# Vertex sedentariness

Hermie Monterde (she/her)

University of Manitoba

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# Graphs as mathematical models

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**Schrödinger's equation** dictates that the evolution of $\mathbf{e}_u$ is given by

$$\psi(t) = U(t)\mathbf{e}_u,$$

where $\psi(t)$ is the state of the quantum system represented by $X$ at time $t$.

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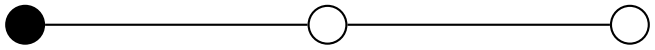
Note: Typically, $H$ is taken to be the adjacency matrix $A$ or the Laplacian matrix $L$ of $X$.

# A visualization



at $t = 0$

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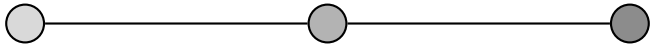

at $0 < t < \frac{\pi}{\sqrt{2}}$

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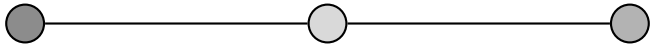

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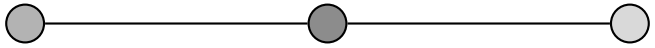

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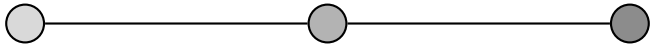

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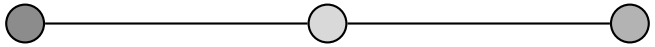

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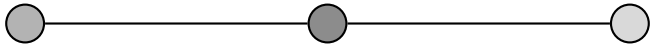

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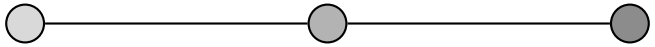

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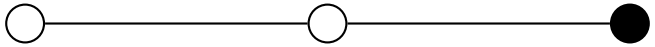

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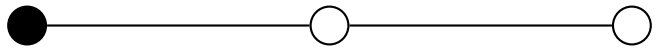

at $t = \frac{2\pi}{\sqrt{2}}$

# Perfect state transfer

## Definition

Vertices $u$ and $v$ admit *perfect state transfer* (PST) in $X$ if for some $\tau > 0$,

$$|U(\tau)_{u,v}| = 1.$$

If $u = v$, then we say that $u$ is periodic.

## Definition

Vertices $u$ and $v$ admit *pretty good state transfer* (PGST) in $X$ if for some $\{\tau_k\} \subseteq \mathbb{R}^+$,

$$\lim_{k \to \infty} |U(\tau_k)_{u,v}| = 1.$$

# Spectral decomposition

Let $\lambda_1, \ldots, \lambda_r$ are the distinct real eigenvalues of $A$ and $E_j$ be the orthogonal projection matrix associated to $\lambda_j$. Then

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Each $E_j = \sum_{j=1}^{k} \mathbf{w}_i \mathbf{w}_i^T$, where $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an ON basis for the eigenspace of $\lambda_j$. The $E_j$'s are real, symmetric, idempotent, pairwise multiplicatively orthogonal and sum to identity.

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## Definition

1. The *eigenvalue support* of vertex $u$ in $X$ is the set $\sigma_u(A) = \{\lambda_j : E_j \mathbf{e}_u \neq \mathbf{0}\}$.
2. Vertices $u$ and $v$ are *strongly cospectral* in $X$ if $E_j \mathbf{e}_u = \pm E_j \mathbf{e}_v$ for each $j$.
3. Vertices $u$ and $v$ are *cospectral* if $(E_j)_{u,u} = \pm (E_j)_{v,v}$ for each $j$.

# Paths and complete graphs

**Theorem (Christandl et al., Godsil et al.)**

1. *$P_2$ and $P_3$ are the only paths that exhibit PST.*
2. *$P_n$ exhibits PGST between end vertices if and only if $n = p - 1$, $n = 2p - 1$ and $n = 2^m - 1$, where $p$ is prime and $m$ is an integer.*

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- $U(t)_{u,u} = \frac{1}{n}\exp(it(n-1)) + \frac{n-1}{n}\exp(-it) \implies |U(t)_{u,u}| = \frac{1}{n}|\exp(itn) + n - 1| \geq 1 - \frac{2}{n}$.

# Sedentariness

## Definition

A vertex $u$ is *$C$-sedentary* if there exists a constant $0 < C \leq 1$ such that for all $t$,

$$|U(t)_{u,u}| \geq C,$$

and in case $|U(t_0)_{u,u}| = C$ for some $t_0 \in \mathbb{R}$, we say that $u$ is *tightly $C$-sedentary*.

