}}^*(\mathfrak{g}^*) \arrow[r] & \Omega^*(G;\mathbb{C}). \end{tikzcd} $$

In [CW21] it is shown that if G has a left invariant J then the lower horizontal map induces an injection $({}^{L}E_r(\mathfrak{g},J), d_r) \to (E_r(G,J), d_r)$ for all $r \geq 0$, and an isomorphism for $r = \infty$. Next, we show that for invariant almost complex structures on G/H, the analogous result holds for the upper horizontal map.

Proposition 2.9. Let G be a compact connected Lie group with closed connected subgroup H, and left J be a left-invariant almost complex structure on G/H. For all $r \geq 0$, the inclusion

$$ C(\mathfrak{g},\mathfrak{h};\mathbb{C}) \to \Omega^*(G/H;\mathbb{C}) $$

induces an injection

$$ {}^{L}E_r(\mathfrak{g},\mathfrak{h};J) \longrightarrow E_r(G/H;J), $$

for all $r \geq 0$, which becomes an isomorphism at the E_∞-page.

Proof. We adapt the proof in [CW21]. Let

$$ I : \Omega^*(G/H) \to C(\mathfrak{g},\mathfrak{h}) \qquad \omega \mapsto I(\pi^*\omega) $$

be the averaging operator. Since the almost complex structure J on G/H is left G-invariant, I commutes with the projections onto the (p,q) spaces of $C(\mathfrak{g},\mathfrak{h};\mathbb{C})$, and therefore I commutes with the components of d.

Since the inclusion $C(\mathfrak{g},\mathfrak{h};\mathbb{C}) \to \Omega^*(G/H;\mathbb{C})$ preserves bigradings, we have well defined maps of spectral sequences

$$ {}^{L}E_r(\mathfrak{g},\mathfrak{h};J) \longrightarrow E_r(G/H;J), $$

and the isomorphism at E_∞ follows from the fact that both spectral sequences converge to $H^*(G/H,\mathbb{C})$.

To prove that the above maps are injective for all $r \geq 1$, we use that there are formulas for the differentials d_r on E_r, which are all expressible in terms of the components of $d = \bar{\mu} + \bar{\partial} + \partial + \mu$, so that I commutes with d_r as well. Then if $\omega \in C(\mathfrak{g},\mathfrak{h};\mathbb{C})$ is a d_r-coboundary in ${}^{L}E_r(G/H,J)$, say $\omega = d_r\eta$ for $\eta \in {}^{L}E_r(G/H,J)$, then

$$ \omega = I(\omega) = I(d_r\eta) = d_r I(\eta), $$

so that ω is a coboundary in $E_r^I(\mathfrak{g},\mathfrak{h};J)$. Therefore $[\omega]$ is zero in $E_r^I(\mathfrak{g},\mathfrak{h};J)$. □

Recall that the Frölicher spectral sequence for maximally non-integrable almost complex 6–manifolds converges at E_2 [CW21, Theorem 6.6]. Combining this with Proposition 2.9, we have:

Corollary 2.10. For any homogeneous maximally non-integrable almost complex 6-manifold $(G/H, J)$ (in particular homogeneous nearly Kähler 6-manifolds) the complex de Rham cohomology with its naturally defined bigrading defined by J can be computed using the relative complex $C(\mathfrak{g}, \mathfrak{h}) \otimes \mathbb{C}$ via the isomorphism ${}^{L}E_2(\mathfrak{g}, \mathfrak{h}) \to E_2(G/H; J)$.

3. Examples

We now compute the invariant Dolbeault cohomology for several homogeneous manifolds.

In [B10, Theorem 1], Butruille showed that every simply connected homogeneous nearly Kähler manifold G/H is isometric to one of the following four (equipped with a naturally induced metric):

- $SU(2) \times SU(2) \times SU(2)/SU(2)$, where the last $SU(2)$ is embedded diagonally. Such a manifold is diffeomorphic to $S^3 \times S^3$.
- $G_2/SU(3)$, diffeomorphic to S^6.
- $SU(3)/\left(U(1) \times U(1)\right)$, which is the full flag variety in $\mathbb{C}^3$.
- $Sp(2)/\left(U(1)Sp(1)\right)$, diffeomorphic to $\mathbb{CP}^3$.

