COMPUTATIONS IN CARTAN-DE RHAM HOMOTOPY THEORY

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Abstract. The sizable differential graded algebra of forms on a smooth manifold admits a tractable model which contains more homotopy information than the real cohomology algebra. We will determine this model for several manifolds, compute some higher homotopy groups modulo torsion, and discuss how to model fiber bundles. These are expanded notes for a talk given at the Stony Brook Graduate Student Seminar in February 2018.

1. Differential graded algebras and models

Consider the algebra of differential forms $\Omega(M)$ on a closed smooth manifold M. We know that by taking the cohomology $\frac{\ker(d)}{\mathrm{image}(d)}$ of this algebra with respect to the exterior derivative d, we get a useful invariant of the underlying homotopy type of the manifold. A result of de Rham tells us that this cohomology is isomorphic to any reasonable (singular, simplicial, ...) real-valued cohomology $H^*(M,\mathbb{R})$ you might consider for your manifold.

In transitioning from the forms $\Omega(M)$ to the cohomology $H^*_{dR}(M)$, we end up with a graded vector space with a multiplication (induced by cup product) which is commutative in the graded sense. That is, $a \cdot b = (-1)^{\deg(a)\deg(b)} b \cdot a$. What did we start with? The forms $\Omega(M)$ comprise a graded-commutative graded algebra as above, along with a differential. Taking cohomology with respect to this differential transformed the unwieldy algebra $\Omega(M)$ into a finite-dimensional algebra which, of course, now has no (non-trivial induced) differential. We have converted the infinite-dimensional real algebra of forms into a finite dimensional object, at the expense of losing d. One could wish for a similar "reduction" of the algebra of forms in a way that lets us keep a differential around.

Definition 1.1. A **dga** (differential graded algebra) (A, d) is a non-negatively graded vector space $A = \oplus_{i\geq 0} A^i$ such that $A^0 = \mathbb{R}$, with a multiplication $A^i \otimes A^j \to A^{i+j}$ and a differential $A^i \xrightarrow{d} A^{i+1}$ satisfying

- graded-commutativity, $a \cdot b = (-1)^{\deg(a)\deg(b)} b \cdot a$ (on homogeneous elements),
- $d^2 = 0$,
- the Leibniz rule $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$.

We will only consider "simply-connected" dga's, i.e. those with

- $A^1 = \{0\}$.

Our wish is to "model" $\Omega(M)$ with a small and manageable dga (A, d_A). Let us take that to mean that we seek a dga (A, d) with a map of dga's $(A, d_A) \xrightarrow{f} (\Omega(M), d)$ which induces an isomorphism on the level of cohomology,

$$H^*(A, d_A) \xrightarrow{f^*} H^*(\Omega(M), d).$$

(Note that what we are calling cohomology is in fact *homology* with respect to a positive degree differential, and so f^* is covariant.) We require of this modelling dga (A, d_A) to be finite-dimensional in every degree ("small"), free *as a graded-commutative associative algebra* (when forgetting the differential), and such that its differential d_A takes degree k elements into the subalgebra generated by elements of degree $\leq k-1$ for all k (call these last two properties "manageable").

First let us see some examples of small and manageable dga's, and then we will see what it means for one to model an algebra of forms on a manifold.

Example 1.2.

- Consider the free graded-commutative algebra on two generators x_2 and y_3, where the subscripts denote degrees. Let us denote this by $\Lambda(x, y)$. Place a differential on this algebra by setting $dx = 0, dy = x^2$. This algebra is certainly finite-dimensional in each degree, and its differential takes degree k elements into sums of products of lesser degree elements. Note that the algebra as a whole is *not* finite-dimensional. Indeed, all the powers $x^2, x^3, x^4, \ldots$ are non-zero due to the freeness of the algebra, but they are also in different degrees. Observe that $y^2 = 0$ simply due to graded-commutativity. Indeed,
$$y \cdot y = (-1)^{3 \cdot 3} y \cdot y,$$
and since our ground field is $\mathbb{R}$, this implies $y^2 = 0$.

- A non-example is given by $\Lambda(x_2, y_3, z_4)$ with differential
$$dx = 0, dy = x^2 + z, dz = 0.$$
The differential applied to y contains a z term, and so is not contained in the subalgebra generated by elements of degree ≤ 2.

