ON ALMOST COMPLEX RATIONAL QUATERNIONIC AND OCTONIONIC PROJECTIVE SPACES

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Abstract. We consider the question of which quaternionic projective spaces have the rational homotopy type of a closed almost complex manifold, partially generalizing a theorem of Hirzebruch. We further comment on the case of rational octonionic projective spaces.

1. Introduction

In the 1950's, Hirzebruch [Hi54] proved that the quaternionic projective spaces $\mathbb{HP}^n$, with their standard smooth structures, do not admit almost complex structures for $n \neq 2, 3$. Together with Milnor, this was later generalized [HiICM58] in Hirzebruch's 1958 ICM address to the statement that no $\mathbb{HP}^n$ with its standard smooth stucture, for $n \geq 2$, admits a stable almost complex structure. Massey [Ma62] subsequently gave an independent proof of the latter (see the Remark at the end of loc. cit.).

In this note we will partially generalize the first statement, regarding the non-existence of almost complex structures, to closed manifolds with the rational cohomology ring of $\mathbb{HP}^n$. We emphasize that we do not fix an orientation on our manifold; by saying that an orientable closed manifold does not admit an almost complex structure, we mean that it does not admit one which is compatible with either possible orientation.

More concretely, our generalization is the following:

Theorem. *If a closed smooth $4n$–manifold M with $H^*(M;\mathbb{Q}) \cong H^*(\mathbb{HP}^n;\mathbb{Q})$ admits an almost complex structure, then $n \equiv 0, 3, 8,$ or $11 \bmod 12$.*

Using Sullivan's closed manifold realization result for rational homotopy types [Sull77, Theorem 13.2] adapted to almost complex manifolds [M22, Theorem 2.4], one can see that there is indeed a simply connected closed almost complex manifold with the rational cohomology of $\mathbb{HP}^3$ [M22, Theorem 7.3]. In this note we will also extend the latter and exhibit a one-parameter family of complex cobordism classes all represented by a rational $\mathbb{HP}^3$. We will comment on the realization problem for an almost complex rational $\mathbb{HP}^8$; this and the further realization problems for $n = 11, 12, 15, \ldots$ are at the moment far too computationally complex for us to hope to solve.

To place our investigation into context, it is known that a closed almost complex manifold with sum of rational Betti numbers equal to three has the rational cohomology of $\mathbb{CP}^2$ [Hu21, Theorem 1.2], [Su22, Corollary 14]. That is, if $\mathbb{Q}[x]/(x^3)$ is the rational cohomology of some closed almost complex manifold, then $\deg(x) = 2$. We contrast this with the case of general closed orientable smooth manifolds, where there are in fact simply connected manifolds with this rational cohomology in dimensions $4, 8, 16, 32, 128, 256$ and possibly beyond (though there are less than ten further possibilities up to dimension one million) [KeSu19, Theorem A], [Zag17]. Note, the dimensions $4, 8, 16$ comprise a complete list for integral cohomology $\mathbb{Z}[x]/(x^3)$ by Adams' resolution of the Hopf invariant one problem.

Furthermore, $\mathbb{Q}[x]/(x^3)$ is realized by a closed almost complex manifold precisely when $\deg(x) = 2, 6$. It is natural then to consider the two-parameter family of algebras $\mathbb{Q}[x]/(x^n)$, with varying $\deg(x)$, and ask for which $\deg(x)$ and n this is realized as the rational cohomology algebra of a closed almost complex manifold.

We focus on $\deg(x) = 4$ and comment on the case of $\deg(x) = 8$, i.e. almost complex "rational quaternionic and octonionic projective spaces". It is well known that a closed manifold with integral cohomology $\mathbb{Z}[x]/(x^n)$ and $\deg(x) = 8$ exists only in dimension 8 and 16, but this conclusion does not extend to the rational setting. Indeed, from Sullivan's realization theorem [Sull77, Theorem 13.2], one sees that $\mathbb{Q}[x]/(x^n)$ with $\deg(x) = 8$ is the cohomology of a closed simply connected manifold for even n (i.e. "rationally, odd octonionic projective spaces $\mathbb{OP}^3, \mathbb{OP}^5, \ldots$ exist"). Namely, since the middle degree rational cohomology is trivial, one can set all the Pontryagin classes to be trivial in the construction of loc. cit. Furthermore, by [KeSu19, Corollary 20], rational $\mathbb{OP}^4, \mathbb{OP}^{16}, \mathbb{OP}^{32}$ exist as well; generally it is unknown for which n they exist. We show that most of these cannot admit almost complex structures:

Proposition. *If a closed smooth $8n$–manifold M with $H^*(M;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$, where $\deg(x) = 8$, admits an almost complex structure, then*

$$n \equiv 0, 80, 95, 144, 224, 239, 320, 335, 464, 479, 495, 560, 575, 639, 704, \text{or } 719 \bmod 720.$$

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2. Stong congruences and almost complex realization

We recall the rational homotopy type realization theorem for simply connected closed (stably) almost complex manifolds [M22, Theorem 2.4]; for simplicity we only include the statement for dimensions divisible by four.

Theorem 2.1. *Let X be a formally n–dimensional simply connected rational space of finite type satisfying rational Poincaré duality, $n \equiv 0 \bmod 4$ and $n \geq 8$, and let $[X] \in H_n(X;\mathbb{Q})$ be a non-zero element. Furthermore, let $c_i \in H^{2i}(X;\mathbb{Q})$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ be cohomology classes. Then there is a closed simply connected stably almost complex manifold M and a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $c_i(TM) = f^*(c_i)$ if the following hold:*

- *the quadratic form on $H^{\frac{n}{2}}(X;\mathbb{Q})$ given by $q(\alpha,\beta) = \langle \alpha\beta, [X]\rangle$ is equivalent over $\mathbb{Q}$ to one of the form $\sum_i \pm y_i^2$,*
- *if we define $p_i = (-1)^i \sum_j (-1)^j c_j c_{i-j}$, then $\langle L(p_1, \ldots, p_{n/4}), [X]\rangle = \sigma(X)$, where L is Hirzebruch's L–polynomial,*
- *the numbers $\langle c_{i_1} c_{i_2} \cdots c_{i_r}, [X]\rangle$, for any partition $\{i_1, \ldots, i_r\}$ of $n/2$, are integers that satisfy the following Stong congruences* [St65, Theorem 1]*: denoting by e_i the elementary symmetric polynomials in the variables $e^{x_j} - 1$, where the x_j are given by formally writing $1 + c_1 + c_2 + \cdots = \prod_j (1 + x_j)$, we have*

$$\langle z \cdot \mathrm{td}(X), [X]\rangle \in \mathbb{Z} \text{ for every } z \in \mathbb{Z}[e_1, e_2, \ldots].$$

Here $\mathrm{td}(X)$ denotes the Todd polynomial evaluated on $c_1, c_2, \ldots$.

- *if* $c_1 = 0$ *and* $n \equiv 4 \bmod 8$, *the numbers* $\langle p_{i_1} p_{i_2} \cdots p_{i_r}, [X]\rangle$ *are integers that satisfy a further set of Stong congruences* [St66, p.134], *for any partition* $\{1, \ldots, i_r\}$ *of* $n/4$*: denoting by* e^p_i *the elementary symmetric polynomials in the variables* $e^{x_j} + e^{-x_j} - 2$, *where the* x_j *are given by formally writing* $1 + p_1 + p_2 + \cdots = \prod_j (1 + x_j^2)$, *we require*

$$\langle z \cdot \hat{A}(X), [X]\rangle \in 2\mathbb{Z} \text{ for every } z \in \mathbb{Z}[e^p_1, e^p_2, \ldots].$$

  Here $\hat{A}(X)$ *denotes the* $\hat{A}$ *polynomial evaluated on* $p_1, p_2, \ldots$*. Note that the above are conditions on* $c_1, c_2, \ldots$*, as they determine* $p_1, p_2, \ldots$*.*

If $\langle c_{n/2}, [X]\rangle$ *equals the Euler characteristic of* X*, and the conditions above are satisfied, then the stable almost complex structure on the obtained manifold* M *is induced by an almost complex structure.*

