# THE RATIONAL HOMOTOPY TYPE OF THE CLASSIFYING SPACE OF SELF-HOMOTOPY EQUIVALENCES

Abstract. We review differential graded Lie algebras and calculate the rational homotopy type of the classifying space of self-homotopy equivalences for some spaces, following Infinitesimal Computations in Topology. Examples include complex projective space, spheres, Eilenberg–MacLane spaces and products thereof, $SO(8)/U(4)$, and a simply-connected manifold with a nontrivial triple Massey product.

## 1. Differential graded Lie algebras

We recall that a differential graded Lie algebra (dgla) is a graded vector space $\bigoplus_{i\geq 0} V^i$ with a bilinear bracket $V^i \otimes V^j \xrightarrow{[-,-]} V^{i+j}$ and differential $V^i \xrightarrow{d} V^{i-1}$ satisfying

- $[y,x] = (-1)^{xy+1}[x,y]$,
- $(-1)^{xz}[x,[y,z]] + (-1)^{zy}[z,[x,y]] + (-1)^{yx}[y,[z,x]] = 0$,
- $d[x,y] = [dx,y] + (-1)^x[x,dy]$.

The differential here is of degree $-1$ and the bracket of degree 0; the definition can be generalized to accommodate different degrees for the differential and bracket, but these are the most common degrees and we focus on them only.

The type of dgla we will be interested in will be obtained in the following manner: Start with a dga $(A,d)$ modelling the rational homotopy type of some space $X$. We associate to $(A,d)$ a dgla $(L,D)$ by setting $L^k$ to be the vector space of derivations on $A$ of degree $-k$. Recall that $x$ is a derivation of degree $k$ if $x(ab) = (xa)b + (-1)^{ka}x(ab)$, i.e. derivations in general satisfy the formula

$$x(ab) = (xa)b + (-1)^{xa}x(ab).$$

We define a bracket on $LA$ by setting

$$[x,y] = xy - (-1)^{xy}yx,$$

where $xy$ denotes the composition of $x$ and $y$.

Note that the first bullet point above is satisfied; indeed,

$$[y,x] = yx - (-1)^{xy}xy = -(-1)^{xy}(xy - (-1)^{xy}yx) = (-1)^{xy+1}[x,y].$$

**Lemma 1.1.** *The bracket $[x,y] = xy - (-1)^{xy}$ applied to $x \in (LA)^k$ and $y \in (LA)^l$ is an element in $(LA)^{k+l}$, i.e. it sends a pair of derivations to a derivation and has degree 0 as a bilinear map $LA \otimes LA \to LA$.*

*Proof.* Take $a, b \in A$, and consider $[x, y](ab)$. We have

$$\begin{aligned}
[x, y](ab) &= (xy)(ab) - (-1)^{xy}(yx)(ab) \\
&= x((ya)b + (-1)^{ya}ayb) - (-1)^{xy}y((xa)b + (-1)^{xa}axb) \\
&= x((ya)b) + (-1)^{ya}x(ayb) - (-1)^{xy}y((xa)b) - (-1)^{xy+xa}y(axb) \\
&= (x(ya))b + (-1)^{x(y+a)}(ya)\cdot(xb) + (-1)^{ya}(xa)\cdot(yb) + (-1)^{ya+xa}a\cdot x(yb) \\
&\quad - (-1)^{xy}(y(xa))\cdot b - (-1)^{xy+y(x+a)}(xa)\cdot(yb) - (-1)^{xy+xa}(ya)\cdot(xb) \\
&\quad - (-1)^{xy+xa+ya}a\cdot y(xb) \\
&= (x(ya))\cdot b + (-1)^{ya+xa}a\cdot(x(yb)) - (-1)^{xy}(y(xa))\cdot b - (-1)^{xy+xa+ya}a\cdot y(xb) \\
&= (x(ya))\cdot b - (-1)^{xy}(y(xa))\cdot b + (-1)^{(x+y)a}(a\cdot x(yb) - (-1)^{xy}a\cdot y(xb)) \\
&= ((xy - (-1)^{xy}yx)a)\cdot b + (-1)^{(x+y)a}a\cdot(xy - (-1)^{xy}yx)b \\
&= ([x, y]a)\cdot b + (-1)^{x+y}a\cdot[x, y]b.
\end{aligned}$$

□

Next we check that the bracket $[-,-]$ is indeed a bracket, i.e. that it satisfies the graded Jacobi identity under the second bullet point above.

**Lemma 1.2.** *The bracket $[-,-]$ on $LA$ satisfies the graded Jacobi identity.*

*Proof.* For $x, y, z \in LA$ (taking $x, y, z$ to be homogeneous, always) we have

$$\begin{aligned}
&(-1)^{xz}[x,[y,z]] + (-1)^{zy}[z,[x,y]] + (-1)^{yx}[y,[z,x]] \\
&= (-1)^{xz}xyz - (-1)^{xy}yzx - (-1)^{yz+xz}xzy + (-1)^{xy+yz}zyx \\
&\quad + (-1)^{yx}yzx - (-1)^{yz}zxy - (-1)^{zx+yx}yxz + (-1)^{yz+zx}xzy \\
&\quad (-1)^{zy}zxy - (-1)^{zx}xyz - (-1)^{xy+zy}zyx + (-1)^{zx+xy}yxz \\
&= 0.
\end{aligned}$$

The terms in the last expression cancel out pairwise. □

We now place a differential $D$ on $LA$ to make it into a dgla. For a homogeneous element $x$ we set

$$Dx = dx - (-1)^{x}xd = [d, x].$$

As we saw above, $Dx$ is a derivation if $x$ is, and its degree is $-1$ (since $d$ is of degree $+1$). Now we verify that $D$ satisfies the graded Leibniz rule and that $D^2 = 0$.

