Graphs that Admit Orthogonal Matrices

Shaun M. Fallat

Department of Mathematics and Statistics
University of Regina

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Outline

- **Introduction**
  - Pattern Constrained Orthogonal Matrices - History
  - Setting
  - IEP-G
  - Minimum # of Distinct Eigevalues
  - Examples & Basic Facts:
- **Graphs with $q = 2$**
  - Observations
- **Graph Joins and $q = 2$**
  - Threshold graphs
- **$q = 2$ and Regular Graphs**
  - Strongly Regular Graphs
- **Graphs that Allow (or Require) $q = 2$**
- **Ending Remarks**

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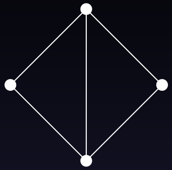
Ending Remarks

A Simple Question



Example

Does the graph above describe a pattern of a 4x4 symmetric orthogonal matrix? Sure... Consider:

$$\left(\frac{1}{\sqrt{3}}\right)\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

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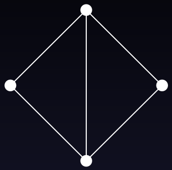

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Existing Works on 'Sparse' Orthogonal Matrices

Brief Literature Review:

1. (Fiedler '91) Conjectured that an $n \times n$ fully indecomposable orthogonal matrix has at least $4n - 2$ nonzero entries. This was proved by Beasley, Brualdi, & Shader in '93 and later a short proof was given by Shader '97.
2. (Craigen '93) Developed a 'product' called weaving that was used to construct weighing matrices.
3. (Cheon & Shader '99) Determined the fewest number of nonzero entries in fully indecomposable row-orthogonal matrices.
4. (Cheon, Johnson, Lee, & Pribble '99) Proved the existence of an $n \times n$ fully indecomposable orthogonal matrix with k zero entries whenever $0 \leq k \leq (n-2)^2$.

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3. (Ahmadi, Alinaghipour, Cavers, F, Meagher, & Nasserasr '13) Established graphs that admit orthogonal matrices: K_n, $K_{n,n}$ and the hypercube...(diamond from pg. 2 appeared in Duarte & Johnson '02).
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Graphs & Matrices...

Central CMT Problem

Given a simple graph $G = (V, E)$, we consider various properties (rank, nullity, spectrum, etc...) for a given collection of matrices “associated" to G.

- Set of $n \times n$ real symmetric matrices $S(G)$, in which for $i \neq j$ the (i, j) entry is nonzero iff $i \sim j$, while entries on the main diagonal are free to be chosen;
- Important subset: $S_+(G)$ denote the PSD subset in $S(G)$ - connected to faithful orthogonal labelling for graphs;
- The set $S(G)$ includes the classical matrices associated with graphs: adjacency, Laplacian (and its variants), and others...

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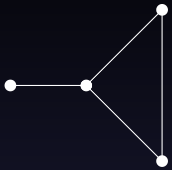

Figure: A graph G

Then the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3.1 & -1.5 & 2 \\ 0 & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ belongs to $S(G)$.

Inverse Eigenvalue Problem

IEP-G

The inverse eigenvalue problem for a graph G is to determine if a given multi-set of real numbers is the spectrum of a matrix in $S(G)$.

Two Extreme Examples:

The only graph G that realizes a single eigenvalue is the empty graph (scalar matrix), and for the complete graph, any list of real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ is realizable whenever $\lambda_1 < \lambda_n$.

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IEP-G for paths

Fiedler's Tridiagonal Matrix Theorem, 1969

If A is a real symmetric $n \times n$ matrix such that for all real diagonal matrices D, $\text{rank}(A + D) \geq n - 1$, then A is irreducible and there is a permutation matrix P such that $P^T AP$ is tridiagonal.

Observations...

- The only graph that requires distinct spectra (i.e., nullity is 1) is the path;
- Use orthogonal polynomials, for example, to deduce that any distinct spectra can be realized by some real tridiagonal matrix;
- Work of Leal Duarte on interlacing also implies that any collection of distinct spectra can be realized by any tree (not just a path);
- More recent work by Monfared/Shader extends Duarte's work to any connected graph.

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Facts about $q(G)$

Definitions & Basic Facts:

- For a square matrix A, $q(A)$ denotes the number of distinct eigenvalues of A.
- The minimum number of distinct eigenvalues of G, $q(G)$, is defined
$$q(G) = \min\{q(A) : A \in S(G)\}.$$
    - $1 \leq q(G) \leq n$, and $q(G) = 1$ iff G is empty,
    - Further, $q(G) = n$ iff $\mathrm{M}(G) = 1$ (ie, G is a path) [F69].

q & adjacency matrix

Diameter

The *length* of a path P is the # of edges in P. The *distance* between two vertices is the length of the shortest path between them, and the *diameter of G* is the maximum distance in G.