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Example: each $u \in V(K_n)$ is tightly $(1 - \frac{2}{n})$-sedentary since

$$|U(t)_{u,u}| = \frac{|n-1+\exp(itn)|}{n} \geq 1 - \frac{2}{n} \text{ for all } t.$$

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## Proposition

1. *If $u$ is $C$-sedentary, then $u$ is $C'$-sedentary for all $0 < C' \leq C$.*
2. *If $u$ and $v$ are cospectral ($U_A(t)_{u,u} = U_A(t)_{v,v}$), then $u$ is $C$-sedentary if and only if $v$ is.*

# Neither sedentary nor involved in PGST

**Proposition (M., 2023)**

*A sedentary vertex cannot be involved in pretty good state transfer. Moreover, a vertex involved in pretty good state transfer cannot be sedentary.*

Follows from: $|U(t)_{u,u}|^2 + |U(t)_{u,v}|^2 + \sum_{j \neq u,v} |U(t)_{u,j}|^2 = 1$

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## Example (M., 2024)

- For each $n \geq 3$, consider $K_{1,n}$ with central vertex $u$. Then $U_A(\frac{\pi}{2\sqrt{n}})_{u,u} = 0$, and so $u$ is not sedentary. Since $u$ is not strongly cospectral with any vertex in $X$, it cannot exhibit PGST.

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- For each $n \geq 3$, consider $X = K_2 \times K_n$ and let $V(K_2) = \{1, 2\}$. Then $X$ is not sedentary at every vertex. For each $u \in V(K_n)$, $(1, u)$ is strongly cospectral only to $(2, u)$ in $X$, and PST occurs between them if and only if $n$ is even. Consequently, each vertex of $K_2 \times K_n$ for each odd $n$ is periodic, sedentary, and is involved in strong cospectrality but not PST.

# Twins

**Definition**

Two vertices are **twins** if they have the same neighbours. A maximal subset $T$ of $V(X)$ is a **twin set** if each pair of vertices in $T$ are twins.

**Lemma (M., 2022)**

*Let $T$ be a set of twins in $X$. Then $u, v \in T$ if and only if $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector of $A$ corresponding to 0 (resp., $-1$) whenever $u$ and $v$ are non-adjacent (resp., adjacent).*

$$A(\mathbf{e}_1 - \mathbf{e}_2) = \begin{bmatrix} 0 & \eta & \\ \eta & 0 & \\ A_{3,1} & A_{3,2} & * \\ \vdots & \vdots & \\ A_{n,1} & A_{n,2} & \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -\eta \\ \eta \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -\eta(\mathbf{e}_1 - \mathbf{e}_2)$$

# A dichotomy

**Theorem (Kirkland, M., Plosker, 2023)**

*Let $T$ be a twin set in $X$. Then for any two vertices $u$ and $v$ in $T$,*

$$|U(t)_{u,u}| + |U(t)_{u,v}| \geq 1 \quad \textit{for all } t \in \mathbb{R}.$$

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*If $u \in T$ is not involved in strong cospectrality, then each $u \in T$ is sedentary.*

# Twins are mostly sedentary

## Theorem (M., 2024)

*Let $\theta \in \{0, -1\}$. Each vertex in a twin set $T$ is sedentary if and only if:*

1. *Either (i) $|T| \geq 3$ or (ii) $T = \{u, v\}$ and there is a $\theta$-eigenvector $\mathbf{w} \notin \operatorname{span}\{\mathbf{e}_u - \mathbf{e}_v\}$ of $A$ such that $\mathbf{w}^T \mathbf{e}_u \neq 0$ or $\mathbf{w}^T \mathbf{e}_v \neq 0$ (i.e., $u \in T$ is not involved in strong cospectrality)*
2. *$T = \{u, v\}$ and $\mathbf{w}^T \mathbf{e}_u = \mathbf{w}^T \mathbf{e}_v = 0$ for all $\theta$-eigenvectors $\mathbf{w} \notin \operatorname{span}\{\mathbf{e}_u - \mathbf{e}_v\}$ of $A$ (i.e., $u$ and $v$ are strongly cospectral), and there are integers $m_j$ such that*

$$\sum_{\sigma_u(A)\backslash\{\theta\}} m_j(\lambda_j - \theta) = 0 \quad \text{and} \quad \sum_{\sigma_u(A)\backslash\{\theta\}} m_j \text{ is odd.}$$

*If we add that $\phi(A, t) \in \mathbb{Z}[t]$ and $u$ is periodic, then the latter statement is equivalent to each eigenvalue $\lambda_j \in \sigma_u(A)\backslash\{\theta\}$ is of the form $\lambda_j = \theta + b_j\sqrt{\Delta}$, where $b_j$ is an integer and either $\Delta = 1$ or $\Delta > 1$ is a square-free integer and the $\nu_2(b_j)$'s are not all equal.*

# A necessary condition

**Lemma (M., 2023)**

*Let $u$ be a vertex of $X$ with $\sigma_u(A) = \{\lambda_1, \ldots, \lambda_r\}$. If $S$ is a non-empty proper subset of $\sigma_u(A)$, say $S = \{\lambda_1, \ldots, \lambda_s\}$, such that*

$$\sum_{j=1}^{s} (E_j)_{u,u} = a$$

*for some $\frac{1}{2} \leq a < 1$, then*

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**Remark.** If $a = \frac{1}{2}$, then $F(t) \leq 0$ for all $t$. Thus, to establish sedentariness, it suffices to find $\varnothing \neq S \subseteq \sigma_u(A)$ with $\frac{1}{2} < a < 1$ such that $F(t)$ is bounded away from 0 for all $t$.

