THE BETTI NUMBERS OF A COMPLEX MANIFOLD GENERALLY DO NOT BOUND THE HODGE NUMBERS

Abstract. We consider differences of the form $(h^{k,0} + h^{k-1,1} + \cdots + h^{0.k}) - b_k$ between Hodge numbers and Betti numbers for complex manifolds. For complex curves and surfaces, these differences are always zero. In any fixed dimension three or greater, these differences can be arbitrarily large. We conclude with a relative bound between these differences on an arbitrary complex threefold.

1. A sequence of complex nilmanifolds

On a closed complex manifold we can consider the Betti numbers b_k and the Hodge numbers $h^{i,j} = \dim_{\mathbb{C}} H^{i,j}_{\bar\partial}(X)$. Denote $h_k = \sum_{i+j=k} h^{i,j}$. We consider the question of whether there is a universal bound on the differences $h_k - b_k$ among all complex manifolds. Since the Frölicher spectral sequence of a given complex manifold converges to the (complexified) de Rham cohomology, and the dimensions of its entries on the first page are the Hodge numbers, we have that $(\sum_{i+j=k} h^{i,j}) - b_k$ is non-negative for all k. We first give an easy construction of a sequence of complex nilmanifolds of increasing dimensions such that $h_1 - b_1$ is unbounded. Next we state a construction of LeBrun [1] from which we obtain sequences of complex manifolds in any fixed dimension such that the differences $h_2 - b_2$ are unbounded.

Consider the following sequence X_n of complex nilmanifolds of dimension n, for $n \geq 3$, assigned via their minimal models,

$$\begin{aligned}\mathrm{Model}(X_n) = \Lambda(&x, \bar x, y, \bar y, z_1, \bar z_1, \ldots, z_{n-2}, \bar z_{n-2},\\ &dx = dy = 0, dz_1 = xy, dz_2 = xz_1, \ldots, dz_{n-2} = xz_{n-3}).\end{aligned}$$

The generators $x, y, z_1, \ldots, z_{n-2}$ are of bidegree $(1,0)$ and their conjugates are of bidegree $(0,1)$. We immediately see from the model that $b_1(X_n) = 4$ for all n. Computing $\bar\partial$ cohomology, we have

$$\begin{aligned} H^{1,0}_{\bar\partial}(X_n) &= \mathrm{span}(x, y, z_1, \ldots, z_{n-2}),\\ H^{0,1}_{\bar\partial}(X_n) &= \mathrm{span}(\bar x, \bar y),\end{aligned}$$

and so $h^{1,0}(X_n) + h^{0,1}(X_n) - b_1(X_n) = n$.

Let us note that the Frölicher spectral sequence for all of the above complex nilmanifolds degenerates on the second page (see [1, Theorem 9]). (Degeneration does not happen on the first page since the $(2,0)$-form xy is $\bar\partial$-closed and ∂-exact, but not $\bar\partial$-exact.) So (non-)degeneracy of the Frölicher spectral sequence generally gives no bound on the differences $h_k - b_k$.

2. Unbounded differences in any fixed dimension

A harder question is whether the differences $h_k - b_k$ are bounded among all closed complex manifolds *of a fixed dimension*. A construction of LeBrun provides a sequence of complex threefolds X_m such that $h_2 - b_2 \geq 2m - 24$. Indeed, consider the following:

Theorem 2.1. *(LeBrun [2], Theorem 1) For any $m > 0$ there is a complex threefold X_m with underlying smooth manifold $K3 \times S^2$ satisfying*

$$\int_{X_m} c_1 c_2 = 48m.$$

Applying Hirzebruch-Riemann-Roch to the constant sheaf $\mathcal{O}$, we have

$$\chi(X_m, \mathcal{O}) = 1 - h^{0,1} + h^{0,2} - h^{0,3} = \int_{X_m} \frac{c_1 c_2}{24} = 2m,$$

and so

$$h^{2,0} + h^{1,1} + h^{0,2} \geq h^{0,2} \geq 2m + h^{0,1} + h^{0,2} - 1 \geq 2m - 1.$$

On the other hand,

$$b_2(X_m) = b_2(K3 \times S^2) = b_1(K3)b_1(S^2) + b_2(K3) + b_2(S^2) = 23,$$

and so the difference $h_2 - b_2$ on X_m is greater than $2m - 24$.

To obtain a sequence of complex n-folds Y_m with unbounded increasing $h_2 - b_2$ for $n \geq 4$, simply take $Y_m = X_m \times \mathbb{CP}^1$ with product complex structure. Then we have $b_2(X_m \times \mathbb{CP}^1) = 24$, and by the Künneth formula for $\bar{\partial}$-cohomology on a product complex manifold,

$$h^{2,0}(Y_m) = h^{2,0}(X_m) + h^{2,0}(\mathbb{CP}^1) + h^{1,0}(X_m)h^{1,0}(\mathbb{CP}^1) = h^{2,0}(Y_m).$$

Therefore, we have

$$h_2 - b_2 \geq h^{2,0}(X_m) - 24 \geq 2m - 25$$

on the complex n-fold Y_m.

3. A relative bound on complex threefolds

We carry out a calculation showing that on a complex threefold X, we have

$$h_2 - b_2 = h_1 - b_1 + \tfrac{1}{2}(h_3 - b_3).$$

Recall that all of the numbers $h_k - b_k$ are non-negative, and so this equation gives us that a bound on $h_2 - b_2$ also bounds the other two differences.