Proposition 1.3. *For M a simply connected smooth manifold, the dga $\Omega(M)$ is modelled by a small and manageable dga. That is, there exists a dga (A, d_A) with the above desirable properties and a dga-map f (i.e. an algebra map that commutes with the differential) $(A, d_A) \xrightarrow{f} (\Omega(M), d)$ that induces an isomorphism on cohomology.*

The proof is constructive, and this construction is best illustrated through examples. The assumption of simple connectivity is not necessary, although it makes the construction somewhat easier.

2. Examples

Example 2.1. Let us model the forms on a two-sphere S^2. We build our model dga (A, d_A) and map f inductively by degree. In degree 0, we set $A^0 = \mathbb{R}$ as required, and map 1 to the constant function 1 on S^2. Since closed functions on S^2 are just constant functions, this will induce an isomorphism on H^0. Next up, we look at the volume form ω on S^2. We introduce a generator x in degree 2 in A, set $d_A x = 0$, and set $f(x) = 0$. This makes f an isomorphism on H^0, H^1, H^2. We might think we are done now, but recall that we require A to be free as an algebra. So, there are the non-zero elements $x^2, x^3, \ldots$ in degrees 4,6,8. Since $d_A(x^k) = kx^{k-1} d_A(x) = 0$ by the Leibniz rule (which we require of d_A), the powers of x induce non-vanishing cohomology classes above degree 2 in A, and so f will not be injective on $H^4, H^6, \ldots$. To remedy this, we would like to set $x^2 = 0$. But we cannot do that since we require freeness of A. We really only want $x^2 = 0$ *in cohomology*, which we can achieve by **introducing a new generator** y in degree 3 and setting $dy = x^2$. Also set $f(y) = 0$ (we have no other choice as there are no non-trivial 3–forms on S^2). Now observe that $x^3, x^4, \ldots$ are exact as well, since $d(x^k y) = x^{k+2}$. So, with $(A, d_A) = \Lambda(x_2, y_3,\ dx = 0, dy = x^2)$ mapping to $\Omega(S^2)$ via f defined by $f(x) = \omega,\ f(y) = 0$ (and extended to all of A) we have achieved our goal of modelling $\Omega(S^2)$ by a small and manageable dga A.

Definition 2.2. Call such a "small and manageable" dga with a map to $\Omega(M)$ a **minimal model of** the manifold$M$.

As promised, the minimal model of a manifold contains more information than just the real cohomology algebra. (It certainly contains the information of the real cohomology algebra as its cohomology is isomorphic to that of the manifold being modelled, by construction.) Let us content ourselves with the following tip of the iceberg:

Theorem 2.3. *For a closed simply-connected smooth manifold M with minimal model (A, d_A), we have the following equality for all $k \geq 2$:*

$$\dim(\pi_*(M) \otimes \mathbb{R}) = \#\ \textit{generators in degree } k \textit{ in } A.$$

(Recall that the higher homotopy groups are abelian, so the tensor product, taken over $\mathbb{Z}$, makes sense. The dimension is to be taken as a real vector space.)

Example 2.4. As we saw, a minimal model of S^2 is given by $\Lambda(x_2, y_3,\ dx = 0, dy = x^2)$. By the theorem we have $\pi_2(S^2) \otimes \mathbb{R} = \pi_3(S^2) \otimes \mathbb{R} = \mathbb{R}$ and $\pi_k(S^2) \otimes \mathbb{R} = 0$ for $k \geq 4$. These are reflections of classically known results. The second homotopy group of S^2 is free abelian generated by the identity map, and the third homotopy group is free abelian generated by the Hopf map $(z_1, z_2) \mapsto [z_1, z_2] \in \mathbb{CP}^1 \cong S^2$, where we interpret S^3 as the unit sphere in $\mathbb{C}^2$. By a result of Serre, the remaining homotopy groups of S^2 are all torsion, as seen in $\pi_k(S^2) \otimes \mathbb{R} = 0$ for $k \geq 4$.