The third and fourth nearly Kähler manifolds arise as the twistor spaces of S^4 and $\mathbb{CP}^2$ respectively, equipped with their standard metrics, where the twistor space is equipped with the Eells–Salamon almost complex structure.

We will describe the invariant Dolbeault cohomology for each of these examples, together with that of an invariant integrable structure on the third, by describing the corresponding complexified relative complexes. Only for $SU(2) \times SU(2)$ will the Frölicher spectral sequence not immediately degenerate.

Example 3.1. First let us consider $SU(2) \times SU(2) \times SU(2)/SU(2)$. As a specific representative, we consider $S^3 \times S^3 = SU(2) \times SU(2)$ with a left-invariant nearly–Kähler structure, explicitly described in [CW21, Example 5.12].

Namely, the complexified (relative) complex of invariant forms is given by

$$\wedge(x, y, z, \bar{x}, \bar{y}, \bar{z}),$$

where x, y, z are in bidegree $(1, 0)$, and

$$\begin{aligned}
dx &= -(1+i)\bar{y}\bar{z} + (i-1)(y\bar{z} + \bar{y}z) - (1+i)yz, \\
dy &= (1+i)\bar{x}\bar{z} + (i-1)(z\bar{x} + \bar{z}x) + (1+i)xz, \\
dz &= (i-1)\bar{x}\bar{y} + (i-1)(x\bar{y} + \bar{x}y) - (1+i)xy.
\end{aligned}$$

The dimensions of the terms in the invariant Frölicher spectral sequence are given in [CW21, Example 5.12]. The invariant spectral sequence degenerates at E_2; note that this example shows that the E_2 degeneration bound in Corollary 2.10 is sharp.

Example 3.2. Consider S^6 with its standard nearly Kähler structure. In [CZ18, Section 7], this example is presented as a homogeneous almost complex manifold $G_2/SU(3)$ with an explicit invariant almost complex structure J on the exceptional Lie group G_2, i.e. the group of real algebra automorphisms of the octonions. Following the notation of [CZ18], we denote by $\{h_1, \ldots, h_8\}$ a basis for $\mathfrak{su}(3)$, and by $\{f_1, \ldots, f_6, h_1, \ldots, h_8\}$ an extension to a basis for $\mathfrak{g}_2$, where the Lie brackets are given in Table 1 (following [CZ18, Section 8], where some brackets within $\mathfrak{su}(3)$ are omitted).

	f_1	f_2	f_3	f_4	f_5	f_6	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
f_1		h_1+h_2	$2f_6$	f_5	$-f_4$	$-2f_3$	$-f_2+h_8$	$-h_8$	$-f_4-h_6$	f_3	f_6	$-f_5+h_3$	0	h_2
f_2			f_5-h_3	$-h_4$	$-f_3+h_5$	h_6	f_1+h_7	$-h_7$	f_3-h_5	f_4	$-f_5+h_3$	$-f_6$	h_2	0
f_3				h_2	h_8	$2f_1$	f_4+h_6	$-f_4$	$-f_2+h_8$	$-f_1$	0	$-h_1-h_2$	f_6	$-f_5$
f_4					$2f_1+2h_7$	h_8	$-f_3+h_5$	f_3	f_1+h_7	$-f_2$	$-h_1-h_2$	0	$-f_5$	$-f_6$
f_5						$-h_2$	h_4	f_6	0	$-h_1$	f_2-h_8	f_1+h_7	f_4	f_3
f_6							h_3	$-f_5$	$-h_1$	0	$-f_1$	f_2	$-f_3$	f_4
h_1								0	$-2h_4$	$2h_3$	$-h_6$	h_5	h_8	$-h_7$
h_2									h_4	$-h_3$	$-h_6$	h_5	$-2h_8$	$2h_7$
h_3										$-2h_1$	$-h_8$	h_7	$-h_6$	h_5
h_4											$-h_7$	$-h_8$	h_5	h_6
h_5												$-2h_1-2h_2$	$-h_4$	$-h_3$
h_6													h_3	$-h_4$
h_7														$-2h_2$
h_8														

