TOPOLOGY – ON THE EXTENSION OF THE STRUCTURE GROUP OF A FIBER SPACE.

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ABSTRACT. The problem of the existence of an extension of a fiber space relative to an extension of structure groups is reduced to the problem of the existence of a section in an associated fiber space; we are thus led to an obstruction which is explicitly determined in some particular cases.

Let $E(B,G)$ and $E'(B,H)$ be two principal fiber spaces with topological structure groups G and H, and the same base B. We say that E' is an extension of E if there exists a representation[1] φ from E' to E which projects to the identity on B; such a representation determines a homomorphism φ from H to G. For every principal fiber space[2] $E'(B,H)$ and for every homomorphism φ from H to G, there is a corresponding canonical fiber space $E(B,G)$ of which E' is an extension (which will be said to be associated to φ). Conversely, given a fiber space $E(B,G)$ and a homomorphism φ from H to G, Mr. Ehresmann ([1]) posed the problem of the existence of an extension of E associated to φ ([2]).

In what follows, we will only consider groups G for which there exists a universal fiber space EG with base BG, and fiber spaces whose base is a cellular complex. We will further assume that φ determines on H a fibered structure with base G.

The method used here is based on the following remark made by A. Borel and J.P. Serre ([3]): the homomorphism φ determines a homotopy class of maps ρ from B_H to B_G, and we can choose B_H, B_G, and ρ (which are defined up to homotopy) so that B_H is a fiber space with basis B_G, projection ρ, and fiber B_N, where N is the kernel of the homomorphism ρ. Indeed, the quotient E_H/N of E_H by the equivalence relation defined by the action of N is none other than B_N; we can then construct the fibered space associated with E_G of fiber E_H/N on which G acts naturally; it has the same homotopy type as B_N and its projection is identified with ρ.

The fiber space $E(B,G)$ is determined by a map f from B to B_G; for there to exist an extension of E associated with ρ, it is necessary and sufficient that f admit a lift $\overline{f}\colon B \to B_H$ relative to ρ (that is to say, $\rho\overline{f} = f$) or, which amounts to the same thing, that the fiber space induced by f from $B_H(B_6, B_N)$ has a section. Two lifts define equivalent extensions if and only if they are homotopic.

We now turn to the study of some special cases.

1. *H is connected and N is a discrete subgroup of H.* – N is then abelian and H is a covering of G with respect to φ. The classifying space B_N has the same homotopy type as an Eilenberg–MacLane complex $K(N,1)$; let $\alpha \in H^2(B_G, N)$ be the first obstruction to constructing a section of the fiber space $\rho\colon B_H \to B_G$. The map $f\colon B \to B_G$ admits a lift relative to ρ if and only if $f^*(\alpha) = 0$, where f^* is the homomorphism $H^2(B_G,N) \to H^2(B,N)$ induced by f. If $H^1(B,N) = 0$, two lifts of f are always homotopic, and all extensions are equivalent.

The covering $\varphi\colon H \to G$ is determined by a homomorphism ψ from the fundamental group π of G to N. The group $\mathrm{Hom}(\pi, N)$ is identified with $H^1(G,N)$ [resp. with $H^2(B_G,N)$] and ψ corresponds to a class $\omega \in H^1(G,N)$ called the fundamental class of the covering [resp. it corresponds to α]. We then have the

Proposition 1. – *Let $\varphi\colon H \to G$ be a connected covering of a topological group G, and N the kernel of φ; for a fiber space $E(B,G)$ to admit an extension associated to φ, it is necessary and sufficient*

[1] A map of vector bundles

[2] There is a typo here in the original paper; instead of $E'(B,G)$ as written in the paper, it should be $E'(B,H)$ as written above.

that a certain characteristic class belonging to $H^2(B,N)$ *vanishes* ([2])*; this class is the image under the transgression in* $E(B,G)$ *of the fundamental class* $\omega \in H^1(G,N)$ *of the covering* $H \to G$.

Corollary 1. – *To be able to extend the structure group of a fiber space* $E(B,\mathrm{SO}(n))$ *to* $\mathrm{Spin}(n)$*, it is necessary and sufficient that the Stiefel–Whitney class* $w^2 \in H^2(B,Z_2)$ *is zero.*

As a consequence, for example, one cannot globally define spinors on the complex projective plane.

Corollary 2. – *To be able to extend the structure group* $U(n)$ *of a fiber space* $E(B,U(n))$ *to its universal covering [resp. to a covering with* m *sheets], it is necessary and sufficient that the Chern class* $c^2 \in H^2(B,Z)$ *[resp.* c^2 *reduced mod* m*] is zero.*

2. *H is a compact connected Lie group.* – The extension H of G by N is determined by a class of homomorphisms of the fundamental group π of G to the center of N ([4]); we can always choose a homomorphism ψ in this class such that the image $\psi(\pi)$ is a finite subgroup $\tilde{N}$ of N. There is a covering $\tilde{G}$ of G associated to ψ, and $\tilde{G}$ is identified with a subgroup of H; if we can extend the structure group G of $E(B,G)$ to $\tilde{G}$, we can also extend it to H by the inclusion of G in H, and proposition 1 gives

Proposition 2. – *The vanishing of a certain characteristic class* $\alpha \in H^2(B,\tilde{N})$ *is a necessary and sufficient condition for there to exist an extension of* $E(B,G)$ *associated to* φ.

If N is connected and abelian, it is a torus T^m (a product of m circles); B_N therefore has the homotopy type of a $K(Z^m,2)$, where Z^m denotes the direct sum of m free cyclic groups. The extension is possible if and only if an obstruction class $\beta \in H^3(B,Z^m)$ is zero. We can show that it is equal, up to sign, to $\delta\alpha$, where δ denotes the connecting homomorphism of the exact sequence in cohomology of B associated to the exact sequence of coefficients $0 \to \pi_1(T^m) \to \pi_1(T^m/\tilde{N}) \to \tilde{N} \to 0$.

Note that if H is a connected Lie group, we can reduce to case 2 by considering the maximal compact of H.

(*) Session of 30 July 1955.

(1) C. Ehresmann, *Colloque de Topologie*, Bruxelles, 1950, p. 51.

(2) Cf. the article (1) and J. Frenckel, *Comptes rendus*, 240, 1955, p.2368; P. Dedecker, *Colloque de Topologie*, Strasbourg, 1955; A. Grothendieck, *A general theory of fiber spaces with structure sheaf*, University of Kansas, 1955.

(3) Amer. J. Math., 75, 1953, p. 410-412.

(4) A. Shapiro, *Ann. Math.*, 50, 1949, p. 581-586.