Similarly as for S^2, we would obtain that a minimal minimal for any even sphere S^{2n} is given by $\Lambda(x_{2n}, y_{4n-1},\ dx = 0, dy = x^2)$. Again, this tells us that $\pi_{2n}(S^{2n}) \otimes \mathbb{R} = \mathbb{R}$ (as expected since $\pi_{2n}(S^{2n}) = \mathbb{Z}$), but also that $\pi_{4n-1}(S^{2n}) \otimes \mathbb{R} = \mathbb{R}$. So, $\pi_{4n-1}(S^{2n})$ contains exactly one factor of $\mathbb{Z}$. In general it is not true that $\pi_{4n-1}(S^{2n}) = \mathbb{Z}$. For example, $\pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}$.

Example 2.5. Let us now consider modelling S^3. As before for S^2, we introduce a variable x, now in degree 3, to map to the volume form on S^3. Now recall that $x^2 = 0$ due to graded-commutativity. So, there is no need to introduce an extra variable y to make the powers of x exact as we did before. We conclude that a minimal model for S^3, and more generally for any odd sphere S^{2n+1} is given by $\Lambda(x_{2n+1})$. Using the theorem we conclude that odd spheres have only one non-torsion homotopy group (also classically known to Serre).

The theorem above holds for a (much) wider class of manifolds called *nilpotent*, but we will not go into that now. In fact, the theorem above holds for any *space* that is "nilpotent", as long as its real cohomology is of finite dimension in every degree.

Example 2.6. Let us now consider some slightly more complicated examples. First of all, by an analogous procedure we can model $\mathbb{CP}^2$ by $\Lambda(x_2, y_5,\ dx = 0, dy = x^3)$. Now we try to model the connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$. Inside $\mathbb{CP}^2 \# \mathbb{CP}^2$ there are two copies of $\mathbb{CP}^1$ meeting at a point. Consider volume forms ω_1 and ω_2 on these two $\mathbb{CP}^1$'s, considered as 2–forms on $\mathbb{CP}^2 \# \mathbb{CP}^2$ by extending the forms to tubular neighborhoods using bump functions. Now in cohomology we have $[\omega_1] \cdot [\omega_2] = 0$

since we can represent these classes by forms with disjoint support. Both forms ω_1^2 and ω_2^2 represent volume forms, and we can normalize ω_1 and ω_2 so that each square integrates to 1. This tells us that $\int_{\mathbb{CP}^2\#\mathbb{CP}^2}\omega_1^2-\omega_2^2=0$, and so this is an exact form. These observations inform us that in our minimal model and map f, we should have two closed degree 2 generators a and b mapping to ω_1 and ω_2, along with two 3–forms ϵ and η such that $d\epsilon=ab$, $d\eta=a^2-b^2$, with $f(\epsilon)$ equal to any 3–form ξ such that $d\xi=ab$, and $f(\eta)$ equal to any 3–form ξ' such that $d\xi'=a^2-b^2$. A slightly tedious check shows that this dga

$$\Lambda(a_2,b_2,\epsilon_3,\eta_3,\ da=db=0,d\epsilon_3=ab,d\eta_3=a^2-b^2)$$

along with the map f described above is indeed a minimal model for $\mathbb{CP}^2\#\mathbb{CP}^2$ (i.e. no further generators are needed).

Observe in particular that $\pi_3(\mathbb{CP}^2\#\mathbb{CP}^2)\neq 0$, while $\pi_3(\mathbb{CP}^2)=0$. Further, $\pi_5(\mathbb{CP}^2\#\mathbb{CP}^2)$ is torsion while $\pi_5(\mathbb{CP}^2)=\mathbb{Z}$.

Example 2.7. We can also model the blowup of $\mathbb{CP}^2$ at a point, which is smoothly $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. The difference here from the case of $\mathbb{CP}^2\#\mathbb{CP}^2$ is that the degree 2 generator b in the second summand now squares to something that integrates to a negative value. Normalizing cannot switch this sign, so now the η_3 we introduce is such that $d\eta_3=a^2+b^2$ instead. All else remains the same as in the previous example. A minimal model of $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is given by

$$\Lambda(a_2,b_2,\epsilon_3,\eta_3,\ da=db=0,d\epsilon_3=ab,d\eta_3=a^2+b^2).$$

We note that the minimal models of $\mathbb{CP}^2\#\mathbb{CP}^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ are *not* isomorphic. Indeed, the intersection form on H^2 of the minimal model of $\mathbb{CP}^2\#\mathbb{CP}^2$ is given by $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ while that of the minimal model of $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is given by $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Note that the signatures (invariant under real cohomology algebra isomorphisms) do not coincide.