TABLE 1. Lie brackets for $\mathfrak{g}_2$ and its Lie subalgebra $\mathfrak{su}(3)$

The almost complex structure is given by $Jf_1 = -f_2, Jf_3 = -f_4, Jf_5 = -f_6, Jh_1 = -h_2, Jh_3 = -h_4, Jh_5 = -h_6, Jh_7 = -h_8$, and one checks that it satisfies the conditions of Proposition 2.3.

Here and later, we will use superscripts to denote the one-forms dual to the given basis of $\mathfrak{g}$. With the help of Maple, we get that the relative complex in positive degrees is spanned by

$\alpha_2 := f^1f^2 + f^3f^4 + f^5f^6$ in degree 2,

$\alpha_3 := -f^1f^3f^5 + f^1f^4f^6 + f^2f^3f^6 + f^2f^4f^5$ and $\alpha_3' := -f^1f^3f^6 - f^1f^4f^5 - f^2f^3f^5 + f^2f^4f^6$ in degree 3,

$\alpha_4 := f^1f^2f^3f^4 + f^1f^2f^5f^6 + f^3f^4f^5f^6$ in degree 4,

$\alpha_6 := f^1f^2f^3f^4f^5f^6$ in degree 6.

The differentials are given by $d\alpha_2 = 3\alpha_3, d\alpha_3 = 0, d\alpha_3' = 4\alpha_4$, and the ring structure is determined by

$$\alpha_2^2 = 2\alpha_4, \alpha_3^2 = 0, \alpha_3'^2 = 0, \alpha_3\alpha_3' = 4\alpha_6, \alpha_2\alpha_4 = 3\alpha_6.$$

Now we complexify $\mathfrak{g}_2$ and write the above α_i in terms of ϕ^i and $\overline{\phi^i}$, where $\{\phi^i\}$ is a basis for $\Lambda^{1,0}(\mathfrak{g}_2^* \otimes \mathbb{C})$, given by $\phi^1 = f^1 - if^2, \dots, \phi^3 = f^5 - if^6, \phi^4 = h^1 - ih^2, \dots, \phi^7 = h^7 - ih^8$.We get

$$\begin{aligned}
\alpha_2 &= -\tfrac{i}{2}(\phi^1\overline{\phi^1} + \phi^2\overline{\phi^2} + \phi^3\overline{\phi^3}),\\
\alpha_3 &= -\tfrac{1}{2}(\phi^1\phi^2\phi^3 + \overline{\phi^1\phi^2\phi^3}),\\
\alpha_3' &= \tfrac{i}{2}(-\phi^1\phi^2\phi^3 + \overline{\phi^1\phi^2\phi^3}),\\
\alpha_4 &= \tfrac{1}{4}(\phi^1\phi^2\overline{\phi^1\phi^2} + \phi^1\phi^3\overline{\phi^1\phi^3} + \phi^2\phi^3\overline{\phi^2\phi^3}).
\end{aligned}$$

Note that $\alpha_3 + i\alpha_3'$ is a $(0,3)$-form, and $\alpha_3 - i\alpha_3'$ is a $(3,0)$-form. We thus see that the double complex for the relative complex has the below form; compare this with [V08, Theorem 4.2], with $\omega = \alpha_2$, $\Omega = \alpha_3 - i\alpha_3'$.