**Lemma 1.3.** *The operator $D$ satisfies the graded Leibniz rule, i.e. $D[x, y] = [Dx, y] + (-1)^{x}[x, Dy]$.*

*Proof.* On the one hand, we have

$$\begin{aligned}
D[x, y] &= d[x, y] - (-1)^{x+y}[x, y]d \\
&= d(xy - (-1)^{xy}yx) - (-1)^{x+y}(xy - (-1)^{xy}yx)d \\
&= dxy - (-1)^{xy}dyx - (-1)^{x+y}xyd + (-1)^{xy+x+y}yxd.
\end{aligned}$$

On the other hand,

$$\begin{aligned}[Dx,y]+(-1)^x[x,Dy] &= [dx-(-1)^x xd,y]+(-1)^x[x,dy-(-1)^y yd]\\ &= [dx,y]-(-1)^x[xd,y]+(-1)^x[x,dy]-(-1)^{x+y}[x,yd]\\ &= dxy-(-1)^{(x+1)y}ydx-(-1)^x xdy+(-1)^{x+(x+1)y}yxd\\ &\quad+(-1)^x xdy-(-1)^{x+x(y+1)}dyx-(-1)^{x+y}xyd\\ &\quad+(-1)^{x+y+x(y+1)}ydx\\ &= dxy-(-1)^{xy}dyx-(-1)^{x+y}xyd+(-1)^{x+y+xy}yxd.\end{aligned}$$

□

**Lemma 1.4.** *The derivation $D$ is a differential, i.e. $D^2=0$*

*Proof.* Recall that $D=[d,-]$. For any homogeneous derivation $x$ by the graded Jacobi rule we have

$$(-1)^x[d,[d,x]]+(-1)^x[x,[d,d]]-[d,[x,d]]=0.$$

Note that $[d,d]=d^2+d^2=0$ and $[x,d]=(-1)^{x+1}[d,x]$, so the above equation becomes

$$(-1)^x[d,[d,x]]+(-1)^{x+2}[d,[d,x]]=0,$$

i.e. $2(-1)^x[d,[d,x]]=0$. □

Note that in $LA$ we have elements of negative degree (for example, $d$). We remove all negative degree elements from $LA$ (by setting $(LA)^{-k}=0$) and in degree 0 only take those derivations that are $D$-closed. Now $(LA,D)$ is truly a dgla in accordance with the above definition.

## 2. The model for $BAut(X)$

Now for any nilpotent space $X$ with model $(A,d)$, we have an associated dgla $(LA,D)$. Given the dgla $LA$, we can form a dga $\Lambda LA$ by the following standard construction: The degree $k+1$ algebra generators of $\Lambda LA$ are (some choice of) degree $k$ vector space generators of $LA$, and the differential $\bar{D}$ is defined to have only a linear term and a quadratic term. The linear term is the dual of $D$, and the quadratic term is the dual of $[-.-]$. Note that the bracket $[-,-]$ becomes a degree $-1$ operator with the shift of degrees, and so its dual is a degree $+1$ quadratic operator.

For the following, we make a more careful choice of algebra generators for $\Lambda LA$. As a basis for the vector space of $(LA)^k$, we split $(LA)^k$ into $\ker D\oplus\overline{\operatorname{im} D^*}$ and choose a basis that respects this splitting.

**Lemma 2.1.** *The homology of $(LA,D)$ is the homotopy of $(\Lambda LA,\bar{D})$.*

*Proof.* Indeed, a generator $g$ in $\Lambda LA$ has no linear part in its differential if and only if $D^*g=0$, which is (by the choice of basis made above) equivalent to $g\notin\operatorname{im} D$. The generator $g$ is not the linear part of $\bar{D}$ of some other element in $\Lambda LA$ if and only if $g\notin\operatorname{im} D^*$, which is equivalent (again by the choice of basis) to $g\in\ker D$. □

Another dga of interest alongside $\Lambda LA$ will be $\widetilde{\Lambda LA}$, which is obtained from $\Lambda LA$ by setting all the degree 0 elements to 0 and taking only the $D$-closed elements in degree 1. (The elements in higher degrees remain the same.)

**Theorem 2.2** (Infinitesimal Computations in Topology, Chapter 11)**.** *The dga $\Lambda LA$ is a model for $BAut(X)$, where $A$ models $X$ and $Aut(X)$ denotes the group of homotopy equivalences $X \to X$. Denoting the connected component of the identity map in $Aut(X)$ by $Aut_1(X)$, we have that a model for $BAut_1(X)$ is given by $\widetilde{\Lambda LA}$.*

Note that the space $BAut_1(X)$ is simply connected, and is in fact the universal cover of $BAut(X)$. Fibrations with fiber $X$ up to fiber homotopy equivalence are classified by $BAut(X)$, while $BAut_1(X)$ classifies *oriented* $X$-fibrations.

**Example 2.3.** We find the minimal model of $BAut_1(\mathbb{CP}^n)$ and observe that it is isomorphic to that of $BPGL(n+1,\mathbb{C})$, whence we conclude that the problem of classifying $\mathbb{CP}^n$-bundles is rationally the same as that of classifying principal $PGL(n+1,\mathbb{C})$-bundles.

We start by taking the minimal model $A = \Lambda(x_2, y_{2n+1}; dx = 0, dy = x^{n+1})$ for $\mathbb{CP}^n$. Note that by the Leibniz rule, a derivation is determined by its value on the generators $x, y$. For convenience we consider the degree 0 element 1 as a generator as well. Observe that any homogeneous derivation is of the following form: it takes one of these three generators to some prescribed admissible value, and the other two generators to 0. We denote (same as in Infintesimal) by $(a, b)$ the derivation taking $a$ to $b$, and taking $b$ and taking every unmentioned generator to 0.

In degree 1, we have the derivation $(y, x^n)$. In degree 2, we have $(x, 1)$; in degree 3 we have $(y, x^{n-1})$. In general in even degrees $\geq 4$ there are no derivations, and in degrees $3, 5, \ldots, 2n-1, 2n+1$ we have the derivations $(y, x^{n-1})$, $(y, x^{n-2})$, …, $(y, x)$, $(y, 1)$. We calculate the differential $D$ and the brackets $[-,-]$.

Note that the derivations in degrees $1, 5, 7, 9, \ldots, 2n+1$ are closed by degree reasons. Next, in degree 2 we have $(x, 1)$ and

$$D(x,1)x = (d(x,1) - (x,1)d)x = 0,$$
$$D(x,1)y = (d(x,1) - (x,1)d)y = -(x,1)x^{n+1} = -(n+1)x^n(x,1)x = -(n+1)x^n.$$

Therefore, $D(x,1) = -(n+1)(y, x^n)$. Since $(x,1)$ spans the derivations in degree 2, we conclude that $D(y, x^{n-1}) = 0$ since $D(x,1) \neq 0$.