# Bounds

**Theorem (M., 2024)**

*Let $T$ be a twin set in $X$ and fix $u \in T$. Let $\mathcal{B}_1$ be the resulting set after orthonormalizing $\{\boldsymbol{e}_u - \boldsymbol{e}_v : v \in T \backslash \{u\}\}$,*

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**Corollary (M., 2024)**

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# Complete multipartite graphs: Laplacian case

**Corollary (M., 2024)**

*Let $X = K_{n_1,\ldots,n_k}$, $n = \sum_{j=1}^{k} n_j$ and $u$ be a vertex in partite set of size $n_\ell$.*

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- If (i) $n \not\equiv 0$ or (ii) $n \equiv 0 \pmod 4 \wedge n_j \neq 2$ for each $j$, then each vertex in $X$ is sedentary.
- $CP(2k)$ for even $k$ are the only complete multipartite graphs with no sedentary vertex.

# Cartesian products

**Theorem (M., 2023)**

*Let $Z = X_1 \square X_2 \square \cdots \square X_n$ and $u = (u_1, \dots, u_n)$.*

1. *If each $X_j$ is $C_j$-sedentary at $u_j$, then $Z$ is $(\prod_{j=1}^{n} C_j)$-sedentary at $u$. In particular, if each $X_j$ is tightly $C_j$-sedentary at $u_j$, then $Z$ is tightly $C'$-sedentary at $u$, where $C' \geq \prod_{j=1}^{n} C_j$.*
2. *If $Z$ is $C$-sedentary at $u$, then each $X_j$ is $C_j$-sedentary at $u_j$ for some $0 < C_j < 1$.*

**Theorem (M., 2023)**

*Let $n_1, \dots, n_m \geq 2$ and $X = K_{n_1} \square K_{n_2} \square \dots \square K_{n_m}$. The following hold.*

1. *If $n_j = 2$ for some $j$, then $X$ is not sedentary at any vertex.*
2. *If each $n_j \geq 3$, then $X$ is $C$-sedentary at $u$, where $C = \prod_{j=1}^{m}(1 - \frac{2}{n_j})$. If we add that all $\nu_2(n_j)$'s are equal, then $X$ is tightly $C$-sedentary at $u$ at time $t = \frac{\pi}{2^{\nu_2(n_1)}}$.*

# Direct products

**Theorem (M., 2024)**

*Let $u \in V(K_m)$ and $v \in V(Y)$. The following hold.*

1. *If $m \geq 3$ and $Y$ is $C$-sedentary at vertex $v$, where $C > \frac{1}{m-1}$, then $K_m \otimes Y$ is $(C - \frac{C+1}{m})$-sedentary at vertex $(u, v)$ for any vertex $u$ of $K_m$. In particular, if $Y = K_n$ and $n \geq 3$, then $(u, v)$ is sedentary.*
2. *$K_2 \times Y$ is $C$-sedentary at $(u, v)$ if and only if $|\mathrm{Re}(U_{A(Y)}(t)_{v,v})| \geq C$ for all $t$. In particular, if $Y = K_n$, then $(u, v)$ is not sedentary.*

# Direct products

**Theorem (M., 2024)**

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**Theorem (M., 2024)**

*Let $Z$ be a $d$-regular graph on $n$ vertices such that $d > 0$ is an integer and $n = \frac{1}{2}s(d+s)$ for some even integer $s$ satisfying $\nu_2(s) \geq \nu_2(d)$. If $v$ is an apex of $Y = O_2 \vee Z$, then $K_2 \times Y$ is tightly $C$-sedentary at vertex $(u, v)$ for some $C > 0$.*

# Joins

**Lemma (Kirkland and M., 2023)**

*Let* $M \in \{A, L\}$ *and* $n = |V(X)|$. *For all* $u, v \in V(X)$ *and for all* $t$, *we have*

$$\Big|\, |U_{M(X \vee Y)}(t)_{u,v}| - |U_{M(X)}(t)_{u,v}| \,\Big| \leq \frac{2}{n}.$$

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**Theorem (M., 2024)**

*If $u$ is $C$-sedentary in $X$ with $C > \frac{2}{n}$, then $u$ is $\left(C - \frac{2}{n}\right)$-sedentary in $X \vee Y$ for any graph $Y$, where we require that $X$ and $Y$ are both regular whenever $M = A$.*

By assumption, $|U_{M(X)}(t)_{u,u}| - \frac{2}{n} \ \geq \ C - \frac{2}{n} > 0$ for all $t$. By the lemma,

$$|U_{M(X \vee Y)}(t)_{u,u}| \geq \ |U_{M(X)}(t)_{u,u}| - \frac{2}{n} \ \geq \ C - \frac{2}{n} > 0.$$

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6. Is a dominating vertex in a graph sedentary relative to the adjacency matrix?

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# Contact

Thank you for your time! ♡

Email: monterdh@myumanitoba.ca

Google scholar: Hermie Monterde

Googlesite: hermiemonterde

Twitter: Hermie Monterde