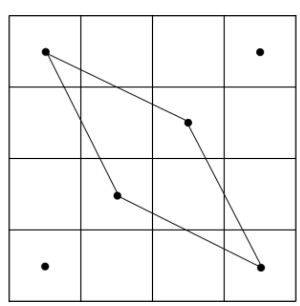

FIGURE 1. The complexified relative complex for the homogeneous nearly Kähler S^6.

The invariant Dolbeault cohomology is thus represented by the constant function 1, and the volume class α_6.

Example 3.3. Consider now the flag manifold $SU(3)/(U(1)\times U(1))$. The Lie algebra for $\mathfrak{su}(3)$ is given by the h_i entries in Table 1; we can take $\{h_1,h_2\}$ to be the subalgebra corresponding to $\mathfrak{u}(1)\oplus\mathfrak{u}(1)$. We have that the relative complex $C(\mathfrak{su}(3),\mathfrak{u}(1)\oplus\mathfrak{u}(1))$ is spanned in positive degrees by

$\alpha_2 := h^3h^4, \alpha_2' := h^5h^6, \alpha_2'' := h^7h^8$ in degree 2,

$\alpha_3 := h^3h^5h^7+h^3h^6h^8-h^4h^5h^8+h^4h^6h^7, \alpha_3' := h^3h^5h^8-h^3h^6h^7+h^4h^5h^7+h^4h^6h^8$ in degree 3,

$\alpha_2\alpha_2' = h^3h^4h^5h^6, \alpha_2\alpha_2'' = h^3h^4h^7h^8, \alpha_2'\alpha_2'' = h^5h^6h^7h^8$ in degree 4,

$\alpha_2\alpha_2'\alpha_2'' = h^3h^4h^5h^6h^7h^8$ in degree 6.

The almost complex structure determined by $Jh_1=-h_2, Jh_3=-h_4, Jh_5=-h_6, Jh_7=-h_8$ induces a left-invariant J on $SU(3)/(U(1)\times U(1))$ which is integrable (as follows from a direct computation of the Nijenhuis tensor). Taking the basis $\phi_1=h^1-ih^2, \phi_2=h^3-ih^4, \phi_3=h^5-ih^6, \phi_4=h^7-ih^8$ for $\Lambda^{1,0}(\mathfrak{su}(3)^*\otimes\mathbb{C})$, we have

$$\alpha_2 = -\tfrac{i}{2}\phi^2\overline{\phi^2},\ \alpha_2' = -\tfrac{i}{2}\phi^3\overline{\phi^3},\ \alpha_2'' = -\tfrac{i}{2}\phi^4\overline{\phi^4},$$
$$\alpha_3 = -\tfrac{1}{2}\left(\phi^2\phi^4\overline{\phi^3}+\phi^3\overline{\phi^2\phi^4}\right),\ \alpha_3' = \tfrac{i}{2}\left(\phi^3\overline{\phi^2\phi^4}-\phi^2\phi^4\overline{\phi^3}\right).$$

Note that $\beta=-\alpha_3+i\alpha_3'$ is of bidegree $(2,1)$, and $\{\beta,\overline{\beta}\}$ spans the degree 3 part of the relative complex. We calculate $d\alpha_2' = -d\alpha_2 = -d\alpha_2'' = \frac{1}{2}\left(\beta+\overline{\beta}\right)$, $d\alpha_3=0$, and $d\alpha_3' = 4(\alpha_2\alpha_2''-\alpha_2\alpha_2'-\alpha_2'\alpha_2'')$. Setting $\gamma = 4i(\alpha_2\alpha_2''-\alpha_2\alpha_2'-\alpha_2'\alpha_2'')$, we thus have $d\beta=\overline{\partial}\beta=\gamma$, and the relative complex is given by the following double complex:

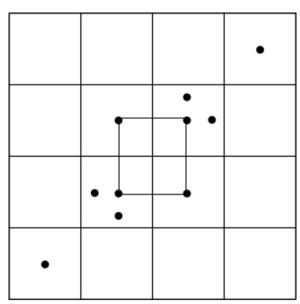

FIGURE 2. The double complex for the described invariant integrable structure on $SU(3)/\left(U(1)\times U(1)\right)$.

