AN EXAMPLE OF A SMOOTH MANIFOLD WITH TWO HOMOTOPIC ALMOST COMPLEX STRUCTURES, ONLY ONE OF WHICH IS INTEGRABLE

Abstract. We give an example of a nilmanifold and two explicit almost complex structures on it. These almost complex structures are connected by a path of such structures on the manifold, though only one of them is integrable.

Consider the simply-connected Lie group G whose Lie algebra is spanned by the vectors X, Y, Z, W with $[X,Y] = Z$ and $[X,Z] = [X,W] = [Y,Z] = [Y,W] = [Z,W] = 0$. Since the structure constants are rational numbers, by a criterion of Malcev [1] we have that there exists a cocompact subgroup Γ of G such that the quotient G/Γ is a smooth 4–manifold M. This manifold is parallelizable, with global vector fields that descend from G; we denote these vector fields by X, Y, Z, W as well.

We define two almost complex structures J_0 and J_1 on M by

$$J_0X = Y,\ J_0Y = -X,\ J_0Z = W,\ J_0W = -Z,$$
$$J_1X = Z,\ J_1Y = W,\ J_1Z = -X,\ J_1W = -Y.$$

Note that $J_0^2 = J_1^2 = -\mathrm{Id}$ and that J_0 and J_1 induce the same orientation on M, namely the orientation provided by the 4–form $xyzw$ composed of 1–forms dual to the spanning vector fields. Indeed,

$$J_0(xyzw) = yxwz = xyzw,$$
$$J_1(xyzw) = zwxy = xyzw.$$

A direct evaluation of the Nijenhuis tensor N shows that J_0 is integrable, the only not-immediately-trivial equation being

$$\begin{aligned} N_{J_0}(X,Y) &= [X,Y] + J_0[J_0X,Y] + J_0[X,J_0Y] - [J_0X,J_0Y] \\ &= Z - Z = 0, \end{aligned}$$

and J_1 is not integrable since

$$N_{J_1}(X,W) = -W \neq 0.$$

Now we observe that J_0 and J_1 are connected by a path of almost complex structures on M. It suffices to find a path connecting J_0 and J_1 at a point $p \in M$, from which the parallelizability of the manifold gives us a global path. The space of endomorphisms of $T_pM \cong \mathbb{R}^4$ squaring to $-\mathrm{Id}$ has the homotopy type of two copies of $SO(4)/U(2) \cong S^2$. The two copies correspond to those endomorphisms preserving and reversing the orientation induced on T_pM respectively. Since both J_0 and J_1 are orientation-preserving on M, it follows that they are connected by a path of such endomorphisms.

Remark 0.1. In real dimension 2, any almost-complex structure is also integrable. So the above example is in the lowest possible dimension in which the exhibited phenomenon can occur. To obtain general $2n$–dimensional examples, we can simply take the Lie algebra spanned by $X_1, X_2, \ldots, X_{2n}$ with Lie bracket given by $[X_1, X_2] = X_3$ and all other brackets among spanning vectors equal to zero. Invoking Malcev's criterion on the simply connected Lie group with this as its Lie algebra, we obtain a closed smooth parallelizable $2n$–manifold with global vector fields whose brackets are given by the above. Then the almost complex structure J_0 on this manifold defined by $J_0X_1 = X_2, J_0X_2 = -X_1, J_0X_3 = X_4, J_0X_4 = -X_3, \ldots$ is integrable, while J_1 given by

$$J_1X_1 = X_3, J_1X_2 = X_4, J_1X_3 = -X_1, J_1X_4 = -X_2, J_1X_5 = X_6, J_1X_6 = -X_5, \ldots$$

is not integrable. They both induce the orientation on the manifold given by the product of dual 1–forms $x_1x_2 \cdots x_{2n}$, and so lie in the same connected component of $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.

References

[1] Malcev, A., 1951. On a class of homogeneous spaces. Izvestiya Akad. Nauk SSSR Set. Math. 13 (1949). AMS Translation, (39).