Now we calculate the brackets. Note that for degree reasons, we only have to calculate brackets of the form $[(x,1),(y,x^k)]$. We have

$$[(x,1),(y,x^k)]x = (x,1)(y,x^k)x - (y,x^k)(x,1)x = 0,$$
$$[(x,1),(y,x^k)]y = (x,1)(y,x^k)y - (y,x^k)(x,1)y = (x,1)x^k = kx^{k-1}.$$

In the first equation above we used that a derivation $\varphi$ that takes $x^k$ to 0 also takes $x$ to 0. Indeed $0 = \varphi(x^k) = kx^{k-1}\varphi(x)$, and so we conclude $\varphi(x) = 0$ because the algebra is free. Therefore

$$[(x,1),(y,x^k)] = k(y,x^{k-1}).$$

A model for $\widetilde{\Lambda LA}$ is now given by

$$\begin{aligned}&\Lambda(\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_8, \dots, \alpha_{2n+2};\\ &\quad \bar{D}\alpha_2 = -(n+1)\alpha_3, \bar{D}\alpha_4 = \alpha_3\alpha_2, \bar{D}\alpha_6 = \alpha_3\alpha_4, \dots, \bar{D}\alpha_{2n+2} = \alpha_3\alpha_{2n}).\end{aligned}$$

By setting $\beta_4 = \alpha_4 - \frac{1}{2}\alpha_2^2$, $\beta_6 = \alpha_6 - \alpha_4\alpha_2$, ..., we see that the minimal model of $\widetilde{\Lambda LA}$ is given by

$$\Lambda(\beta_4, \beta_6, \dots, \beta_{2n+2}).$$

The differential on a free dga generated in even degrees is necessarily trivial for degree reasons.

Now we observe that $\Lambda(\beta_4, \dots, \beta_{2n+2})$ is in fact the minimal model of the classifying space for projective linear bundles $BPGL(n+1, \mathbb{C})$. This can be seen in the following way: We have the short exact sequence of groups

$$0 \to \mathbb{C}^* \to GL(n+1, \mathbb{C}) \to PGL(n+1, \mathbb{C}) \to 0,$$

from which we obtain the fibration on classifying spaces (using $B\mathbb{C}^* = BS^1 = K(\mathbb{Z}, 2)$)

$$K(\mathbb{Z}, 2) \to BGL(n+1, \mathbb{C}) \to BPGL(n+1, \mathbb{C}).$$

From the long exact sequence in homotopy tensored with $\mathbb{Q}$ we conclude that

$$\pi_2(BPGL(n+1, \mathbb{C}) \otimes \mathbb{Q} \cong \pi_3(BPGL(n+1, \mathbb{C}) \otimes \mathbb{Q}$$

and

$$\pi_k(BPGL(n+1, \mathbb{C}) \cong \pi_k(BGL(n+1, \mathbb{C})) \text{ for } k \geq 4.$$

Since the rational homotopy of $BPGL(n+1, \mathbb{C})$ is concentrated in even degrees (since the rational homotopy of Lie groups is concentrated in odd degrees), we conclude that $\pi_2(BPGL(n+1, \mathbb{C}) \otimes \mathbb{Q} \cong \pi_3(BPGL(n+1, \mathbb{C}) \otimes \mathbb{Q} = 0$ and so the minimal model of $BPGL(n+1, \mathbb{C})$ is given by $\Lambda(\beta_4, \beta_6, \dots, \beta_{2n+2})$; the same as that of $BAut(\mathbb{CP}^n)$ as we calculated above.

**Corollary 2.4.** *The map $BPGL(n+1, \mathbb{C}) \to BAut_1(\mathbb{CP}^n)$ induced by the inclusion $PGL(n+1, \mathbb{C}) \hookrightarrow Aut_1(\mathbb{CP}^n)$ is a rational homotopy equivalence.*

*Proof.* The isomorphism of models follows from the previous discussion and the fact that the map $BPGL(n+1, \mathbb{C}) \to BAut_1(\mathbb{CP}^n)$ is a cohomological embedding. □

Now suppose we are given a $\mathbb{CP}^n$-fiber bundle (classified by a map to $BDiff_+(\mathbb{CP}^n)$ for simplicitiy) over some space $X$; for example the $\mathbb{CP}^3$ bundle over $S^6$ whose sections correspond to almost complex structures on $S^6$. We have the following diagram that we would like to lift through:

$$\begin{tikzcd} & BGL(n+1, \mathbb{C}) \arrow[d] \\ & BPGL(n+1, \mathbb{C}) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] \arrow[uur, dashed] & BDiff_+(\mathbb{CP}^n) \end{tikzcd}$$

Suppose we have a lift to $BPGL(n+1,\mathbb{C})$. In order to further lift to $BGL(n+1,\mathbb{C})$ (i.e. to conclude that the bundle is the projectivization of some complex vector bundle) we see that the only obstruction lies in $H^3(X;\pi_2(K(\mathbb{Z},2))) = H^3(X;\mathbb{Z})$.

**Lemma 2.5.** *The only obstruction to lifting from a PGL bundle to a GL bundle over $X$ is a torsion class in $H^3(X;\mathbb{Z})$, the "Schur class".*

*Proof.* Since the obstruction to lifting on $X$ is the pullback of the universal obstruction to finding a section of $BGL \to BPGL$, we see that the universal obstruction lies in $H^3(BPGL;\mathbb{Z})$. From the minimal model of $BPGL$ obtained above we see that $H^3(BPGL;\mathbb{Q}) = 0$ and so the obstruction is a torsion class. □

Before that, however, we have the problem of lifting from $BDiff_+(\mathbb{CP}^n)$ to $BPGL(n+1,\mathbb{C})$. As we saw, the map $BPGL(n+1,\mathbb{C}) \to BAut_1(\mathbb{CP}^n)$ is a rational homotopy equivalence, and so the homotopy fiber is rationally contractible. Therefore the obstructions to a lift of the composite map $X \to BDiff_+(\mathbb{CP}^n) \to BAut_1(\mathbb{CP}^n)$ to $X \to BPGL(n+1,\mathbb{C})$ are all torsion. If the space $X$ admits self-maps of sufficiently divisible degree (as does $S^n$), then the pullback fibration via this self-map admits a lift to $PGL$. Note however that the intermediate map $X \to BDiff_+(\mathbb{CP}^n)$ might change in this lifting procedure.

**Remark 2.6.** A fiber bundle of interest, considered above, is the $\mathbb{CP}^3$-fibration over a six–manifold $X$ whose sections correspond to almost complex structures on $X$. This bundle is obtained by pulling back the universal $\mathbb{CP}^3$-bundle $\mathbb{CP}^3 = SO(6)/U(3) \to BU(3) \to BSO(6)$. Since this universal fibration is orientable (the base $BSO(6)$ being simply connected), the pullback fibrations are orientable as well.

It should be noted that $\mathbb{CP}^n$ rationally admits a multitude of self-homotopy equivalences not homotopic to the identity. Namely, the minimal model is generated in degree 2 by a single generator $\alpha$, and any map $\alpha \mapsto \lambda\alpha$ for $\lambda \in \mathbb{Q}^\times$ determines a rational self-homotopy equivalence, all mutually non-homotopic. Recall that as a group, $\mathbb{Q}^\times = \{\pm 1\} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}\langle p\rangle$. This is reflected in the existence of a degree one generator in $\Lambda LA$, the model for $BAut(\mathbb{CP}^n)$, which in turn corresponds to a degree 0 closed derivation on the model $\Lambda(x_2, y_{2n+1}; dy = x^{n+1})$. The degree 0 derivation $(n+1)(y,y) - (x,x)$ is closed and spans $(LA)^0$.

