Eigenpolytopes

Let X be a graph on n vertices. Assume θ is an eigenvalue of $A = A(X)$, with multiplicity d.

Let U be an $n \times d$ matrix whose columns form an orthogonal basis for the θ-eigenspace.

Then

(a) $U^T U = I_d$

(b) $AU = \theta U$

Further

$$uu^T \cdot uu^T = uu^T$$

and so

(c) uu^T represents projection onto the θ-eigenspace.

We set $E_\theta = uu^T$, it is the spectral idempotent belonging to θ.

Assume $V(X) = \{1, \dots, n\}$. We define $u_i(\theta) := e_i^T U$.

The convex hull of $\{u_i : i = 1, \dots, n\}$ is the

θ-eigenpolytope.

Lemma $\theta u_i = \sum_{j \sim i} u_j$. $\square$

Lemma Each automorphism of X induces an automorphism of each eigenpolytope.

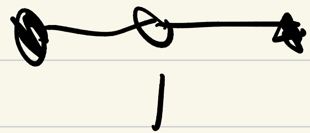
Proof. $P \in \mathrm{Aut}(X) \Leftrightarrow PA = AP$. So P fixes each eigenspace. $\square$

As $A^k = \sum_r \theta_r^k E_r = \sum_r \theta_r^k u_r u_r^T$, we have

$$(A^k)_{i,j} = \sum_r \theta_r^k \langle u_i(\theta_r), u_j(\theta_r)\rangle$$

and so the inner products $\langle u_i(\theta_r), u_j(\theta_r)\rangle$, for all r, determine $(A^k)_{i,j}$ for all k.

Theorem If X is distance-regular, $\langle u_i(\theta_r), u_j(\theta_r)\rangle$ is determined by $\mathrm{dist}_X(i,j)$. $\square$



$\mathbb{R}^2$

nbl n vx

Lemma If X is walk-regular, $\langle u_i(\theta_r), u_i(\theta_r)\rangle = \dfrac{\mathrm{mult}(\theta_r)}{n}$

Lemma If X is 1-walk regular with valency k and $i \sim j$, then

$$\frac{\langle u_i(\theta_r), u_j(\theta_r)\rangle}{\langle u_i(\theta_r), u_i(\theta_r)\rangle} = \frac{\theta_r}{k}.$$ (cosine of angle)

Proof As $\theta_r u_i(\theta_r) = \sum_{j \sim i} u_j(\theta_r)$, we have

$$\theta_r \langle u_i(\theta_r), u_j(\theta_r)\rangle = \sum_{j \sim i} \langle u_i(\theta_r), u_j(\theta_r)\rangle = k \langle u_i(\theta_r), u_j(\theta_r)\rangle$$

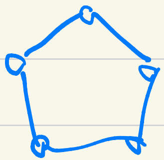
and the result follows. $\square$

Examples

Examples
C_n
Q_n
Pete
$J(v,k)$

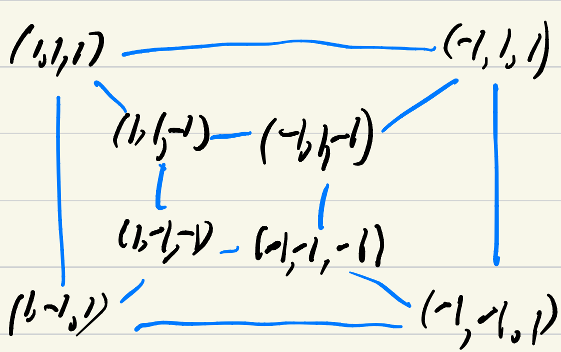
Eigenvalues : $\theta_0 = k, \theta_1, \dots, \overset{\tau}{\theta_d}$