Result

The number of distinct eigenvalues of the adjacency matrix is at least the diameter of G plus 1.

- The proof uses the degree of the minimal polynomial
- The proof applies verbatim to nonnegative matrices in $S(G)$

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Unique shortest paths...

Question

A natural question that arises: Is there still a relationship between $q(G)$ and $diam(G)$?

Simple Lower Bound

If there are vertices u, v in G at distance d and the path of length d from u to v is unique, then $q(G) \geq d + 1$.

Notes:

1. For any tree T, $q(T) \geq diam(T) + 1$,
2. For general trees, it is known that q can be much larger than $diam(T) + 1$,
3. The hypercube, Q_n, satisfies $q(Q_n) = 2$ and $diam(Q_n) = n$

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Example – Trees

Facts:

- If G is a tree, then $\mathrm{M}(G) = P(G) = Z(G)$ ([JL99], [AIM08]).
- Trees T with diameter at most 5 are known to satisfy $q(T) = \mathrm{diam}(T) + 1$. However, the tree T_1 with diameter 6 satisfies $q(T_1) = 8$ [BF04].

Figure: BF-tree T_1

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Graphs with $q = |V(G)| - 1$

Theorem [FRG 2017] - Conjecture from [DMRG '13]

A graph G has $q(G) \geq |V(G)| - 1$ if and only if G is one of the following:

(a) a path,

(b) the disjoint union of a path and an isolated vertex,

(c) a path with one leaf attached to an interior vertex,

(d) a path with an extra edge joining two vertices at distance 2.

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Graphs with $q = 2$

Observation 1:
$S(G)$ contains an orthogonal matrix iff $q(G) = 2$.

Observation 2:
$q(G) = 2$ iff $\exists\ A \in S(G)$ such that A^2 is in $\mathrm{span}\{A, I\}$.

Observation 3:
$q(G) = 2$. Then, for any independent set of vertices $\{v_1, v_2, \ldots, v_k\}$, that satisfies for each $i = 1, 2, \ldots, k$ there exists a $j \neq i$ for which $N(v_i) \cap N(v_j) \neq \emptyset$, we have

$$\left| \bigcup_{i \neq j} (N(v_i) \cap N(v_j)) \right| \geq k.$$

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Hypercube

Theorem [DMRG '13]

For $n \geq 1$, we have $q(Q_n) = 2$. In fact this result follows from a slightly stronger statement of the form: for any graph G, $q(G \square K_2) \leq 2q(G) - 2$.

Notes:

- This result is tied to the so-called 'sensitivity conjecture' of Nisan and Szegedy '92 that was resolved by Huang in '19...
- Recently, Ahmad, F. proved that $q(K_s \square K_2) = 2$ for $s \geq 3$ and that there exists an SSP matrix realization in $S(K_s \square K_2)$ with two distinct eigenvalues.

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Theorem [DMRG '13]

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Other Sporadic Results on $q = 2$

Facts:

- $q(K_{p_1,\ldots p_l;q_1,\ldots q_{l'}}) = 2$ for $l, l' \geq 2$, if $\sum p_i = \sum q_j$ [DMRG '19].
- $q(K_n \setminus M) = 2$ $(n \geq 3)$ M - perfect matching [Johnson & Zhang '18 or Bailey & Craigen '19].
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Introduction
- Pattern Constrained Orthogonal Matrices - History
- Setting
- IEP-G
- Minimum # of Distinct Eigevalues
- Examples & Basic Facts:

Graphs with $q = 2$
- Observations

Graph Joins and $q = 2$
- Threshold graphs

$q = 2$ and Regular Graphs
- Strongly Regular Graphs

Graphs that Allow (or Require) $q = 2$

Ending Remarks

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Studying $q(G \vee H)$

Known Results on Joins of Graphs

- G connected, $q(G \vee G) = 2$ [DMRG '13].
- If G, H connected & $|G| = |H|$, then $q(G \vee H) = 2$ [Monfared & Shader '16].
- If G, H connected, & $|H| \leq |G| + 2$, then $q(G \vee H) = 2$ [AIM ARC Bordering Group '23]
- If $q(G \vee H) = 2$, then G and H have compatible multiplicity matrices. Further, if G is generically realizable & H is sane, then $q(G \vee H) = 2$, iff G and H have compatible multiplicity matrices. [Levene, Oblak, Smigoc '22].