We compute the Euler characteristics of the constant sheaf $\mathcal{O}$ and the sheaf of one-forms Ω on an arbitrary threefold X using Hirzbruch-Riemann-Roch. First, we have as before

$$\chi(X, \mathcal{O}) = 1 - h^{0,1} + h^{0,2} - h^{0,3} = \int_X \mathrm{Td}(TX)\mathrm{ch}(\mathcal{O}) = \int_X \tfrac{1}{24} c_1(TX)c_2(TX).$$

For the sheaf of one-forms, we have

$$\chi(X, \Omega) = \sum_{i=0,1,2,3} (-1)^i \mathrm{dim}(H^i(X, \Omega)) = \sum_{i=0,1,2,3} (-1)^i \mathrm{dim}(H^{1,i}(X))$$
$$= h^{1,0} - h^{1,1} + h^{1,2} - h^{1,3} = \int_X \mathrm{Td}(TX)\mathrm{ch}(\Omega).$$

Let us denote the Chern classes of the tangent bundle $c_i(TX)$ simply by c_i. Now using the splitting principal, we suppose we have a splitting of the tangent bundle $TX = A \oplus B \oplus C$ into line bundles whose first Chern classes we denote by a, b, c. Then $T^*X = A^* \oplus B^* \oplus C^*$, and so

$$\mathrm{ch}(T^*X) = e^{-a} + e^{-b} + e^{-c} = 3 - (a + b + c) + \tfrac{1}{2}(a^2 + b^2 + c^2) - \tfrac{1}{6}(a^3 + b^3 + c^3).$$

Since $c_1(TX) = a + b + c$, $c_2(TX) = ab + ac + bc$, and $c_3(TX) = abc$, we have

$$\mathrm{ch}(T^*X) = 3 - c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) - \tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3).$$

Hirzebruch-Riemann-Roch now gives us

$$\begin{aligned}
\chi(X, \Omega) &= \int_X \mathrm{Td}(X)\mathrm{ch}(T^*X) \\
&= \int_X \big(1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) + \tfrac{1}{24}c_1c_2\big)\big(3 - c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) - \tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3)\big) \\
&= \int_X \tfrac{1}{24}c_1c_2 - \tfrac{1}{2}c_3.
\end{aligned}$$

Therefore $\chi(X, \Omega) = \chi(X, \mathcal{O}) - \frac{1}{2}\chi$ (where χ denotes the topological Euler characteristic), and so

$$\begin{aligned}
h^{1,0} - h^{1,1} + h^{1,2} - h^{1,3} &= 1 - h^{0,1} + h^{0,2} - h^{0,3} - \tfrac{1}{2}(1 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6) \\
&= 1 - h^{0,1} + h^{0,2} - h^{0,3} - \tfrac{1}{2}(2 - 2b_1 + 2b_2 - b_3).
\end{aligned}$$

From here we have

$$h^{1,0} + h^{0,1} - b_1 + h^{1,2} + h^{0,3} - \tfrac{1}{2}b_3 = h^{1,1} + h^{1,3} + h^{0,2} - b_2.$$

Since $h^{3,0} = h^{3,0}, h^{2,1} = h^{2,1}$, and $h^{1,3} = h^{2,0}$ by Serre duality, we can rewrite $h^{1,2} + h^{0,3} - \frac{1}{2}b_3$ as $\frac{1}{2}(h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3} - b_3$ and we have the equation

$$(h_1 - b_1) + \tfrac{1}{2}(h_3 - b_3) = h_2 - b_2$$

as desired.

Example 3.1. Consider the Iwasawa complex nilmanifold X_3, given by its minimal model

$$\Lambda(x, \bar{x}, y, \bar{y}, z, \bar{z},\ dx = dy = 0, dz = xy).$$

We compute some of its de Rham and Dolbeault cohomology,

$$\begin{aligned}
H^1_{dR} &= \mathrm{span}(x, \bar{x}, y, \bar{y}), \\
H^2_{dR} &= \mathrm{span}(xz, yz, x\bar{x}, x\bar{y}, y\bar{x}, y\bar{y}, \bar{x}, \bar{z}, \bar{y}, \bar{z}), \\
H^3_{dR} &= \mathrm{span}(xyz, xz\bar{x}, xz\bar{y}, yz\bar{x}, yz\bar{y}, \bar{x}\bar{z}x, \bar{x}\bar{z}y, \bar{y}\bar{z}x, \bar{y}\bar{z}y, \bar{x}\bar{y}\bar{z}), \\
H^{1,0}_{\bar{\partial}} &= \mathrm{span}(x, y, z), \\
H^{0,1}_{\bar{\partial}} &= \mathrm{span}(\bar{x}, \bar{y}), \\
H^{2,0}_{\bar{\partial}} &= \mathrm{span}(xy, xz, yz), \\
H^{1,1}_{\bar{\partial}} &= \mathrm{span}(x\bar{x}, x\bar{y}, y\bar{x}, y\bar{y}, z\bar{x}, z\bar{y}), \\
H^{0,2}_{\bar{\partial}} &= \mathrm{span}(\bar{x}\bar{z}, \bar{y}\bar{z}), \\
H^{3,0}_{\bar{\partial}} &= \mathrm{span}(xyz), \\
H^{2,1}_{\bar{\partial}} &= \mathrm{span}(xy\bar{x}, xy\bar{y}, xz\bar{x}, xz\bar{y}, yz\bar{x}, yz\bar{y}).
\end{aligned}$$

The remaining Betti and Hodge numbers are determined by Poincaré and Serre duality. We have

$$h_1 - b_1 = 1,$$
$$h_2 - b_2 = 3,$$
$$h_3 - b_3 = 4.$$

References

[1] Cordero, L.A., Fernández, M. and Gray, A., 1991. The Frölicher spectral sequence for compact nilmanifolds. Illinois Journal of Mathematics, 35(1), pp.56-67.

[2] LeBrun, C., 1999. Topology versus Chern numbers for complex 3-folds. Pacific Journal of Mathematics, 191(1), pp.123-131.