Example 2.2. We compute the Stong congruences for a four–dimensional closed stably almost complex manifold. In this case, the Chern classes $c_1 = x_1+x_2$, $c_2 = x_1x_2$ are the elementary symmetric polynomials in the Chern roots x_1, x_2, while $e^{x_1} - 1 = x_1 + \frac{x_1^2}{2}$, $e^{x_2} - 1 = x_2 + \frac{x_2^2}{2}$ up to higher order terms. We find that, up to higher order terms,

$$\begin{aligned} e_1 &= (e^{x_1} - 1) + (e^{x_2} - 1) = x_1 + x_2 + \tfrac{x_1^2+x_2^2}{2} = c_1 + \tfrac{c_1^2-2c_2}{2}, \\ e_2 &= (e^{x_1} - 1)(e^{x_2} - 1) = x_1x_2 = c_2, \\ e_1^2 &= c_1^2. \end{aligned}$$

Again, by degree reasons, the relevant part of the Todd polynomial is

$$\mathrm{td}(X) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12}.$$

Clearly, $\langle z \cdot \mathrm{td}(X), [X]\rangle \in \mathbb{Z}$ for every $z \in \mathbb{Z}[e_1, e_2]$ if and only if

$$\langle z \cdot \mathrm{td}(X), [X]\rangle \in \mathbb{Z} \text{ for } z = 1, e_1, e_2, e_1^2.$$

Pairing with the fundamental class, we have:

$$\begin{aligned} \langle 1 \cdot (1 + \tfrac{c_1}{2} + \tfrac{c_1^2+c_2}{12}), [X]\rangle &= \tfrac{\langle c_1^2,[X]\rangle+\langle c_2,[X]\rangle}{12} \in \mathbb{Z} \\ \langle (c_1 + \tfrac{c_1^2-2c_2}{2})(1 + \tfrac{c_1}{2} + \tfrac{c_1^2+c_2}{12}), [X]\rangle &= \langle c_1^2, [X]\rangle - \langle c_2, [X]\rangle \in \mathbb{Z} \\ \langle c_2 \cdot (1 + \tfrac{c_1}{2} + \tfrac{c_1^2+c_2}{12}), [X]\rangle &= \langle c_2, [X]\rangle \in \mathbb{Z} \\ \langle c_1^2 \cdot (1 + \tfrac{c_1}{2} + \tfrac{c_1^2+c_2}{12}), [X]\rangle &= \langle c_1^2, [X]\rangle \in \mathbb{Z} \end{aligned}$$

Abusing the pairing notation and denoting $\langle c_I, [X]\rangle$ simply by c_I, we have the Stong congruences:

$$\tfrac{c_1^2+c_2}{12} \in \mathbb{Z}, \quad c_1^2 - c_2 \in \mathbb{Z}, \quad c_2 \in \mathbb{Z}, \quad c_1^2 \in \mathbb{Z}.$$

The Stong congruences generally account for the Chern numbers being integers; beyond this in dimension four we therefore only have the integrality of the Todd genus $c_1^2+c_2 \in 12\mathbb{Z}$.

Remark 2.3. In general, the integrality of the Chern numbers and Todd genus will not imply all the Stong congruences. Already in dimension six we have the congruences $\frac{c_1c_2}{24}, \frac{c_1^3}{2}, \frac{c_3}{2} \in \mathbb{Z}$, where the first is the integrality of the Todd genus. Further, even in the case of compact complex manifolds it is not true that the integrality of the holomorphic Euler characteristics of the sheaves of holomorphic k–forms recover all the

Stong congruences. For example, in complex dimension three, the holomorphic Euler characteristics only give integrality of $\frac{c_1c_2}{24}$ (for 0–forms or 3–forms) and $\frac{c_3}{2} - \frac{c_1c_2}{24}$ (for 1–forms or 2–forms) by Hirzebruch–Riemann–Roch.

Example 2.4. The Stong congruences described in Theorem 2.1, which can be interpreted as all the universal relations holding among Chern numbers of stably almost complex manifolds obtained from the Atiyah–Singer index theorem, allow one to rule out the existence of almost complex structures on manifolds with certain rational cohomology algebras. We consider $\mathbb{Q}[x]/(x^2)$ and $\mathbb{Q}[x]/(x^3)$ with $\deg(x) = 4$ as an example. For the former, we can avoid invoking the specific value of the signature (which is not determined, beyond being an integer, by the Stong congruences alone); for the latter however we cannot avoid the signature, as there are closed almost complex eight–manifolds with cohomology concentrated in middle degree (e.g. see [M22, Proposition 7.1]).