The invariant Dolbeault cohomology is thus spanned by $1, \alpha_2-\alpha_2', \alpha_2-\alpha_2'', \gamma-\alpha_2\alpha_2'', \gamma-\alpha_2'\alpha_2'', \alpha_6$.

Example 3.4. The flag manifold $SU(3)/(U(1)\times U(1))$ also carries an $SU(3)$-invariant nearly Kähler structure as the twistor space of $\mathbb{CP}^2$ equipped with the Eells–Salamon almost complex structure. We follow the description in [BILL10,

	f_1	f_2	f_3	f_4	f_5	f_6	h_1	h_2
f_1		$\frac{\sqrt{3}}{6}h_1 - \frac{1}{2}h_2$	$-\frac{1}{2\sqrt{3}}f_5$	$\frac{1}{2\sqrt{3}}f_6$	$\frac{1}{2\sqrt{3}}f_3$	$-\frac{1}{2\sqrt{3}}f_4$	$-\frac{1}{2\sqrt{3}}f_2$	$\frac{1}{2}f_2$
f_2			$\frac{1}{2\sqrt{3}}f_6$	$\frac{1}{2\sqrt{3}}f_5$	$-\frac{1}{2\sqrt{3}}f_4$	$-\frac{1}{2\sqrt{3}}f_3$	$\frac{1}{2\sqrt{3}}f_1$	$-\frac{1}{2}f_1$
f_3				$\frac{1}{2\sqrt{3}}h_1 + \frac{1}{2}h_2$	$-\frac{1}{2\sqrt{3}}f_1$	$\frac{1}{2\sqrt{3}}f_2$	$-\frac{1}{2\sqrt{3}}f_4$	$-\frac{1}{2}f_4$
f_4					$\frac{1}{2\sqrt{3}}f_2$	$\frac{1}{2\sqrt{3}}f_1$	$\frac{1}{2\sqrt{3}}f_3$	$\frac{1}{2}f_3$
f_5						$-\frac{1}{\sqrt{3}}h_1$	$\frac{1}{\sqrt{3}}f_6$	0
f_6							$-\frac{1}{\sqrt{3}}f_5$	0
h_1								0
h_2								

TABLE 2. Lie brackets for $\mathfrak{su}(3)$ and its Lie subalgebra $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ in the new basis given in [BILL10, (4.3), (4.4)].

Section 4]; the almost complex structure is easy to describe in a different basis for $\mathfrak{su}(3)$ than the one above, given by

$$f_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ f_2 = \frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ f_3 = \frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_4 = -\frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ f_5 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ f_6 = -\frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$h_1 = -\frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ h_2 = \frac{i}{6}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The Lie brackets are given in Table 2. The almost complex structure is given by (see [BILL10, (3.14)ff.])

$$Jf_1 = -f_2, \quad Jf_3 = -f_4, \quad Jf_5 = -f_6.$$

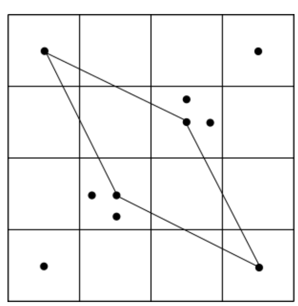

FIGURE 3. The relative complex for the homogeneous nearly Kähler structure on $SU(3)/(U(1) \times U(1))$.