The minimal model of a manifold contains "all" the non-torsion homotopy information in a precise sense. We have the following result (which can be interpreted as an equivalence of the categories of dga's and the category of simply connected spaces with real cohomology isomorphisms inverted).

Theorem 2.8. *Two simply connected manifolds M and N have isomorphic minimal models if and only if there is a sequence of spaces $S_1,\ldots,S_{r+1}$ (not necessarily manifolds) and maps $f_1,\ldots,f_r$ which induce isomorphisms on real cohomology*

$$M\xleftrightarrow{f_1}S_1\xleftrightarrow{f_2}\cdots\xleftrightarrow{f_r}S_r\xleftrightarrow{f_{r+1}}N.$$

Here $\longleftrightarrow$ denotes that the corresponding map could be pointing in either direction.

We have also been mixing the expressions "a minimal model" with "*the* minimal model". This is justified by the fact that **any two minimal models for a fixed manifold are isomorphic**.

3. Modelling fiber bundles

These dga models work nicely when considering models of fiber bundles. Suppose we have a fiber bundle $F\to E\to B$, and suppose we know the models of F and B, which we will denote simply by $(F,d_F),(B,d_E)$. A model for the total space E of the fiber bundle is given by

$$(F\otimes B,d),$$

where $F \otimes B$ denotes the tensor product (over $\mathbb{R}$) of the underlying algebras, and d is such that

- applied to a generator coming from the base B, it is equal to d_B. More precisely, $d(b \otimes 1) = d_B(b) \otimes 1$ for $b \in B$.
- applied to a generator f coming from the fiber (i.e. $1 \otimes f$), it is of the form
$$df = d_F f + \text{ mixed terms }.$$
Here by "mixed terms" we mean some polynomial in generators coming from the fiber or the base. Mixed terms can also consist purely of polynomials in generators coming from the base.

How exactly the differential on the total space acts on the fiber generators encodes the difference between one F–fiber bundle over B and another.

Example 3.1. Let us consider S^1–bundles over S^2. (We have been assuming our manifolds are simply connected up to now, but S^1 and tori in general are simple enough to be well-behaved in this theory.) We fix a model $\Lambda(\alpha_1,\ d\alpha = 0)$ for S^1 and $\Lambda(x_2, y_3,\ dx = 0, dy = x^2)$ for S^2. Now, any total space E of such a fibration will have a model whose underlying algebra is $\Lambda(\alpha, x, y)$, and the differential d in this model satisfies $dx = 0$ and $dy = x^2$. The only degree of freedom is in assigning what $d\alpha$ will be in the total space. For degree reasons, the only possibilities are $d\alpha = cx$ for $c \in \mathbb{R}$. We distinguish two cases:

- $c = 0$. In this case, the model of the total space E is just the tensor product of the models of S^1 and S^2. This is the same as the model of $S^1 \times S^2$.
- $c \neq 0$. We can redefine α to be $\frac{1}{c}\alpha$, and so we can assume $c = 1$. Our total space therefore has model
$$\Lambda(\alpha, x, y,\ dx = 0, dy = x^2, d\alpha = x).$$
Note that this dga is *not minimal*, since $d\alpha = x$. We see that we can model it by a minimal dga mapping a single degree three generator $\tilde{y}$ to the closed element $y - \alpha x$. Our minimal model for the total space is then $\Lambda(\tilde{y}_3,\ d\tilde{y} = 0)$, which is the model of S^3.