A closed manifold M with the rational cohomology of $\mathbb{HP}^1$, i.e. of S^4 (e.g., other than S^4, connected sums of the unoriented Grassmannian of real two–planes in $\mathbb{R}^4$), does not admit an almost complex structure. Namely, if it did, we would have $\int_M c_1^2 - 2c_2 = -4$. On the other hand, $\int_M c_1^2 + c_2$ is divisible by 12, and hence $\int_M c_1^2 - 2c_2$ is divisible by 3, a contradiction.

We remark that, by contrast, every orientable four–manifold admits a stable almost complex structure, as the only obstruction is the integral Stiefel–Whitney class W_3, which is known to vanish in this case.

Example 2.5. Moving up in dimension, there is no closed stably almost complex manifold with the rational cohomology of $\mathbb{HP}^2$. Namely, since c_1 and c_3 would have to be torsion classes, we see that the following congruences must hold (here and from now on we tacitly assume the Chern classes to be paired with the fundamental class):

$$3c_2^2 - c_4 \equiv 0 \bmod 720,$$
$$c_4 \equiv 0 \bmod 6.$$

Adding $15c_4$ to the first congruence, we have that $3c_2^2 + 14c_4 \equiv 0 \bmod 90$. However, $3c_2^2 + 14c_4$ is 45 times the signature, by expressing Hirzebruch's L-genus in terms of Chern classes. We conclude that the signature must be even, a contradiction.

Example 2.6. In the direction of existence results, we note how the Hirzebruch–Milnor theorem of non-existence of *stable* almost complex structures on $\mathbb{HP}^n$, for $n \geq 2$, can be immediately seen not to extend to the rational setting. Namely, for odd $n \geq 3$, setting all $c_i = 0$ in Theorem 2.1 one obtains a stable almost complex manifold with the rational homotopy type of $\mathbb{HP}^n$, as the first two points are trivially satisfied since $H^{2n}(\mathbb{HP}^n; \mathbb{Q}) = 0$ in this case.

3. Lemmas on e_i

For a closed almost complex $2n$-dimensional manifold consider its Chern classes $c_1, \ldots, c_n$ being the elementary symmetric polynomials in the Chern roots $x_1, \ldots, x_n$, and $e_1, \ldots, e_n$ being the elementary symmetric polynomials in the variables $e^{x_1} - 1, \ldots, e^{x_n} - 1$. Then we have the following observations/computations:

- We have explicitly computed e_{n-k} for $k = 0, 1, 2, 3, 4$ in terms of $c_1, \ldots, c_n$ and n (from this point onwards we assume that $n > 2k$).
- In general, when k is a fixed number, the coefficients in front of the top-degree products of $c_1, \ldots, c_n$ are polynomials in the variable n (with rational coefficients).

- Denote by $e_{n,i}$ the expression of e_n in terms of $c_1, \dots, c_n$ where n, as a symbol, is replaced with $n-i$. For example, $e_n = e_{n,0} = c_n$, and $e_{n,1} = c_{n-1}$. Then it holds $e_{n-k} = [\text{terms of degree } n] + e_{n-k-1,1}$.
- Therefore, when computing e_{n-k} recursively, starting with $k=0$, at each step it is sufficient to compute only the top degree terms in e_{n-k}, which is a purely combinatorical problem formulated in terms of partitions of the number n.

