Quantum algorithms (CO 781) Winter 2008

ASSIGNMENT 3 due Thursday 27 March (in class)

Problem 1 (*The triangle problem*).

In the *triangle problem*, you are asked to decide whether an n-vertex graph G contains a triangle (a complete subgraph on 3 vertices). The graph is specified by a black box that, for any pair of vertices of G, returns a bit indicating whether those vertices are connected by an edge in G.

a. What is the classical query complexity of the triangle problem?

b. We say that an edge of G is a *triangle edge* if it is part of a triangle in G. What is the quantum query complexity of deciding whether a particular edge of G is a triangle edge?

c. Now suppose you know the vertices and edges of some m-vertex subgraph of G. Explain how you can decide whether this subgraph contains a triangle edge using $O(m^{2/3}\sqrt{n})$ quantum queries.

d. Consider a quantum walk algorithm for the triangle problem (or, equivalently, deciding whether a graph contains a triangle edge). The walk takes place on a graph $\mathcal{G}$ whose vertices correspond to subgraphs of G on m vertices, and whose edges correspond to subgraphs that differ by changing one vertex. A vertex of $\mathcal{G}$ is marked if it contains a triangle edge. How many queries does this algorithm use to decide whether G contains a triangle? (Hint: Be sure to account for the queries used to initialize the walk, the queries used to move between neighboring vertices of $\mathcal{G}$, and the queries used to check whether a given vertex of $\mathcal{G}$ is marked. To get a nontrivial result, you should use the search framework mentioned in class that takes many steps according to the walk on $\mathcal{G}$ with no marked vertices before performing a phase flip at marked vertices.)

e. Choose a value of m that minimizes the number of queries used by the algorithm. What is the resulting upper bound on the quantum query complexity of the triangle problem?

f. *Challenge problem:* Generalize this algorithm to decide whether G contains a k-clique. How many queries does the algorithm use?

Problem 2 (*Grover's algorithm by formula evaluation*).

Grover's algorithm computes the OR of n bits using $O(\sqrt{n})$ quantum queries to those bits. In this problem you will give an alternative algorithm for computing OR by evaluating a NAND formula.

Since $\mathrm{OR}(x_1, \ldots, x_n) = \mathrm{NAND}(\bar{x}_1, \ldots, \bar{x}_n)$, we can represent the OR formula by a NAND tree in which the root has n children, and each of those children has one child, which is a leaf. Given an input $x_1, \ldots, x_n$, we modify the tree by deleting every leaf in the original tree corresponding to an index i for which $x_i = 1$.

We will start our quantum algorithm from the root, so you can restrict your attention to the subspace $\mathcal{S} := \mathrm{span}\{H^j|\mathrm{root}\rangle : j = 0, 1, 2, \ldots\}$, where H is a weighted adjacency matrix of the tree (with weights to be determined).

a. First consider the input $x_1 = \cdots = x_n = 0$, for which the formula evaluates to 0. Define the weighted adjacency matrix H of the corresponding tree by assigning a weight of α to the edges connected to the root and a weight of 1 to the remaining edges. Compute the spectrum (both eigenvalues and eigenvectors) of H within the subspace $\mathcal{S}$.

b. For what values of α does H (as defined in part a) have an eigenstate of eigenvalue 0 with overlap $\Omega(1)$ on the root?

c. Now consider an input with $x_i = 1$ for precisely one index i. Compute the spectrum of H within the subspace $\mathcal{S}$.

d. For what values of α does H (as defined in part c) have a minimum eigenvalue of $\Omega(1/\sqrt{n})$ (in absolute value)? Choose a value of α so that this condition and the one from part b are satisfied simultaneously.

e. Compute the spectrum of H for an arbitrary input, and show that the minimum eigenvalue of H (again in absolute value) can only be larger than in part c if there is more than one index i for which $x_i = 1$.

f. Explain why your calculations imply a discrete-time quantum walk algorithm for computing the OR of n bits using $O(\sqrt{n})$ queries. (Hint: Refer to problem 5 from assignment 2.)

g. *Challenge problem:* Describe a simulation of the continuous-time quantum walk generated by H that computes OR using $O(\sqrt{n})$ queries. (Notice that the root of the tree has high degree, so you cannot use results on the simulation of sparse Hamiltonians.)

Problem 3 (*Adiabatic evolution of a qubit*).

Consider a spin in a magnetic field that is rotated from the $-x$ direction to the $-z$ direction in a total time T. Such a spin is described by the Hamiltonian

$$H(t) = -\cos\left(\frac{\pi t}{2T}\right)\sigma_x - \sin\left(\frac{\pi t}{2T}\right)\sigma_z.$$

Suppose that at time $t = 0$, the spin is in the ground state of $H(0)$. Plot the behavior of the x, y, and z components of the spin as a function of time from $t = 0$ to $t = T$, where $T = 5$, 10, or 50. Comment on the results in light of the adiabatic theorem.