The relative complex for $U(1) \times U(1) \subset SU(3)$ is spanned in degree two by $f^1 f^2, f^3 f^4, f^5 f^6$, and in degree three by

$$-f^1f^3f^5 + f^1f^4f^6 + f^2f^3f^6 + f^2f^3f^6 + f^2f^4f^5, \quad f^1f^3f^6 + f^1f^4f^5 + f^2f^3f^5 - f^2f^4f^6.$$

	f_1	f_2	f_3	f_4	f_5	f_6	h_1	h_2	h_3	h_4
f_1		$2f_5 - 2h_2$	$2f_6 - 2h_3$	$2h_1 - 2h_4$	$-f_2$	$-f_3$	$-f_4$	f_2	f_3	f_4
f_2			$2h_1 + 2h_4$	$-2f_6 - 2h_3$	f_1	f_4	$-f_3$	$-f_1$	f_4	$-f_3$
f_3				$2f_5 + 2h_2$	$-f_4$	f_1	f_2	$-f_4$	$-f_1$	f_2
f_4					f_3	$-f_2$	f_1	f_3	$-f_2$	$-f_1$
f_5						$2h_1$	$-2f_6$	0	0	0
f_6							$2f_5$	0	0	0
h_1								0	0	0
h_2									$2h_4$	$-2h_3$
h_3										$2h_2$
h_4										

TABLE 3. Lie brackets for $\mathfrak{sp}(2)$ and its Lie subalgebra $\mathfrak{u}(1) \oplus \mathfrak{sp}(1)$

Denoting $\phi^1 = f^1 - if^2, \phi^2 = f^3 - if^4, \phi^3 = f^5 - if^6$, from here we see that the degree two part of the complexified relative complex is concentrated in bidegree $(1,1)$, spanned by $\phi^1\bar{\phi}^1, \phi^2\bar{\phi}^2, \phi^3\bar{\phi}^3$.

We further calculate that a basis for the degree 3 part of the complexified relative complex is given by $\phi^1\phi^2\phi^3, \bar{\phi}^1\bar{\phi}^2\bar{\phi}^3$. This is enough to see that the complexified relative complex has the form displayed in Figure 3.

Example 3.5. We now consider the nearly Kähler $\mathbb{CP}^3 = Sp(2)/U(1)Sp(1)$, see [B10, Section 3]. Take the following basis for $\mathfrak{sp}(2)$, where $\{1, i, j, k\}$ is the standard basis for the quaternions:

$$f_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ f_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \ f_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \ f_5 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix},$$

$$f_6 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \ h_1 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \ h_2 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \ h_3 = \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \ h_4 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.$$

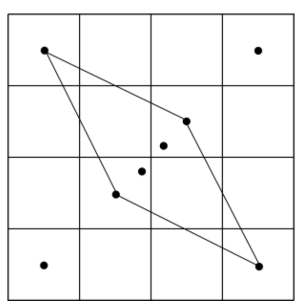

FIGURE 4. The relative complex for the homogeneous nearly Kähler structure on $\mathbb{CP}^3 = Sp(2)/\left(U(1)Sp(1)\right)$.

Here h_1 spans a copy of $\mathfrak{u}(1)$, and h_1, h_2, h_3, h_4 span a Lie subalgebra isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{sp}(1)$. The relative complex for $(\mathfrak{sp}(2), \mathfrak{u}(1) \oplus \mathfrak{sp}(1))$ is spanned by $f^5f^6, f^1f^4 + f^2f^3$ in degree 2, and by

$$f^1f^2f^5 + f^1f^3f^6 - f^2f^4f^6 + f^3f^4f^5, \quad f^1f^2f^6 - f^1f^3f^5 + f^2f^4f^5 + f^3f^4f^6$$

in degree 3. From here we can argue abstractly what the complexified relative complex must look like for any homogenous nearly Kähler structure on this structure.

Namely, as we have already noted before, by [V08, Theorem 4.2], there must be a "diamond" in the complexified relative complex, with vertices in bidegrees $(1,1),(0,3),(3,0),(2,2)$. From here, since the relative complex is two-dimensional in both degrees 2 and 3, and since it has a real structure, there must be one more dot in bidegree $(1,1)$, and the additive structure of the complex is given in Figure 4.

Compare with Example 3.4, where it was necessary to fix J and compute the complexified relative complex in order to determine Figure 3, since the two dots in total degree 2 not participating in the "diamond" could a priori have lied in bidegrees $(2,0)$ and $(0,2)$.

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