For example,

$$\begin{aligned}
e_n &= c_n, \\
e_{n-1} &= c_{n-1} + \frac{c_1c_{n-1} - nc_n}{2}, \\
e_{n-2} &= c_{n-2} + \frac{c_1c_{n-2} - (n-1)c_{n-1}}{2} + \frac{1}{6}c_1^2c_{n-2} - \frac{1}{12}c_{n-2}c_2 - \frac{(3n-4)}{12}c_1c_{n-1} + \frac{n(3n-5)}{24}c_n, \\
e_{n-4} &= \left(\frac{1}{384}n^4 - \frac{5}{192}n^3 + \frac{97}{1152}n^2 - \frac{251}{2880}n\right)c_n \\
&+ \left(\frac{-1}{48}n^3 + \frac{1}{6}n^2 - \frac{19}{48}n + \frac{1}{4}\right)c_{n-1} + \left(\frac{1}{8}n^2 - \frac{17}{24}n + \frac{11}{12}\right)c_{n-2} + \left(\frac{-1}{2}n + \frac{3}{2}\right)c_{n-3} \\
&+ c_{n-4} + \left(\frac{1}{720}\right)c_4c_{n-4} + -\frac{1}{240}c_3c_{n-3} + \left(-\frac{1}{96}n^2 + \frac{17}{288}n - \frac{13}{180}\right)c_2c_{n-2} \\
&+ \left(\frac{1}{24}n - \frac{1}{8}\right)c_2c_{n-3} + -\frac{1}{12}c_2c_{n-4} + \frac{1}{360}c_2^2c_{n-4} \\
&+ \left(-\frac{1}{96}n^3 + \frac{3}{32}n^2 - \frac{37}{144}n + \frac{1}{5}\right)c_1c_{n-1} + \left((\frac{1}{16}n^2 - \frac{19}{48}n + \frac{7}{12}\right)c_1c_{n-2} \\
&+ \left(-\frac{1}{4}n + \frac{5}{6}\right)c_1c_{n-3} + \frac{1}{2}c_1c_{n-4} - \frac{1}{720}c_1c_3c_{n-4} + \left(\frac{1}{48}n - \frac{47}{720}\right)c_1c_2c_{n-3} \\
&- \frac{1}{24}c_1c_2c_{n-4} + \left(\frac{1}{48}n^2 - \frac{5}{36}n + \frac{13}{60}\right)c_1^2c_{n-2} + \left(-\frac{1}{12}n + \frac{7}{24}\right)c_1^2c_{n-3} + \frac{1}{6}c_1^2c_{n-4} \\
&- \frac{1}{80}c_1^2c_2c_{n-4} + \left(-\frac{1}{48}n + \frac{3}{40}\right)c_1^3c_{n-3} + \frac{1}{24}c_1^3c_{n-4} + \frac{1}{120}c_1^4c_{n-4}.
\end{aligned}$$

We sketch the computation of e_{n-2}. First, we introduce more notation. Fix n. Order all symmetric polynomials in $x_1, \dots, x_n$ of degree n lexicographically in the following way. Each such polynomial is determined by a sequence of exponents, for example, c_n is given by $[1, \dots, 1]$ with n ones. We then order these sequences lexicographically (from lower to higher) in the obvious way, setting $s_0^n := [1, \dots, 1]$, $s_1^n := [2, 1, \dots, 1, 0]$, $s_2^n := [2, 2, \dots, 0, 0]$, $s_3^n := [3, 1, \dots, 0, 0]$, and so on.

By definition, e_{n-2} is the second elementary symmetric polynomial in $e^{x_1} - 1, \dots, e^{x_n} - 1$. Since terms of degree higher than n vanish, it will consist only of monomials in $x_1, \dots, x_n$ of degrees $n-2$, $n-1$, and n. Thus, e_{n-2} is a sum of three homogeneous symmetric polynomials of degrees $n-2$, $n-1$, and n, respectively. By inspecting, for example, the first product $(e^{x_1} - 1) \dots (e^{x_{n-2}} - 1)$, it becomes clear that

$$e_{n-2} = s_0^{n-2} + \frac{s_1^{n-1}}{2} + \frac{s_2^n}{4} + \frac{s_3^n}{6}.$$

Expressing each s_i^{n-k} in terms of $c_1, \dots, c_n$ yields the result.

Moreover, if one starts expressing the (lexicographically) highest term s_3^n first by writing $s_3^n = c_3c_{n-3} - \dots$, one sees that the dots in the expression consist of the (lexicographically) lower terms than s_3^n, namely, s_2^n, s_1^n, and s_0^n. In other words, recursion is possible.

4. Rational quaternionic projective spaces

Remark 4.1. When restricting to simply connected spaces, being rationally homotopy equivalent to $\mathbb{HP}^n$ is equivalent to having rational cohomology isomorphic to $H^*(\mathbb{HP}^n;\mathbb{Q})$.[1] Namely, take a space X with the latter property. A minimal model for $H^*(\mathbb{HP}^n;\mathbb{Q})$ is given by $\big(\Lambda(x,y); dx = 0, dy = x^{n+1}\big)$ with $\deg(x) = 4, \deg(y) = 4n+3$. We can construct a map from this minimal model to $A_{PL}(X)$ by sending x to a representative of a generator α of $H^*(X;\mathbb{Q})$, and sending y to any element whose differential in A_{PL} is α^{n+1}.