**Example 2.7.** Similarly we can calculate the rational homotopy type of $BAut_1(S^n)$. We consider separately the cases of even and odd spheres.

- For $n$ even, we model $S^n$ by $\Lambda(x_n, y_{2n+1}; dx = 0, dy = x^2)$, and we obtain the dgla of derivations spanned by $(y,x)$ in degree $n-1$, $(x,1)$ in degree $n$, and $(y,1)$ in degree $2n-1$. We see that $D(x,1) = -2(y,x)$ and $[(y,x),(x,1)] = (y,1)]$. A model for $BAut_1(S^n)$ is therefore given by $\Lambda(\alpha_n, \alpha_{n+1}, \alpha_{2n}; d\alpha_{n-1} = -2\alpha_{n+1}, d\alpha_{n+1} = 0, d\alpha 2n = \alpha_n\alpha_{n+1})$, with minimal model given by
$$\Lambda(\beta_{2n}; d = 0).$$
Compare this with the classical result that $Diff(S^2) \cong SO(3)$, and so $BDiff_+(S^2) = BSO(3) =_{\mathbb{Q}} K(\mathbb{Q},4)$
- For $n$ odd, we model $S^n$ by $\Lambda(x_n)$ and immediately conclude that the model of $BAut_1(S^n)$ is given by $\Lambda(\alpha_{n+1})$. Compare this with $BDiff_+(S^1) = BS^1 =$

$K(\mathbb{Z}, 2)$. On the other hand, Hatcher proved that $Diff_+(S^3) \cong SO(4)$, and so $BDiff_+(S^3) =_{\mathbb{Q}} K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 4)$. Meanwhile, $BAut_1(S^3) =_{\mathbb{Q}} K(\mathbb{Q}, 4)$ by this calculation.

**Remark 2.8.** We note that $[\phi, \phi] = 0$ for any additive generator of $LA$ of the type we consider (namely, a derivation of the form $(x_i, p(x_J))$, where $x_i$ and the elements of $x_J$ are generators of $A$, and $p$ is a polynomial. Indeed, if $\phi$ is an even degree derivation, then this follows immediately by the axioms of a dgla. Otherwise, if $\phi$ is of odd degree, then $[\phi, \phi] = 2\phi\phi$. Applying $\phi\phi$ to any generator $x$ of $A$ evidently yields 0, and so $\phi\phi$ is the zero derivation.

**Example 2.9.** Consider the 12–manifold $SO(8)/U(4)$ (where the quotient is taken via the inclusion $U(4) \hookrightarrow SO(8)$). It is classically known that the first $2n - 2$ homotopy groups of $SO(2n)/U(n)$ are stable, and so $\pi_1(SO(8)/U(4)) = \pi_1(SO(4)/U(2)) = \pi_1(S^2) = 0$. In particular, $SO(8)/U(4)$ is nilpotent and so its minimal model directly calculates its rational homotopy groups. Its minimal model is given by

$$A = \Lambda(c_1, c_3, \eta_7, \eta_{11}; dc_1 = dc_3 = 0, d\eta_7 = \tfrac{1}{4}c_1^4 - 2c_1c_3, d\eta_{11} = c_3^2),$$

obtained by a standard calculation of the rational homotopy type of the homotopy fiber of $BU(n) \to BSO(2n)$. Here $c_1$ and $c_2$ should be thought of as corresponding (up to a non-zero factor) to the universal Chern classes; these are generators of degrees 2 and 6 respectively. The generators $\eta_7, \eta_{11}$ are in degrees 7 and 11 respectively.

We list a vector space basis for the negative degree derivations on $A$. Recall that since $A$ is a free algebra, a derivation is assigned by prescribing its values on the generators $c_1, c_3, \eta_7, \eta_{11}$. (There are no degree 0 derivations since all of the algebra generators of $A$ are in different degrees.) Denote by $(a, b)$ the derivation sending the generator $a$ to $b \in A$, and all other generators to 0.

$$\begin{aligned}
(LA)^1 &= \mathrm{span}((\eta_7, c_3), (\eta_{11}, c_1^5), (\eta_{11}, c_1^2c_3), (\eta_7, c_1^3)),\\
(LA)^2 &= \mathrm{span}((c_1, 1), (c_3, c_1^2), (\eta_{11}, c_1\eta_7)),\\
(LA)^3 &= \mathrm{span}((\eta_7, c_1^2), (\eta_{11}, c_1^4), (\eta_{11}, c_1c_3)),\\
(LA)^4 &= \mathrm{span}((c_3, c_1), (\eta_{11}, \eta_7)),\\
(LA)^5 &= \mathrm{span}((\eta_7, c_1), (\eta_{11}, c_3), (\eta_{11}, c_1^3)),\\
(LA)^6 &= \mathrm{span}((c_3, 1)),\\
(LA)^7 &= \mathrm{span}((\eta_7, 1), (\eta_{11}, c_1^2)),\\
(LA)^8 &= 0,\\
(LA)^9 &= \mathrm{span}((\eta_{11}, c_1)),\\
(LA)^{10} &= 0,\\
(LA)^{11} &= \mathrm{span}((\eta_{11}, 1)).
\end{aligned}$$

Now we calculate the values of $D$ that do not vanish trivially for degree reasons:

$$
\begin{aligned}
D(c_1,1) &= -4(\eta_7,c_1^3)+2(\eta_7,c_3), \quad D(c_3,c_1^2)=2(\eta_7,c_1^3)+2(\eta_{11},c_1^2c_3),\\
D(\eta_{11},c_1\eta_7) &= \tfrac{1}{4}(\eta_{11},c_1^5)-2(\eta_{11},c_1^2c_3),\\
D(\eta_7,c_1^2) &= 0, \quad D(\eta_{11},c_1^4)=0, \quad D(\eta_{11},c_1c_3)=0,\\
D(c_3,c_1) &= 2(\eta_7,c_1^2)-2(\eta_{11},c_1c_3), \quad D(\eta_{11},\eta_7)=\tfrac{1}{4}(\eta_{11},c_1^4)-2(\eta_{11},c_1c_3),\\
D(\eta_7,c_1) &= 0,\\
D(\eta_{11},c_3) &= 0, \quad D(\eta_{11},c_1^3)=0,\\
D(c_3,1) &= 2(\eta_7,c_1)-2(\eta_{11},c_3),\\
D(\eta_7,1) &= 0, \quad D(\eta_{11},c_1^2)=0,\\
D(\eta_{11},c_1) &= 0,\\
D(\eta_{11},1) &= 0.
\end{aligned}
$$