Example 3.2. Analogously to the previous example, we can consider fiber bundles with fiber S^3 and fiber S^4. The $c = 1$ case gives us a total space with the same minimal model as that of S^7. Similarly there is an S^7 fibration over S^8 whose total space has the same minimal model as S^{15}. The Hopf fibrations in the quaternion and octonion cases tell us that there are fibrations $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$ where the total spaces are honest spheres, and not just spaces with the same minimal model as these spheres. For example, consider modelling S^{15}–fibrations over S^{16}. The $c = 1$ case again gives us a total space whose minimal model is that of S^{31} (and a result in the theory tells us that this space can be taken to be a smooth manifold). However, there is no fibration $S^{15} \to S^{31} \to S^{16}$. Indeed, if there was, the long exact sequence in homotopy groups would yield $\pi_{30}(S^{16}) = \pi_{29}(S^{15})$, but $\pi_{30}(S^{16}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\pi_{29}(S^{15}) = \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Example 3.3. We can ask ourselves how many S^1–bundles there *really* are over S^2. It turns out that every circle bundle over a manifold can be realized by the unit sphere bundle of a complex line bundle. Over S^2, the clutching construction shows us that there are $\pi_1(\mathrm{GL}(1, \mathbb{C}) = \mathbb{Z}$ many such bundles. By rotating the fiber S^1 by an angle of π, we get a bundle isomorphism between the bundles corresponding to the integers k and $-k$, for any k. So let us say that our bundles are indexed by $k = 0, 1, 2, \ldots$. Using the Gysin sequence in cohomology along with Poincaré duality and the universal coefficient theorem, we can see that the total space E_k of the S^1–bundle corresponding to $k \in \pi_1(GL_{\mathbb{C}}(1))$ has integral cohomology

$$\begin{aligned} &H^0(E_k, \mathbb{Z}) = \mathbb{Z}, \\ &H^1(E_k, \mathbb{Z}) = 0 \text{ if } k \neq 0 \text{ and } \mathbb{Z} \text{ if } k = 0, \\ &H^2(E_k, \mathbb{Z}) = \mathbb{Z}_k \text{ if } k \neq 0 \text{ and } \mathbb{Z} \text{ if } k = 0, \\ &H^3(E_k, \mathbb{Z}) = \mathbb{Z}. \end{aligned}$$

Note how all of the integral cohomology rings for $k \neq 0$ are identified after tensoring with $\mathbb{R}$. Using minimal models we can only tell apart $k = 0$ and $k \neq 0$. In particular, $\mathbb{RP}^3$ (corresponding to $k = 2$) is identified with S^3 (corresponding to $k = 1$). Though this is not as bad as it may seem. Here, the class $d\alpha$ is the Euler class of the circle bundle. Minimal models retain not only the information about whether this class is zero or not, but also which "line" the Euler class lies on in cohomology.

Example 3.4. We can complicate things a little by considering S^2–fiber bundles over S^2. Model the fiber sphere by $\Lambda(\alpha_2, \beta_3,\ d\alpha = 0, d\beta = \alpha^2)$ and the base sphere by $\Lambda(x_2, y_3,\ dx = 0, dy = x^2)$. The total space E of such a fiber bundle will be modelled by

$$\Lambda(x_2, y_3, \alpha_2, \beta_3,\ dx = 0, dy = x^2, d\alpha = 0, d\beta = \alpha^2 + c_1\alpha x + c_2 x^2),$$

where $c_i \in \mathbb{R}$. Note that $d\alpha$ cannot contain a y term even though the degrees match up, since we should have $dd\alpha = 0$, but $dy = x^2$.

The case of $c_1 = 0, c_2 = 0$ corresponds to the trivial fibration $S^2 \times S^2$. Let us consider $c_1 = 1, c_2 = 0$, i.e. the total space E modelled by

$$\Lambda(x_2, y_3, \alpha_2, \beta_3,\ dx = 0, dy = x^2, d\alpha = 0, d\beta = \alpha^2 + \alpha x).$$

Note that this dga is already minimal, and it is in fact isomorphic to a minimal dga we have already seen. An isomorphism Φ between

$$\Lambda(\tilde{x}_2, \tilde{y}_3, \tilde{\alpha}_2, \tilde{\beta}_3,\ d\tilde{x} = d\tilde{\alpha} = 0, d\tilde{y} = \tilde{x}^2 + \tilde{\alpha}^2, d\tilde{\beta} = \tilde{a}\tilde{x})$$

and

$$\Lambda(x_2, y_3, \alpha_2, \beta_3,\ dx = d\alpha = 0, dy = x^2, d\beta = \alpha^2 + \alpha x)$$

is given by

$$\begin{aligned} \Phi(\tilde{x}) &= \alpha + x, \\ \Phi(\tilde{\alpha}) &= \alpha, \\ \Phi(\tilde{y}) &= y + 2\beta, \\ \Phi(\tilde{\beta}) &= \beta. \end{aligned}$$