Theorem 4.2. *Let M be a closed $4n$–dimensional almost complex manifold with $H^2(M;\mathbb{Q}) = 0$. The Euler characteristic $\chi(M)$ satisfies*

$$12 \mid n \cdot \chi(M).$$

Proof. The Stong congruence $e_{n-2} \cdot \mathrm{td} \in \mathbb{Z}$ and $\langle c_n, [M]\rangle = \chi(M)$ yield the result. □

Corollary 4.3. *If a closed smooth $4n$–manifold M with $H^*(M;\mathbb{Q}) \cong H^*(\mathbb{HP}^n;\mathbb{Q})$ admits an almost complex structure, then $n \equiv 0, 3, 8,$ or $11 \bmod 12$.*

Proof. Since $\chi(\mathbb{HP}^n) = n+1$, we have

$$12 \mid n(n+1),$$

and the statement follows immediately. □

Theorem 4.4. *Let M be a closed $8n$–dimensional almost complex manifold with*

$$H^2(M;\mathbb{Q}) = H^4(M;\mathbb{Q}) = 0.$$

The Euler characteristic $\chi(M)$ satisfies

$$720 \mid n(480n^3 - 1200n^2 + 970n - 251)\chi(M).$$

Proof. The Stong congruence $e_{n-4} \cdot \mathrm{td} \in \mathbb{Z}$ and $\langle c_n, [M]\rangle = \chi(M)$ yield the result. □

Corollary 4.5. *If a closed smooth $8n$–manifold M with $H^*(M;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$, where $\deg(x) = 8$, admits an almost complex structure, then*

$$n \equiv 0, 80, 95, 144, 224, 239, 320, 335, 464, 479, 495, 560, 575, 639, 704, 719 \bmod 720.$$

or, equivalently,

$$n \equiv 0, 15 \bmod 16, \quad n \equiv 0, 2, 5, 8 \bmod 9, \quad \text{and} \quad n \equiv 0, 4 \bmod 5.$$

[1] That is, the rational algebra $H^*(\mathbb{HP}^n;\mathbb{Q})$ is *intrinsically formal*, i.e. there is only one rational homotopy type realizing it.

4.1. **A family of almost complex rational $\mathbb{HP}^3$'s.** In [M22, Theorem 7.3] it was shown that there is a closed almost complex manifold with the rational homotopy type of $\mathbb{HP}^3$. Since the rational cohomology ring of $\mathbb{HP}^3$ is $\mathbb{Q}[x]/(x^4)$, a choice of Chern classes c_2, c_4, c_6 corresponds to a choice of scalars $a, b, c \in \mathbb{Q}$ with $c_2 = ax$, $c_4 = bx^2$, $c_6 = cx^3$. The choice of the fundamental class $[X]$ on our rational homotopy type corresponds to choosing a rational number $d \neq 0$ with $\langle x^3, [X]\rangle = d$. As in loc. cit. we fix $d = 1$. The Stong congruences and signature condition then come down to requiring that $a \in \mathbb{Z}, b \in \mathbb{Q}$ (that a must be an integer follows from $\langle c_2^2, [X]\rangle \in \mathbb{Z}$, and c must be $\chi(\mathbb{HP}^3) = 4$) satisfy the following Diophantine system:

$$\begin{aligned} -a^3 + 4ab &\in 24\mathbb{Z}, \\ ab + 8 &\in 1920\mathbb{Z}, \\ 5a^3 - 36ab &= 248. \end{aligned}$$

We observe that $a = -2, b = 4$ as used in loc. cit. is in fact the only integer solution. Indeed, if b is an integer, expressing it from the last equation yields that $\frac{5a^3-248}{36a}$ is an integer. Hence, a is divisible by 2 and writing $a = 2\hat{a}$ yields again that $\frac{5\hat{a}^3-31}{9\hat{a}}$ is an integer, which implies that $\hat{a}$ divides 31, whence one concludes by direct check.