Now, observe that $d$ is injective on $(LA)^{\text{even}}$, and that $d$ vanishes on all of $(LA)^{\text{odd}}$. We count the dimensions of the homology groups $H_*(LA,D)$ to be given by:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\geq 12$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dim H_k(LA,D)$ | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 0 | 1 | 0 | 1 | 0 |

We now see that the generators of $\widetilde{\Lambda LA}$ will all be in even degree, and so the differential on $\widetilde{\Lambda LA}$ will be trivial. The minimal model of $BAut_1(SO(8)/U(4))$ is therefore given by

$$\Lambda(\beta_2,\beta_4,\beta_6,\beta_6',\beta_8,\beta_8',\beta_{10},\beta_{12};d=0).$$

**Remark 2.10.** If we were to consider the derivations on the cohomology algebra $H^*(SO(8)/U(4);\mathbb{Q})$ instead of those on the minimal model of $SO(8)/U(4)$, we would lose a substantial amount of information. In fact, a negative degree derivation on the cohomology of a simply-connected compact Kähler manifold $X^{2n}$ of real dimension $2n$ is *trivial*. (All the manifolds of the form $SO(2n)/U(n)$ admit Kähler metrics.) Indeed, suppose $\phi$ is a negative-degree derivation on $H^*(X;\mathbb{R})$ (we will use the Kähler form in what follows, and so need to extend to real coefficients to cover the non-projective case). Since $H^1(X;\mathbb{R})=0$, we have $\phi(\omega)=0$, where $\omega\in H^2(X;\mathbb{R})$ denotes the Kähler class. Now take a class $\alpha\in H^k$ in the largest degree $k$ such that $\phi(H^{\leq k-1}(X;\mathbb{R}))=0$. If $k\geq n+1$, then $\phi=0$. Indeed, by the hard Lefschetz property, there exists a $\beta\in H^{2n-k}(X;\mathbb{R})$ such that $\alpha=\beta\omega^{k-n}$ and so $\phi(\alpha)=\phi(\beta)\omega^{k-n}=0$; we used that $2n-k<k$ and so $\phi(\beta)=0$ by assumption. If $k\leq n$, then consider the class $\omega^{n-k}\alpha$ obtained by applying the hard Lefschetz isomorphism to $\alpha$. Multiplying this form by another copy of $\omega$, again by the hard Lefschetz isomorphism we can write

$$\omega^{n-k+1}\alpha=\omega^{n-k+2}\gamma,$$

where $\gamma\in H^{k-2}(X;\mathbb{R})$. By assumption, $\phi(\gamma)=0$ and so applying $\phi$ to the above equation we have

$$(n-k+1)\omega^{n-k}\phi(\alpha)=0.$$

Applying a few more powers of $\omega$ to $\omega^{n-k}\phi(\alpha)$ would bring us to the image of $\phi(\alpha)$ under the hard Lefschetz isomorphism. Since that element in the image is necessarily 0, we conclude that $\phi(\alpha) = 0$.

**Example 2.11.** Let us now consider an example in which the minimal model $A$ of the space $X$ under consideration has multiple generators in the same degree, and we will also keep track of all degree 0 derivations and form the model of $BAut(X)$. We calculate $\Lambda LA$ for $X = S^2 \times S^2$. The minimal model $A$ of $S^2 \times S^2$ is given by

$$\Lambda(x_2, y_3, x_2', y_3'; dx = dx' = 0, dy = x^2, dy' = x'^2).$$

The space of derivations of degree 0 is now eight–dimensional. We have the following:

$$\begin{aligned}
(LA)^0 &= \mathrm{span}((x,x),(x',x'),(x,x'),(x',x),(y,y),(y',y'),(y,y'),(y',y)),\\
(LA)^1 &= \mathrm{span}((y,x),(y,x'),(y',x),(y',x')),\\
(LA)^2 &= \mathrm{span}((x,1),(x',1)),\\
(LA)^3 &= \mathrm{span}((y,1),(y',1)).
\end{aligned}$$

We calculate that a basis of $D$-closed derivations in degree 0 is given by

$$\{(x,x)+2(y,y),(x',x')+2(y',y')\}.$$

The only non-trivial values of $D$ on the higher degree derivations are $D(x,1) = -2(y,x)$ and $D(x',1) = -2(y',x')$.

Denote $Z = (x,x)+2(y,y)$ and $Z' = (x',x')+2(y',y')$. We calculate the brackets to be given by

$$\begin{aligned}
&[(LA)^0,(LA)^0] : [Z,Z'] = 0,\\
&[(LA)^0,(LA)^1] : [(y,x),Z] = (y,x), [(y,x),Z'] = 0, [(y',x'),Z] = 0, [(y',x'),Z'] = (y',x'),\\
&\qquad\qquad\qquad\quad [(y',x),Z] = -(y',x), [(y,x'),Z'] = -(y,x'),\\
&\qquad\qquad\qquad\quad [(y',x),Z'] = 2(y',x), [(y,x'),Z] = 2(y,x'),\\
&[(LA)^0,(LA)^2] : [(x,1),Z] = (x,1), [(x'1),Z] = 0, [(x,1),Z'] = 0, [(x',1),Z'] = (x',1),\\
&[(LA)^0,(LA)^3] : [(y,1),Z] = 2(y,1), [(y',1),Z] = 0, [(y,1),Z'] = 0, [(y',1),Z'] = 2(y',1),\\
&[(LA)^1,(LA)^1] : \text{all brackets } 0,\\
&[(LA)^1,(LA)^2] : [(y,x),(x,1)] = -(y,1), [(y,x),(x',1)] = 0,\\
&\qquad\qquad\qquad\quad [(y',x'),(x,1)] = 0, [(y',x'),(x',1)] = -(y',1).
\end{aligned}$$

For ease of reference we record the matrices of the actions of $Z$ and $Z'$ on the additive basis of $(LA)^{>0}$ given by $\{(y,x),(y,x'),(y',x),(y',x'),(x,1),(x'1),(y,1),(y',1)\}$:

$$[-,Z]=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&2&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&0\end{pmatrix},\quad [-,Z']=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&2&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&2\end{pmatrix}$$

Note that these matrices are *not nilpotent*, and so $BAut(X)$ is not a nilpotent space. However, we see that rationally $\pi_1(BAut(X))$ acts on the homotopy type of $BAut(X)$ by scaling transformations only, i.e. the action is *reductive*.