(1) Cycle



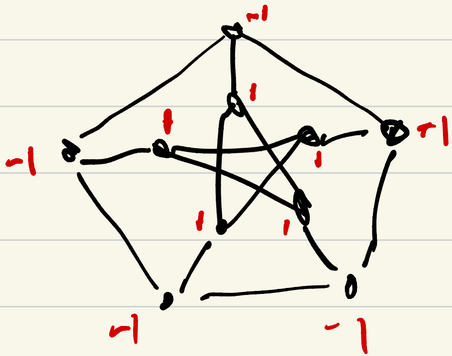
$\lambda = e^{2\pi i/5}$ $\begin{bmatrix} 1 & 1 \\ \lambda & \lambda^4 \\ \lambda^2 & \lambda^3 \\ \lambda^3 & \lambda^2 \\ \lambda^4 & \lambda \end{bmatrix}$

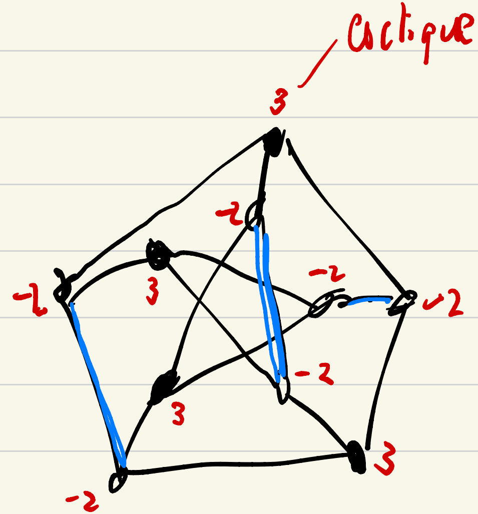
(2) d-cube



$\theta = 1$

(3) Petersen





$\theta_1 = 1$

$m_1 = 5$

$\theta_2 = -2$

$m_2 = 4$

4) **Kneser graphs $K(v,k)$**: k-subsets of $\{1,\dots,v\}$, adjacent if disjoint.

A coclique in $K(v,k)$ is a collection of k-sets, any two of which have a point in common.

Theorem (EKR) Assume $v \geq 2k+1$. The maximum size of a coclique in $K(v,k)$ is $\binom{v-1}{k-1}$. This bound is tight and, if a collection meets this bound, it consists of the k-sets on a point. □

Theorem (Ratio bound) If X is a k-regular graph on n vxs with least eigenvalue τ, then $\alpha(X) \leq \frac{n}{1-k/\tau}$. If S is a coclique of this size, with characteristic vector x_S, then $x_S - \frac{|S|}{n}\mathbf{1}$ is an eigenvector of A with eigenvalue τ.

Proof

(1) A subset S of $V(X)$ with char. vector z is a coclique if & only if $z^T A z = 0$.

(2) If X is k-regular on n vxs with least eigenvalue τ, then $A - \tau I - \frac{k-\tau}{n} \underset{\sim}{1}\, \underset{\sim}{1}^T$ is positive semidefinite. exer

(3) $0 \leq z^T (A - \tau I - \frac{k-\tau}{n} \underset{\sim}{1}\, \underset{\sim}{1}^T) z = z^T A z - \tau |S| - \frac{k-\tau}{n} |S|^2)$ gives bound

4) If $M \succcurlyeq 0$ and $z^T M z = 0$, then $Mz = 0$. (if equality, then $(A - \tau I - \frac{k-\tau}{n} \underset{\sim}{1}\, \underset{\sim}{1}^T) z = 0$)

e.g. Petersen, aka $K(5,2)$

tight

$$n=10,\quad k=3,\quad \tau=-2 \qquad \alpha(\text{Pete}) \leq \frac{10}{1-\frac{3}{-2}} = 4 \checkmark$$

In general $K(v,k)$ has valency $\binom{v-k}{k}$. Its least eigenvalue is $-\binom{v-k-1}{k-1}$. Consequently

$$\alpha(K(v,k)) \leq \frac{\binom{v}{k}}{1+\binom{v-k}{k}\Big/\binom{v-k-1}{k-1}} = \cdots = \binom{v-1}{k-1}$$

This establishes the bound in the EKR theorem.

characterization to come

A Clique Bound

Lemma Assume X is 1-walk regular with valency k on n vertices, with least eigenvalue τ. Then $\omega(X) \leq 1 - \frac{k}{\tau}$.