Problem 4 (*Perturbation theory*).

Let $H(s)$ be a Hermitian matrix depending smoothly on a parameter $s \in \mathbb{R}$. Let $P(s)$ be the projector onto the eigenstate of $H(s)$ with the smallest eigenvalue, which is separated by a gap $\Delta(s) > 0$ from the rest of the spectrum. (In particular, the eigenstate is non-degenerate for all values of s.)

a. Prove that

$$\|\dot{P}(s)\| \le c_1 \frac{\|\dot{H}(s)\|}{\Delta(s)}$$

for some constant $c_1 > 0$, where $\dot{X}(s) := \frac{\mathrm{d}}{\mathrm{d}s}X(s)$, and as usual, $\|X\|$ denotes the spectral norm of X. (Hint: This is a formalization of first-order non-degenerate perturbation theory, as discussed in any introductory textbook on quantum mechanics; you could give a proof along those lines. Alternatively, if you are comfortable with complex analysis, define the *resolvent*, $R(z,s) := (H(s) - z)^{-1}$, in terms of which $P(s) = -\frac{1}{2\pi \mathrm{i}}\int_\Gamma R(z,s)\mathrm{d}z$, where Γ is a contour enclosing only the smallest eigenvalue of $H(s)$; upper bound $\|\dot{P}(s)\|$ by integrating around some circular contour.)

b. Prove that

$$\|\ddot{P}(s)\| \le c_2 \frac{\|\ddot{H}(s)\|}{\Delta(s)} + c_3 \frac{\|\dot{H}(s)\|^2}{\Delta(s)^2}$$

for some constants $c_2, c_3 > 0$, where $\ddot{X}(s) := \frac{\mathrm{d}^2}{\mathrm{d}s^2}X$.

Problem 5 (*Tunneling in the adiabatic algorithm*).

In quantum mechanics, particles can tunnel through a classically impenetrable barrier. In this problem you will see how tunneling allows an adiabatic algorithm to minimize a cost function that could not be minimized by a classical local search algorithm such as simulated annealing.

a. Consider adiabatic optimization of a cost function $h : \{0,1\}^n \to \mathbb{R}$ for which $h(x)$ depends only on $|x| := \sum_i x_i$, the *Hamming weight* of x. In particular, consider the Hamiltonian $H(s) := (1-s)H_B + sH_P$, where the initial and final Hamiltonians are

$$H_B := -\sum_{j=1}^{n} \sigma_x^{(j)} \qquad\qquad H_P := \sum_{x\in\{0,1\}^n} h(x)|x\rangle\langle x|.$$

Show that evolution of the initial state $|u\rangle := \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle$ according to the Hamiltonian $H(t/T)$ remains in the subspace $\mathrm{span}\{|[k]\rangle : k = 0,1,\ldots,n\}$, where $[k]$ denotes the set of n-bit strings of Hamming weight k.

b. Suppose that $h(x) = |x|$, and compute the spectrum of $H(s)$ in the subspace of Hamming weight states as a function of $s \in [0,1]$. In particular, show that the minimum gap between the ground and excited states of $H(s)$ is at least some constant, independent of n.

c. Now suppose that $h(x) = |x| + \Delta(|x|)$, where $\Delta(w)$ is a non-negative function of width $\approx n^\delta$ and height $\approx n^\epsilon$ centered around $w = w_0$. For concreteness, suppose that

$$\Delta(w) = \begin{cases} 0 & w < w_0 - n^\delta \text{ or } w > w_0 + n^\delta \\ n^\epsilon & w_0 - n^\delta \le w \le w_0 + n^\delta. \end{cases}$$

Define a *local search algorithm* as a classical randomized algorithm that works as follows:

- Initialize x to a random bit string.
- For $i = 1$ to $\mathrm{poly}(n)$:
  - Let y_i be some string with $O(1)$ bits equal to 1.
  - If $h(x \oplus y_i) > h(x) + O(1)$, leave x unchanged. Otherwise, leave x unchanged or set x equal to $x \oplus y_i$ according to some specified rule.
- Output x.

Argue that if $\delta, \epsilon > 0$ are constants and $w_0 < cn$ for some constant $c < 1/2$, a local search algorithm is unlikely to find the minimum of $h(x)$.

d. Finally, analyze the performance of the adiabatic algorithm for minimizing $h(x) = |x| + \Delta(|x|)$. Since $\Delta(w) \ge 0$, the eigenvalues of $H(s)$ can only be larger than in part b. Thus, to lower bound the gap between the ground and first excited states of $H(s)$, it suffices to upper bound the perturbed ground state energy. Using the original ground state as an ansatz, give an upper bound on the ground state energy of $H(s)$. What are the conditions on δ, ϵ such that the minimum gap is at least $1/\mathrm{poly}(n)$?