Therefore this fiber bundle has the same minimal model as $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

The space of diffeomorphisms of S^2 has the homotopy type of $GL_+(3, \mathbb{R})$ and so we can see from the clutching construction again that there are indeed only two S^2–bundles over S^2, which are $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

4. Formality of manifolds

A desirable property of a dga A would be that we can form its minimal model directly from the even-more-manageable dga $(H^*(A), 0)$ (where 0 is the trivial differential). We say the dga is **formal** if this is the case, i.e. if there is a map $\text{Model}(A) \xrightarrow{f} H^*(A)$ inducing an isomorphism on cohomology, where $\text{Model}(A)$ is the minimal model of A. If we look at how we modelled the spheres and the connected sums of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$, we can see that only the cohomology rings of these manifolds were important when creating the minimal model. The next example will demonstrate how a manifold can fail to be formal.

Example 4.1. Consider the map $S^2 \times S^2 \xrightarrow{c} S^4$ which crushes the complement of a small disk on $S^2 \times S^2$ to a point, and recall the S^3–fiber bundle over S^4 whose total space is S^7. We can model the fiber S^3 with one generator t_3, and the base S^4 with $\Lambda(x_4, y_7,\ dx = 0, dy = x^2)$. In the total space of the fiber we have $du = x$. Now, consider the pullback over this S^3–bundle via the map c to an S^3–bundle over $S^2 \times S^2$.

$$\begin{array}{ccccccc} S^3 & \longrightarrow & E & \longrightarrow & S^7 & \longleftarrow & S^3 \\ & & \downarrow & & \downarrow & & \\ & & S^2 \times S^2 & \xrightarrow{c} & S^4 & & \end{array}$$

Model $S^2 \times S^2$ by $\Lambda(a_2, b_2, u_3, v_3,\ da = db = 0, du = a^2, dv = b^2)$. On the level of models, c^* takes the volume form x to the volume form ab, and $d(f^*(t)) = f^*(dt) = f^*(x) = ab$. So, our S^3–bundle E over $S^2 \times S^2$ is modelled by

$$\Lambda(a, b, u, v, t,\ da = db = 0, du = a^2, dv = b^2, dt = ab),$$

and observe that this is a minimal model. Note that $c^* y = \frac{1}{2}(a^2 v + ub^2)$. Computing cohomology, we have

$$\begin{aligned} H^2(E) &= \text{span}([a], [b]), \\ H^5(E) &= \text{span}([ub - at], [tb - av]), \\ H^7(E) &= \text{span}([abz - a^2 v] = [abz - b^2 u]), \end{aligned}$$

and all other groups are trivial. Now suppose there was a map f from the minimal model to the cohomology algebra with trivial differential inducing an isomorphism on cohomology (where the cohomology of the cohomology algebra is just itself). For degree reasons, $f(u) = f(v) = 0$, and so the classes $[ub - at]$ and $[tb - av]$ in degree five cannot be hit by f, and so f cannot be surjective.

Non-formality here is detected by the non-trivial *Massey products* $[ub - at]$ and $[tb - av]$, which are non-trivial cohomology classes in the ideal generated by non-closed generators in the model.

Deciding a priori which manifolds are formal is an active area of research. In [5] it is proved that a closed manifold of dimension $\leq 4k + 2$ with vanishing cohomology groups $H^1, H^2, \ldots, H^k$ is formal. If furthermore $\dim_{\mathbb{R}} H^{k+1} = 1$, then dimension $\leq 4k + 4$ implies formality ([6]). There is the now classical result [3] that closed manifolds admitting a Kähler metric are formal. In [7] it is proved that quaternion-Kähler closed manifolds admitting a metric of positive scalar curvature are formal. Therein is stated the following conjecture:

Conjecture 4.2. Manifolds with special holonomy are formal.

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