Proof Assume the vertices $1, \dots, c$ form a clique in X. and let $u_1, \dots, u_c$ be the images of vxs in C in the τ-eigenspace. Set $\hat{u}_i = \frac{1}{\|u_i\|} u_i$. Then if $i \sim j$, $\langle \hat{u}_i, \hat{u}_j \rangle = \frac{\tau}{k}$ and the Gram matrix of $u_1, \dots, u_c$ is $I + \frac{\tau}{k}(J - I)$.

Since Gram matrices are positive semidefinite, the row sums of $I + \frac{\tau}{k}(J - I)$ are non-negative. So

$$0 \leq 1 + (c-1)\frac{\tau}{k}$$

and hence $(c-1)(-\tau) \leq k$. $\square$

Faces of Eigenpolytopes

Let $\mathcal{C}$ be the polytope generated by the rows of U. Assume U is $v \times d$. If $h \in \mathbb{R}^d$, then Uh is a function on the rows of U. The indices i such that $(Uh)_i$ is maximal form a face of $\mathcal{C}$.

The indices i such that $(Uh)_i$ is minimal also form a face.

Claim Each face of P arises in this way. $\square$

If the columns of U are a basis for an eigenspace, the Uh is an eigenvector for each h. So faces of eigenpolytopes arise from eigenvectors

The 1-skeleton of a polytope P is the graph with the vertices of P as its vertices, with vertices u & v adjacent if $\{u, v\}$ is a face of P.

Theorem (Balinski) If P is a polytope with dimension d, then the 1-skeleton of P is d-connected. $\square$

(see Ziegler)

Example Consider Pete. Its least eigenvalue is -2, with multiplicity four. Hence the 1-skeleton of the (-2)-eigenpolytope is 4-connected. Since Pete is distance transitive, it follows that the 1-skeleton is K_{10}.

neighbourly

Let $W_{t,k}$ be the incidence matrix for t-sets vs. k-sets.

Lemma (a) The column space of $W_{t,k}^T$ is an invariant subset for $A(K(n,k))$.

(b) The column space of $W_{1,k}^T$ is the sum of $\langle \underline{1} \rangle$ and the τ-eigenspace. $\square$

Corollary If S is a coclique in $K(v,k)$ of size $\binom{v-1}{k-1}$ with characteristic vector z, then $z \in \mathrm{col}(W_{1,k}^T)$. The vertices in S form a face in the polytope generated by S, as do the vertices not S. $\square$

Our problem now is to determine the faces of the polytope generated by the rows of $W_{1,k}^T$.

Let $\mathcal{P}$ be the polytope generated by the rows of $W_{3,6}^T$.

Theorem The faces of $\mathcal{P}$ are the sets

$$S_{B,C} = \{ S : B \subseteq S \subseteq C,\ |S| = k \}. \qquad \square$$

[Lambeck, Ph.D., 1990]

Theorem (EKR) A coclique in $K(v,k)$ of size $\binom{v-1}{k-1}$ consists of all k-sets containing a point.

Proof. Suppose S is a coclique of size $\binom{v-1}{k-1}$. Then S is a face of Θ and so there exist subsets B, C such that

$$S = \{\beta : B \subseteq \beta \subseteq C\}$$

If $B \neq \emptyset$ the size of S implies that $|B| = 1$, we're done. If $B = \emptyset$, then S is set of all k-subsets of C.