We can now simplify things and consider only $BAut_1(X)$. To form the model $\widetilde{\Lambda LA}$, we remove all degree 0 elements, and restrict only to the closed degree 1 derivations (which in this case is all of them) in $\Lambda LA$. We see that a model for $BAut_1(X)$ is given by

$$\Lambda(\alpha_2,\alpha_2',\beta_4,\beta_4';d=0).$$

**Example 2.12.** We calculate $BAut(X)$ for a simply connected non-formal space $X$. Consider the $S^3$–fibration over $S^4$ whose total space is $S^7$. Pulling back this fibration via the map $S^2\times S^2\to S^4$ that crushes the 3–skeleton, we obtain a fibration with nilpotent total space modelled by

$$A=\Lambda(a_2,b_2,u_3,v_3,t_3;da=db=0,du=a^2,dv=b^2,dt=ab).$$

The degree 0 derivations are spanned by

$$(a,b),(b,a),(u,v),(u,t),(v,u),(v,t),(t,u),(t,v),(a,a),(b,b),(u,u),(v,v),(t,t).$$

We calculate the space of closed degree 0 derivations to be spanned by

$$\begin{aligned}Z&=(b,a)+(t,u)+2(v,t),\\Z'&=(a,b)+2(u,t)+(t,v),\\W&=(b,b)+2(v,v)+(t,t),\\W'&=(a,a)+2(u,u)+(t,t).\end{aligned}$$

The brackets on $(LA)^0$ (the spanned spanned by $Z,Z',W,W'$) are given by

$$\begin{aligned}[][Z,Z']&=W'-W,\\[][W,W']&=0,\\[][Z,W]&=Z,\\[][Z',W]&=Z',\\[][Z,W']&=-Z,\\[][Z',W']&=Z'.\end{aligned}$$

As for derivations of negative degree, we have the following:

$$\begin{aligned}(LA)^1 &= \mathrm{span}((u,a),(v,a),(t,a),(u,b),(v,b),(t,b)),\\ (LA)^2 &= \mathrm{span}((a,1),(b,1)),\\ (LA)^3 &= \mathrm{span}((u,1),(v,1),(t,1)).\end{aligned}$$

The non-zero values of the differential $D$ on these basis elements are given by

$$\begin{aligned}D(a,1) &= -(t,b),\\ D(b,1) &= -(t,a).\end{aligned}$$

The values of the bracket are given by:

$$\begin{aligned}[(LA)^0,(LA)^0] : {}& [Z,Z'] = W' - W, [W,W'] = 0, [Z,W] = Z,\\ & [Z',W] = -Z', [Z,W'] = -Z, [Z',W'] = Z',\end{aligned}$$

$$\begin{aligned}[(LA)^0,(LA)^1] : {}& [Z,(u,a)] - (t,a), [Z',(u,a)] = (u,b), [W,(u,a)] = 0,\\ & [W',(u,a)] = -(u,a), [Z,(v,a)] = 0, [Z',(v,a)] = (v,b) - (t,a),\\ & [W,(v,a)] = -2(v,a), [W',(v,a)] = (v,a), [Z,(t,a)] = -2(v,a),\\ & [Z',(t,a)] = (t,b) - 2(u,a), [W,(t,a)] = -(t,a), [W',(t,a)] = 0,\\ & [Z,(t,b)] = (t,a) - 2(v,b), [Z',(t,b)] = -2(u,b), [W,(t,b)] = -(t,b),\\ & [W',(t,b)] = -(t,b), [Z,(u,b)] = (u,a) - (t,b), [Z',(u,b)] = 0,\\ & [W,(u,b)] = (u,b), [W',(u,b)] = -2(u,b), [Z,(v,b)] = (v,a),\\ & [Z',(v,b)] = -(t,b), [W,(v,b)] = -(v,b), [W',(v,b)] = 0,\end{aligned}$$

$$\begin{aligned}[(LA)^0,(LA)^2] : {}& [Z,(a,1)] = -(b,1), [Z',(a,1)] = 0, [W,(a,1)] = 0,\\ & [W',(a,1)] = -(a,1), [Z,(b,1)] = 0, [Z',(b,1)] = -(a,1),\\ & [W,(b,1)] = -(b,1), [W',(b,1)] = 0,\end{aligned}$$

$$\begin{aligned}[(LA)^0,(LA)^3] : {}& [Z,(u,1)] = -(t,1), [Z',(u,1)] = 0, [W,(u,1)] = 0,\\ & [W',(u,1)] = -2(u,1), [Z,(v,1)] = 0, [Z',(v,1)] = -(t,1),\\ & [W,(v,1)] = -2(v,1), [W',(v,1)] = 0, [Z,(t,1)] = -2(v,1),\\ & [Z',(t,1)] = -2(u,1), [W,(t,1)] = -(t,1), [W',(t,1)] = -(t,1),\\ [(LA)^1,(LA)^1] : {}& \text{all brackets } 0,\end{aligned}$$

$$\begin{aligned}[(LA)^1,(LA)^2] : {}& [(u,a),(a,1)] = -(u,1), [(v,a),(a,1)] = -(v,1), [(t,a),(a,1)] = -(t,1),\\ & [(u,b),(b,1)] = -(u,1), [(v,b),(b,1)] = -(v,1), [(t,b),(b,1)] = -(t,1),\\ & \text{all other brackets } 0.\end{aligned}$$

We now form a model $\Lambda LA$ for $BAut(X)$. The generators are given by $z, z', w, w'$ in degree one, $\eta^{ua}, \eta^{va}, \eta^{ta}, \eta^{ub}, \eta^{vb}, \eta^{tb}$ in degree two, $\alpha, \beta$ in degree 3 (corresponding to

$(a,1)$ and $(b,1)$ respectively, and $\bar{u},\bar{v},\bar{t}$ in degree four. The dga $\Lambda LA$ is given by

$$\begin{aligned}
&\Lambda(z_1, z_1', w_1, w_1', \eta_2^{ua}, \eta_2^{va}, \eta_2^{ta}, \eta_2^{ub}, \eta_2^{vb}, \eta_2^{tb}, \alpha_3, \beta_3, \bar{u}_4, \bar{v}_4, \bar{t}_4;\\
&dz = zw - zw', dz' = z'w' - z'w, dw = -zz', dw' = zz',\\
&d\eta^{ua} = -w'\eta^{ua} - 2z'\eta^{ta} + z\eta^{ub}, d\eta^{va} = -2w\eta^{va} + w'\eta^{va} - 2z\eta^{ta} + z\eta^{vb},\\
&d\eta^{ta} = -\beta - z\eta^{ua} - z'\eta^{va} - w\eta^{ta} + z\eta^{tb}, d\eta^{ub} = z'\eta^{ua} - 2z'\eta^{tb} + w\eta^{ub} - 2w'\eta^{ub},\\
&d\eta^{vb} = z'\eta^{va} - 2z\eta^{tb} - w\eta^{vb}, d\eta^{tb} = -\alpha, +z'\eta^{ta} - w\eta^{tb} - w'\eta^{tb} - z\eta^{ub} - z'\eta^{vb},\\
&d\alpha = -w'\alpha - z'\beta, d\beta = -z\alpha - w\beta,\\
&d\bar{u} = -\eta^{ua}\alpha - \eta^{ub}\beta - 2w'\bar{u} - 2z'\bar{t}, d\bar{v} = -\eta^{va}\alpha - \eta^{vb}\beta - 2w\bar{v} - 2z\bar{t},\\
&d\bar{t} = -\eta^{ta}\alpha - \eta^{tb}\beta - z\bar{u} - z'\bar{v} - w\bar{t} - w'\bar{t}).
\end{aligned}$$