If, however, we do not require b to be an integer, there is an infinite family of solutions, which we now describe. Again, because $k := ab$ must be an integer, we have $a = 2\hat{a}$ for some $\hat{a}$. Writing $k = 2\hat{k}$ and noting that by Fermat's little theorem $(-\hat{a})^3 \equiv -\hat{a} \bmod 3$, we have

$$\begin{aligned} -\hat{a} + \hat{k} &\in 3\mathbb{Z}, \\ \hat{k} + 4 &\in 960\mathbb{Z}, \\ 5\hat{a}^3 - 9\hat{k} &= 31. \end{aligned}$$

Equivalently, we write $\hat{a} = 3n + 2$, $\hat{k} = 960l + 4$, and $(3n + 2)^3 + 1 = 1728l$. We are thus reduced to the question of when $(3n+2)^3+1 = (3n+3)(9n^2+9n+3)$ is divisible by 1728, that is, when $(n + 1)(3n^2 + 3n + 1)$ is divisible by 192. As $192 = 3 \cdot 2^6$, $(3n^2 + 3n + 1)$ is always coprime with 192. Therefore, all solutions are parametrized by $n = 192s - 1$, where s is an arbitrary integer. Resubstituting now gives us the parametrization of solutions

$$a = 1152s - 2, \; b = \frac{105615360s^2 + 960s - 4}{576s - 1}.$$

Since $ab = 211230720s^2 + 1920s - 8$ is the Chern number c_2c_4, we have the following:

Corollary 4.6. *There are infinitely many classes in the bordism group Ω_{12}^U represented by manifolds with the rational homotopy type of $\mathbb{HP}^3$.*

In fact, by construction [M22, Theorem 2.4] all of these manifolds represent classes in Ω_{12}^{SU}; they are simply connected with rationally vanishing c_1, and hence c_1 vanishes integrally.

The next step would be to understand the situation for an arbitrary rational $d \neq 0$, which we will not pursue here. Considering rational Poincaré duality algebras equipped with a choice of fundamental class, up to isomorphism that is compatible with the fundamental class there are $\mathbb{Q}^\times/(\mathbb{Q}^\times)^3$ equivalences classes.

4.2. **Towards an almost complex rational $\mathbb{HP}^8$.** As in the section above, since the rational cohomology ring of $\mathbb{HP}^8$ is $\mathbb{Q}[x]/(x^9)$, a choice of Chern classes $c_2, \dots, c_{16}$ corresponds to a choice of scalars $a_2, \dots, a_{16}$ with $c_i = a_i x^{2i}$.

There are 915 Stong congruences (with redundancies) that have to be satisfied. For example, one of them is

$$\begin{gathered} 162765c_2^8 - 838020c_2^6c_4 + 1246002c_2^4c_4^2 + 579586c_2^5c_6 - 534048c_2^2c_4^3 - 1256864c_2^3c_4c_6 \\ -380686c_2^4c_8 + 31005c_4^4 + 443426c_2c_4^2c_6 + 266117c_2^2c_6^2 + 545341c_2^2c_4c_8 + 227533c_2^3c_{10} \\ -69008c_4c_6^2 - 70810c_4^2c_8 - 174311c_2c_6c_8 - 180346c_2c_4c_{10} - 117033c_2^2c_{12} + 16365c_8^2 \\ +32951c_6c_{10} + 34345c_4c_{12} + 46551c_2c_{14} - 10851c_{16} \in 32011868528640000\mathbb{Z} \end{gathered}$$

At this point, we can neither prove nor disprove the existence of a solution in $a_2, \dots, a_{16}$. However, in attempts of proving any of the latter we discovered the following:

- One can treat the system of Stong congruences as a system of linear equations, where the 22 occurring Chern numbers are treated as independent integer variables. By finding the $m :=$ lcm of moduli of all the congruences we obtain a linear system over $\mathbb{Z}/m\mathbb{Z}$. By using the Smith normal form, one can parametrize the solution space of that system by 22 independent integer parameters. Our result is that we know the parametrization on the level of Chern numbers.
- On the other hand, however, we have shown that *none* of the c_{2i} (and thus a_{2i}) can be zero for a fixed solution. This means that if the solution in $a_2, \dots, a_{16}$ exists, it is highly non-trivial.

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