Since any two elements of S have a vertex in common,

$|C| \le 2k-1$ and therefore $|S| \le \binom{2k-1}{k} = \binom{2k-1}{k-1}$, which leads

to a contradiction. $\left(\binom{2k-1}{k} \le \binom{v-1}{k-1} \right)$ $\square$

A similar polygonal argument works for intersecting permutations.

1- Skeletons of Distance-Regular Graphs

Assume X is distance-regular with diameter d & eigenvalues $\theta_0 = k \geq \theta_1 \geq \cdots \geq \theta_d$. We ask when the 1-skeleton of the θ_1-eigenpolytope is isomorphic to X. We assume $m = \mathrm{mult}(\theta_1)$.

Lemma If $k < m$, X is not isomorphic to the 1-skeleton of the θ_1-eigenpolytope. $\square$

Cosines Assume X is distance regular as before. The **cosine sequence** of X is given by

$$w_j = \frac{\langle u_i, u_j\rangle}{\langle u_i, u_i\rangle}, \quad j = 0, 1, \dots, d$$

We note that $w_0 = 1$ & $w_1 = \frac{\theta_1}{k}$. As X is connected, $w_1 < 1$.

Lemma: The cosine sequence of the θ_1-eigenpolytope is non-increasing.

orthogonal polynomials

If $12 \in E(X)$ then

$$\langle u_1 + u_2, u_1 \rangle = 1 + w_1 = \langle u_1 + u_2, u_2 \rangle$$

and, if $i \notin \{1,2\}$,

$$\langle u_1 + u_2, u_i \rangle = \langle u_1, u_i \rangle + \langle u_2, u_i \rangle \in 2w_1$$

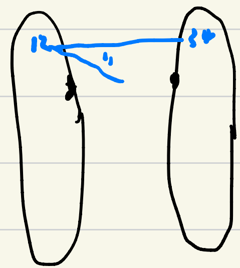
It follows that $\{1,2\}$ is an edge in the eigenpolytope.

With more work, we obtain the following

Theorem Assume X is distance-regular and P is its θ_1-eigenpolytope. Then X is isomorphic to the 1-skeleton of P if & only if it is one of:

(a) $J(v,k)$
(b) $H(n,q)$
(c) halved n-cube
(d) Schläfli
(e) Gosset graph
(f) icosahedron
(g) dodecahedron
(h) rK_2
(i) cycle.

Gosset?



$L(K_8)$ $\qquad$ $L(K_8)$

$$i'j' \sim k'l' \Leftrightarrow ij \cap kl = \emptyset$$

Determining faces is hard.

Lemma The edges in a regular graph Y on n vertices form a face in the θ_1-eigenpolytope of $L(K_n)$; this face is a facet if & only if Y is connected & not bipartite $\square$

Lemma Assume X is strongly regular and λ is an eigenvalue, not the valency. Let $\mathcal{P}$ be the λ-eigenpolytope. If $b_1 := k-a-1 > \lambda+1$ the 1-skeleton of $\mathcal{P}$ is complete; if $b_1 > 2(\lambda+1)$, every triple is a face. $\square$

From Brendan Rooney's Ph.D. thesis: two srgs with parameters (16,6;2,2)

For the Shrikhande graph, the f -vectors of the τ and θ eigenpolytopes are

f (Pτ) = (1, 16, 120, 528, 1440, 2464, 2608, 1622, 524, 64, 1),

f (Pθ) = (1, 16, 96, 236, 272, 144, 28, 1).

1-skeleton not complete

For L(K4,4) the f -vectors of the τ and θ eigenpolytopes are

f (Pτ) = (1, 16, 120, 528, 1392, 2176, 1968, 978, 240, 24, 1),

f (Pθ) = (1, 16, 48, 68, 56, 28, 8, 1).

Problems

(1) For which distance-regular graphs is the θ_1-polytope complete? Strongly regular?

(2) Investigate the τ-eigenpolytopes of distance-regular graphs

(3) Describe the θ_1-eigenpolytope of $J_q(v,k)$, the Grassmann graph.

The End(s)