A model for $BAut_1(X)$ is given by

$$\begin{aligned}
&\Lambda(\eta_2^{ua}, \eta_2^{va}, \eta_2^{ta}, \eta_2^{ub}, \eta_2^{vb}, \eta_2^{tb}, \alpha_3, \beta_3, \bar{u}_4, \bar{v}_4, \bar{t}_4;\\
&d\bar{u} = \eta^{ua}\alpha + \eta^{ub}\beta, d\bar{v} = \eta^{va}\alpha + \eta^{vb}\beta, d\bar{t} = \eta^{ta}\alpha + \eta^{tb}\beta).
\end{aligned}$$

Note that from $d(\bar{u}\beta) = \eta^{ua}\alpha\beta$, $d(\bar{u}\alpha) = -\eta^{ub}\alpha\beta$, $d(\bar{v}\beta) = \eta^{va}\alpha\beta$, $d(\bar{v}\alpha) = -\eta^{vb}\alpha\beta$, $d(\bar{t}\beta) = \eta^{ta}\alpha\beta$, $d(\bar{t}\alpha) = -\eta^{tb}\alpha\beta$, we see some nontrivial triple Massey products in $BAut_1(X)$. For instance, we have the triple Massey product

$$\mathrm{Massey}([\eta^{ua}], [\alpha\beta], [\eta^{ub}]) = [\bar{u}\beta\eta^{ub} + \eta^{ua}\bar{u}\alpha].$$

Similarly we obtain two more triple Massey products,

$$\begin{aligned}
\mathrm{Massey}([\eta^{va}], [\alpha\beta], [\eta^{vb}]) &= [\bar{v}\beta\eta^{vb} + \eta^{va}\bar{v}\alpha]\\
\mathrm{Massey}([\eta^{ta}], [\alpha\beta], [\eta^{tb}]) &= [\bar{t}\beta\eta^{tb} + \eta^{ta}\bar{t}\alpha]
\end{aligned}$$

The non-formality of $X$ itself is detected by two nontrivial triple Massey products in $H^5$, namely $\mathrm{Massey}([a], [a], [b]) = [ub - at]$ and $\mathrm{Massey}([a], [b], [b]) = [tb - av]$.

**Example 2.13.** We calculate the rational homotopy type of $BAut_1(K(\mathbb{Z}, l) \times K(\mathbb{Z}, n))$ where $l \leq n$. First, note that if $l = n$, then the minimal model of $K(\mathbb{Z}, l) \times K(\mathbb{Z}, l)$ is given by $\Lambda(\alpha_l, \beta_l; d = 0)$ and so the space of negative degree derivations is spanned by $(\alpha, 1)$ and $(\beta, 1)$ in degree $l$. It immediately follows that $D = 0$ and $[(\alpha, 1), (\beta, 1)] = 0$, and so the rational homotopy type of $BAut_1(K(\mathbb{Z}, l) \times K(\mathbb{Z}, l))$ is given by $\Lambda(a_{l+1}, b_{l+1}; d = 0)$, which is that of $K(\mathbb{Z}, l+1) \times K(\mathbb{Z}, l+1)$. Analogously,

$$BAut_1(K(\mathbb{Z}, l)^m) =_{\mathbb{Q}} K(\mathbb{Z}, l+1)^m.$$

Now consider the case of $l < n$. The minimal model of $K(\mathbb{Z}, l) \times K(\mathbb{Z}, n)$ is given by $\Lambda(\alpha_l, \beta_n; d = 0)$, and so the space of negative degree derivations is spanned by $(\alpha, 1), (\beta, 1), (\beta, \alpha)$ in degrees $l, n, n-l$ respectively. Note that since $D = [d, -]$ and $d = 0$, we immediately have $D = 0$. The only non-trivial bracket is given by $[(\alpha, 1), (\beta, \alpha)] = (\beta, 1)$. The minimal model of $BAut_1(K(\mathbb{Z}, l) \times K(\mathbb{Z}, n))$ is thus given by

$$\Lambda(a_{l+1}, b_{n+1}, \gamma_{n-l+1}; da = d\gamma = 0, db = a\gamma).$$

Note that this space is non-formal if $l$ and $n$ are odd, detected by the Massey triple product $\mathrm{Massey}(a, \gamma, \gamma)$.

**Example 2.14.** The rational homotopy of $BAut_1$ of higher products of $K(\mathbb{Z},n)'s$ are very computable. We do so for a triple product and a quadruple product, where the degrees of all the generators are different.

First consider $K(\mathbb{Z},k)\times K(\mathbb{Z},l)\times K(\mathbb{Z},n)$ with $k<l<n$. Denote the generators by $\alpha_k$, $\beta_l$, and $\gamma_n$ respectively. The space of negative degree derivations is spanned by $(\alpha,1),(\beta,1),(\gamma,1),(\beta,\alpha),(\gamma,\alpha),(\gamma,\beta)$. We calculate the non-trivial brackets to be given by

$$\begin{aligned}[(\beta,\alpha),(\gamma,\beta)]&=(\gamma,\alpha),\\ [(\alpha,1),(\beta,\alpha)]&=(\beta,1),\\ [(\alpha,1),(\gamma,\alpha)]&=(\gamma,1),\\ [(\beta,1),(\gamma,\beta)]&=(\gamma,1).\end{aligned}$$

Again, $D=0$ since $d=0$, and we obtain the following minimal model for $BAut_1(K(\mathbb{Z},k)\times K(\mathbb{Z},l)\times K(\mathbb{Z},n))$,

$$\begin{aligned}&\Lambda(a_{k+1},b_{l+1},c_{n+1},u_{l-k+1},v_{n-l+1},w_{n-k+1};\\ &\quad da=du=dv=0,db=au,dc=bv+aw,dw=uv).\end{aligned}$$