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Studying $q(G \vee K_1)$

Various Results

- Paths: $q(P_n \vee K_1) = \lceil \frac{n+1}{2} \rceil$.
- $A \in S(G)$, λ an eigenvalue with a nowhere zero eigenvector. Then $\exists\, A' \in S(G \vee K_1)$ such that:
  1. $q(A') = q(A) + 1$, if λ is extreme,
  2. $q(A') = q(A)$, if λ is not extreme,
  3. $q(A') = q(A) - 1$, if λ is extreme and simple.
- Hypercube: $q(Q_n \vee K_1) \leq 3$, and $q(Q_4 \vee K_1) = 3$ [AIM ARC Bordering Group '23]
- Using a fact about join duplicating a vertex, we know that $q(G \vee K_{s+1}) \leq q(G \vee K_s)$.

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Threshold Graphs

Creation Sequence

Any threshold graph G can be represented as a binary sequence, depending on a vertex being isolated or dominating, and the trace T of G, is the number of ones in its creation sequence.

Results for $q = 2$ [F., Mojallal '22]:

- For a threshold graph G, $q(G) = 2$ if and only if there exists a matrix $A \in S(G)$ s.t. $A(\bar{1}, \bar{0})$ is column orthogonal.
- Let G be a connected threshold graph of order n and trace T. If $q(G) = 2$, then $T \geq \lceil \frac{n}{2} \rceil$.
- (Complete Split) Let $G \cong (\underbrace{0, \ldots, 0}_{k_1}, \underbrace{1, \ldots, 1}_{t_1})$, where $t_1, k_1 \geq 1$. If $k_1 \leq t_1$, then $q(G) = 2$ and otherwise if $k_1 > t_1$, then $q(G) = 3$.

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Outline

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r - Regular Graphs

Case $r \leq 3$ **[AIM ARC Q/q Group]**

If G is a connected r-regular graph with $q(G) = 2$ for some $r \leq 3$, then G is one of:

(1) K_2;

(2) K_3 or C_4; or,

(3) K_4, $K_{3,3}$, $K_3 \square K_2$, or Q_3.

Cartesian product of cliques

For $m, n \leq 3$, we have $q(K_m \square K_n) = 3$.

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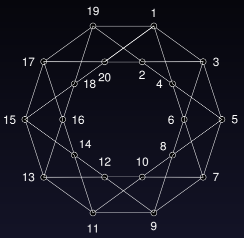

4-regular case [AIM ARC Q/q Group]

If G is a connected 4-regular graph with $q(G) = 2$, then G is either:

(1) K_5;

(2) $K_3 \square C_4$, $K_{3,3} \square K_2$, Q_4,

(3) a closed candle H_k for some $k \geq 3$,

(4) one of 11 other sporadic 4-regular graphs on at most 16 vertices.

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SRGs and q

Observations

- If G is an SRG, then $q(G) \leq 3$.
- If $\mu = 0$, then $q(G) = 2$ and $q(G^c) = 2$.
- If $\mu = 1$, then $q > 2$.
- Line graph of K_n has $q = 2$ [Furst, Grotts '21].
- Open Question: Characterize the SRGs with $q = 2$

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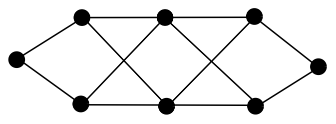
Graphs that Allow $q = 2$

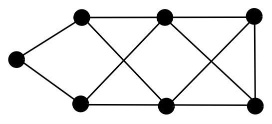
Allows Theorem [AIM ARC Q/q Group]

If G is a connected graph on $n \geq 3$ vertices with $q(G) = 2$, then

$$|E(G)| \geq \begin{cases} 2n-4, & \text{if } n \text{ is even, and} \\ 2n-3, & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, the only graphs that meet this bound with n even are Q_3 and the double-ended candles. The only graphs that meet this bound with n odd are the single-ended candles.





We are currently studying the graphs that 'require' $q = 2$. Such graphs are necessarily dense and we can remove $n-2$ edges from K_n to produce H such that $q(H) > 2$...to be continued!

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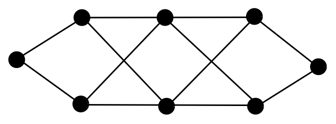

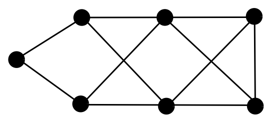

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Outline

Summary and Future Considerations

1. BIG Question: Characterize the graphs G with $q(G) = 2$? There are a number of avenues to explore and work is on-going! One thing to keep in mind: Every graph is an induced subgraph of a graph that admits an orthogonal matrix!!!
2. It seems eigenvectors will play a bigger role in any such characterizations...
3. The bipartite case seems interesting (not just $q = 2$, but what q values are possible), with my PIMS PDF P. Viskwakarma, we are making progresst...imposing other structure constraints is also a direction to consider.
4. SRGs and the distance regular graphs seem to be a natural place to study further!

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The End

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Special thanks to Sabrina M. Lato and the other organizers for the kind invitation to speak in this Algebraic Graph Theory Seminar.

Thank you all for your time and attention...Any questions?

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