Pretty good fractional revival via magnetic fields: theory and examples

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Algebraic Graph Theory Seminar

January 22, 2024

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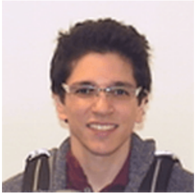
Joint work with Whitney Drazen and Mark Kempton



1. {CCDEGKLTZ} Fundamentals of fractional revival
   LAA **655** (2022) 129–158.
2. {DKL} Pretty good fractional revival via magnetic fields: theory and examples, arXiv:2311.18143, 2023 November

Overview

Overview

Quantum walk

- $G(V, E)$ finite graph
- H is an associated Hamiltonian
- For us $H = A + Q$ where Q is a diagonal matrix describing the potential
- Study evolution of

$$U(t) = \exp(itH)$$

- 1-excitation subspace of a spin network

State Transfer vs Fractional Revival

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- Motivation: quantum information is transferred from u to v.
- Fractional revival: $u, v \in G$, at some time t one looks for

$$U(t)|_{\{u,v\}} = \begin{pmatrix} a & b \\ b & a' \end{pmatrix} = Z$$

  where $|a|^2 + |b|^2 = |a'|^2 + |b|^2 = 1$ so Z is unitary.
- Motivation: at this time nodes u, v are in an entangled state given by Z.
- Note: state transfer is the special case of $a = a' = 0$.

Fractional Revival in subsets

- Let $S \subset G$ be a set of nodes.
- The Hamiltonian H has S-fractional revival at time t if

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- This is clearly a generalization of fractional revival between two nodes.
- Can be used to generate entanglement among more than 2 particles.

Pretty good?!

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- Is there a unitary matrix in the closure?

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- Asymptotic version? PGFR!
- Consider the closure of $\{U(t)|_S : t \in [0, \infty)\}$.
- Is there a unitary matrix in the closure?
- Caveat: identity matrix is always in the closure!
- Need a matrix with ≥ 2 distinct eigenvalues.

In summary...

Definition (PGFR)

We say that a set of nodes $S \subset V(G)$ has pretty good fraction revival if

$$\mathrm{cl}\{exp(itH)_{S\times S} : t \geq 0\} \cap U(S) \not\subseteq \{\rho\, \mathrm{Id}_{S\times S} : \rho \in \mathbb{C}\}$$

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Caveat:

Would a diagonal block be interesting? .... No!

But it turns out, for connected graphs, it never happens!

Overview

1. Introduction
2. Kronecker condition
3. Fractional cospectrality?
4. Examples

Partitioning the spectrum

- Eigenvectors of $H \Longrightarrow$ eigenvectors of Z
- $exp(it\lambda) \approx \rho$
- Identical restrictions
- Partition $\Pi_0, \Pi_1, \ldots, \Pi_s$ of the spectrum of H. It is independent of Z.
- Generated by the eigenvalue support Φ:

$$(\lambda_i, \lambda_j) \in \Phi \Leftrightarrow E_i D_S E_j \neq 0$$

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Non-degenerate partition

- To have PGFR we need a non-degenerate partition:
  1. Unit complex numbers $\rho_1, \ldots, \rho_s$ that aren't all the same.
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- s-Kronecker condition: the partition is non-degenerate if and only if $\exists 1 \leq r_1 < r_2 \leq s$ such that there is no sequence of integers l_j for which
  1. $\sum l_j \lambda_j = 0$
  2. $\sum_{j \in \Pi_r} l_j = 0$ for $r \neq r_1, r_2$
  3. $\sum_{j \in \Pi_{r_1}} l_j = 1$ and $\sum_{j \in \Pi_{r_2}} l_j = -1$.

Kronecker proof I

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Field-trace method

- How to verify the Kronecker condition?
- The only method we know: field trace
- Required setup:
  1. Find polynomials $P_1, \ldots, P_s$ such that the roots of P_j are exactly the elements of Π_j.
  2. Find a field in which these are irreducible.
  3. Find two values r_1, r_2 so that

$$\frac{\operatorname{tr} P_{r_1}}{\deg P_{r_1}} \neq \frac{\operatorname{tr} P_{r_2}}{\deg P_{r_2}}$$

Overview

Z-cospectral subsets

How to find such polynomials? Cospectrality!

Definition

The subset S is Z-cospectral if there is an orthonormal basis $\psi_1, \ldots, \psi_{|G|}$ of eigenvectors of H such that the restrictions of these to S are all eigenvectors of the $S \times S$ matrix Z.

Remark

- *The "strong" analogue would be to require this for every eigenvector, not just a basis.*
- *For* $Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *we recover the classic notion.*

Equivalent characterizations

Let $S \subset G, H, Z$ as above. The following are equivalent:

1. S is Z-cospectral.
2. $Z\widetilde{H^k} = \widetilde{H^k}Z$ for all k.
3. $Z\tilde{E}_i = \tilde{E}_i Z$ for all j.
4. $F_j\tilde{E}_i = \tilde{E}_i F_j$ for all i, j.
5. For any v, w eigenvectors of Z belonging to different eigenvalues, the subspaces $\langle H^k\hat{v} : k = 0, 1, \ldots\rangle$ and $\langle H^k\hat{w} : k = 0, 1, \ldots\rangle$ are orthogonal.

The case of simple eigenvalues

- When Z has only simple eigenvalues, the theory becomes very useful. Important: Z does not have to be symmetric, only normal!
- The relative minimal polynomials of the eigenvectors of Z become the P_j polynomials.
- Adding a generic diagonal potential to the subset preserves cospectrality and makes the polynomials irreducible.
- All that remains is to verify the trace/degree condition.

What happened to “parallel”?!

- $|S| = 2$
- Larger S?

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- Larger S?

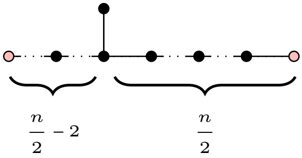
Theorem

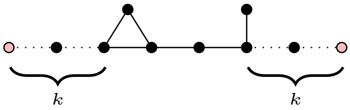
The eigenvalue support Φ is the same as pairs of roots of the P_j polynomials.

Overview

2-vertex

- These examples were first found by computer search.
- They can be easily verified using the minimal polynomial approach.
- Due to their nature, degree and trace are easily computable.





Computing $\tilde{H}^k$

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$\mathbb{Z}_p$-symmetry

- How to get examples of PGFR on bigger subsets?
- Currently our only examples are graphs with a $\mathbb{Z}_p$ symmetry.
- Fractional cospectrality is easy to verify with respect to the A_{DC_p}, the adjacency matrix of the directed cycle graph. This is non-symmetric, but normal, and has simple eigenvalues.
- We need

$$\frac{\operatorname{tr} P_1}{\deg P_1} \neq \frac{\operatorname{tr} P_2}{\deg P_2}$$

  Since trace always contains the transcendental Q, it's enough to have different degrees, or different traces if the degrees are equal.

Examples with different degrees

Examples with different traces

Further directions

Problem

Is there a way to determine the trace/degree of the P_j polynomials? A useful combinatorial theory is still lacking!

Problem

Find examples without automorphisms for $|S| \geq 3$.

The End

Thank you!