Now consider the quadruple product $X=K(\mathbb{Z},k)\times K(\mathbb{Z},l)\times K(\mathbb{Z},m)\times K(\mathbb{Z},n)$ with generators given by $\alpha_k,\beta_l,\gamma_m,\delta_n$. The spanning derivations are $(\alpha,1)$, $(\beta,1)$, $(\gamma,1)$, $(\delta,1)$, $(\beta,\alpha)$, $(\gamma,\alpha)$, $(\delta,\alpha)$, $(\gamma,\beta)$, $(\delta,\beta)$, $(\delta,\gamma)$, whose corresponding generators in $BAut_1(X)$ are given by $a,b,c,d,u,v,w,x,y,z$. The minimal model of $BAut_1(X)$ is given by

$$\begin{aligned}&\Lambda(a,b,c,d,u,v,w,x,y,z;\\ &da=0,db=au,dc=av+bx,dd=aw+by+cz,du=0,dv=ux,\\ &dw=uy+vz,dx=0,dy=xz,dz=0).\end{aligned}$$

**Example 2.15.** We calculate the model for $BAut_1(\mathbb{CP}^2\#\mathbb{CP}^2)$. A minimal model for $\mathbb{CP}^2\#\mathbb{CP}^2$ is given by

$$A=\Lambda(x_1,y_1,x_2,y_2,\eta,\epsilon;dy_1=x_1^3,dy_2=x_2^3,d\eta=x_1x_2,d\epsilon=x_1^2-x_2^2),$$

where $deg(x_i)=2$, $deg(y_i)=5$, $deg(\eta)=3$, $deg(\epsilon)=3$. For our dgla $LA$, in positive degrees we have:

$$\begin{aligned}(LA)^1&=\text{span}((\eta,x_1),(\eta,x_2),(\epsilon,x_1),(\epsilon,x_2),(y_1,x_1x_2),(y_2,x_1x_2),(y_1,x_1^2),(y_1,x_2^2),(y_2,x_1^2),(y_2,x_2^2)),\\ (LA)^2&=\text{span}(y_1,\eta),(y_1,\epsilon),(y_2,\eta),(y_2,\epsilon),(x_1,1),(x_2,1)),\\ (LA)^3&=\text{span}((y_1,x_1),(y_1,x_2),(y_2,x_1),(y_2,x_2),(\eta,1),(\epsilon,1)),\\ (LA)^4&=\{0\},\\ (LA)^5&=\text{span}((y_1,1),(y_2,1)).\end{aligned}$$

The usual calculation now gives us the following (non-minimal) model for $BAut_1(\mathbb{CP}^2\#\mathbb{CP}^2)$:

$$\begin{aligned}
&\Lambda(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9,\alpha_{10},\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\delta_1,\delta_2;\\
&d\alpha_1 = d\alpha_2 = d\alpha_3 = d\alpha_4 = 0,\\
&d\alpha_5 = \beta_1, d\alpha_6 = \beta_3, d\alpha_7 = \beta_2, d\alpha_8 = \beta_5, d\alpha_9 = \beta_3, d\alpha_{10} = \beta_6,\\
&d\beta_1 = d\beta_2 = d\beta_3 = d\beta_4 = d\beta_5 = d\beta_6 = 0,\\
&d\gamma_1 = \alpha_1\beta_1 + \alpha_3\beta_2 - \alpha_5\beta_6 - 2\alpha_7\beta_5 + 10\alpha_8\beta_5 - 2\alpha_8\beta_2 - \alpha_{10}\beta_3,\\
&d\gamma_2 = \alpha_4\beta_2 + \alpha_2\beta_1 - \alpha_5\beta_6 - 2\alpha_7\beta_6 + 10\alpha_8\beta_6 - 2\alpha_{10}\beta_2 - \alpha_8\beta_1,\\
&d\gamma_3 = \alpha_1\beta_3 + \alpha_3\beta_4 - \alpha_6\beta_6 - 2\alpha_9\beta_5 + 10\alpha_{10}\beta_5 - 2\alpha_8\beta_4 - \alpha_{10}\beta_3,\\
&d\gamma_4 = \alpha_2\beta_3 + \alpha_4\beta_4 - \alpha_6\beta_5 - 2\alpha_9\beta_6 + 10\alpha_{10}\beta_6 - 2\alpha_{10}\beta_4 - \alpha_8\beta_3,\\
&d\delta_1 = -\beta_1\gamma_5 - \beta_2\gamma_6 - \beta_5\gamma_1 - \beta_6\gamma_2,\\
&d\delta_2 = -\beta_3\gamma_5 - \beta_4\gamma_6 - \beta_5\gamma_3 - \beta_6\gamma_4),
\end{aligned}$$

where the generators $\alpha_i, \beta_i, \gamma_i, \delta_i$ correspond to the following generators in $\Lambda A$:

| | |
|---|---|
| $\alpha_1$ | $(\eta, x_1)$ |
| $\alpha_2$ | $(\eta, x_2)$ |
| $\alpha_3$ | $(\epsilon, x_1)$ |
| $\alpha_4$ | $(\epsilon, x_2)$ |
| $\alpha_5$ | $(y_1, x_1x_2)$ |
| $\alpha_6$ | $(y_2, x_1x_2)$ |
| $\alpha_7$ | $(y_1, x_1^2 - x_2^2)$ |
| $\alpha_8$ | $-3(y_1, x_1^2) - (\eta_1, x_2) - 2(\epsilon, x_2)$ |
| $\alpha_9$ | $(y_2, x_1^2 - x_2^2)$ |
| $\alpha_{10}$ | $-3(y_2, x_2^2) - (\eta, x_1) - 2(\epsilon, x_2)$ |
| $\beta_1$ | $(y_1, \eta)$ |
| $\beta_2$ | $(y_1, \epsilon)$ |
| $\beta_3$ | $(y_2, \eta)$ |
| $\beta_4$ | $(y_2, \epsilon)$ |
| $\beta_5$ | $(x_1, 1)$ |
| $\beta_6$ | $(x_2, 1)$ |
| $\gamma_1$ | $(y_1, x_1)$ |
| $\gamma_2$ | $(y_1, x_2)$ |
| $\gamma_3$ | $(y_2, x_1)$ |
| $\gamma_4$ | $(y_2, x_2)$ |
| $\gamma_5$ | $(\eta, 1)$ |
| $\gamma_6$ | $(\epsilon, 1)$ |
| $\delta_1$ | $(y_1, 1)$ |
| $\delta_2$ | $(y_2, 1)$ |

From the proof of Theorem 2.2 in [1], we see that a minimal model of $BAut_1(\mathbb{CP}^2\#\mathbb{CP}^2)$ is given by

$$\Lambda(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\delta_1,\delta_2; d = 0),$$

where $deg(\alpha_i) = 2$, $deg(\gamma_i) = 4$, $deg(\delta_i) = 6$.

## References

[1] Sullivan, D., 1977. Infinitesimal computations in topology. Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 47(1), pp.269-331.