Covers of the complete graph, equiangular lines, and the absolute bound

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Algebraic Graph Theory Seminar
University of Waterloo

March 10, 2025

Iowa State University
of Science and Technology

Joint work with

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The Ohio State University

Thanks to

NSF DMS 2220301

AFOSR Summer Faculty Fellowship Program

ARO W911NF-16-1-0008

tl;dw

Godsil & Hensel: cover of complete graph $\rightarrow$ equiangular lines

This talk:

1) DRACKNs can't reach the absolute bound (except once)
2) Maybe "roux" can

Godsil & Hensel, J. Combin. Theory Ser. B, 1992

Coutinho, Godsil, Shirazi & Zhan, Linear Algebra Appl., 2016

Outline

Outline

1: Equiangular lines

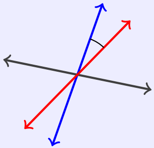
2: DRACKNs

3: The absolute bound

4: Roux

Line packing problem

Find n lines (1-dim subspaces) in $\mathbb{R}^d$ or $\mathbb{C}^d$ without sharp angles



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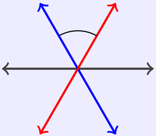

Good

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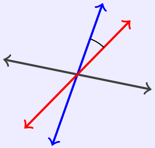

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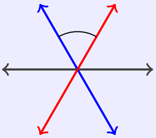

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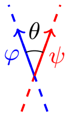

$\cos\theta = |\langle \varphi, \psi \rangle|$

- Given lines, choose unit norm reps

$$\Phi = \begin{bmatrix} \vert & & \vert \\ \varphi_1 & \cdots & \varphi_n \\ \vert & & \vert \end{bmatrix} \in \mathbb{F}^{d \times n}$$

- To avoid sharp angles, minimize **coherence**

$$\mu = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|$$

One application

Uniquely solve an underdetermined system

$$\begin{bmatrix} & & \\ & \Phi & \\ & & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} \\ y \\ \\ \end{bmatrix}$$

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Compressed sensing

$\Phi x = y$ has a unique **sparse** solution if

- x has a lot of zero entries, and
- the columns of Φ are “very different”

In fact, x can be found by a *linear program* (fast)

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- Span lines with wide angles $\implies$ "very different"
- Better coherence $\implies$ more nonzero entries allowed

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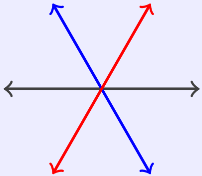
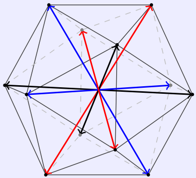
Some optimal line packings



Real 2×3



Real 3×6

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 1 & 0 & -\omega^2 & \omega & 0 & -\omega & \omega^2 \\ 1 & 0 & -1 & \omega & 0 & -\omega^2 & \omega^2 & 0 & -\omega \\ -1 & 1 & 0 & -\omega^2 & \omega & 0 & -\omega & \omega^2 & 0 \end{bmatrix}, \quad \omega = e^{2\pi i/3}$$

Complex 3×9

How do you know it's optimal?

Theorem (Welch/Rankin "relative" bound)

For n unit vectors $\Phi = \left[\begin{array}{ccc} \varphi_1 & \cdots & \varphi_n \end{array} \right]$ in $\mathbb{R}^d$ or $\mathbb{C}^d$,

$$\mu := \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle| \geq \sqrt{\frac{n-d}{d(n-1)}}.$$

Equality holds iff Φ is an **equiangular tight frame** (ETF):

- Equiangular: $|\langle \varphi_i, \varphi_j \rangle| = \mu$ for all $i \neq j$
- Tight frame: $\Phi\Phi^* = \mathrm{const} \cdot I$

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equiangular tight frame (ETF) $\Longrightarrow$ **optimal line packing**

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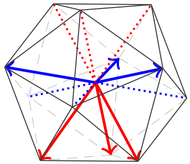
Certifying optimality

Example: Complex 3 × 9

$$\Phi := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 1 & 0 & -\omega & \omega^2 & 0 & -\omega^2 & \omega \\ 1 & 0 & -1 & \omega^2 & 0 & -\omega & \omega & 0 & -\omega^2 \\ -1 & 1 & 0 & -\omega & \omega^2 & 0 & -\omega^2 & \omega & 0 \end{bmatrix}, \qquad \omega = e^{2\pi i/3}$$

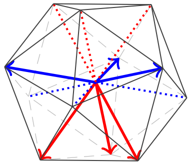
$$\Phi^*\Phi = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 & -1 & -\omega^2 & -\omega & -1 & -\omega & -\omega^2 \\ -1 & 2 & -1 & -\omega & -1 & -\omega^2 & -\omega^2 & -1 & -\omega \\ -1 & -1 & 2 & -\omega^2 & -\omega & -1 & -\omega & -\omega^2 & -1 \\ -1 & -\omega^2 & -\omega & 2 & -\omega & -\omega^2 & -1 & -1 & -1 \\ -\omega & -1 & -\omega^2 & -\omega^2 & 2 & -\omega & -1 & -1 & -1 \\ -\omega^2 & -\omega & -1 & -\omega & -\omega^2 & 2 & -1 & -1 & -1 \\ -1 & -\omega & -\omega^2 & -1 & -1 & -1 & 2 & -\omega^2 & -\omega \\ -\omega^2 & -1 & -\omega & -1 & -1 & -1 & -\omega & 2 & -\omega^2 \\ -\omega & -\omega^2 & -1 & -1 & -1 & -1 & -\omega^2 & -\omega & 2 \end{bmatrix} \qquad \Phi\Phi^* = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Equiangular **Tight Frame**



Equiangular tight frame $\Phi = [\,\varphi_1 \ \cdots \ \varphi_n\,] \in \mathbb{F}^{d\times n}$

$$\Phi\Phi^* = \tfrac{n}{d} I, \qquad\qquad |\langle \varphi_i, \varphi_j\rangle| = \begin{cases} 1 & i = j \\ \mu & i \neq j \end{cases}$$



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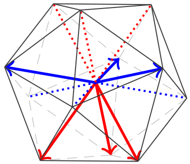
$$\Updownarrow$$

$$G = \frac{1}{\sqrt{5}} \begin{bmatrix} \sqrt{5} & - & - & + & - & - \\ - & \sqrt{5} & - & - & + & - \\ - & - & \sqrt{5} & - & - & + \\ + & - & - & \sqrt{5} & + & + \\ - & + & - & + & \sqrt{5} & + \\ - & - & + & + & + & \sqrt{5} \end{bmatrix}$$

Gram matrix $G = \Phi^*\Phi \in \mathbb{F}^{n\times n}$



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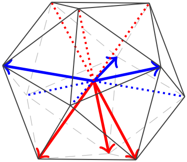

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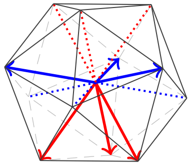

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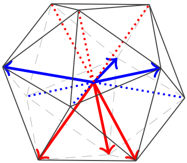

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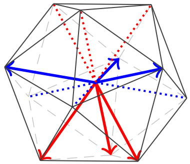

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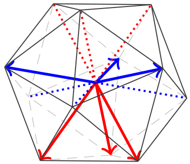

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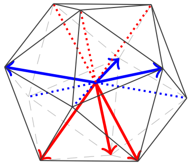

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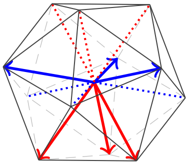
Theorem (folk)

ETFs come in **Naimark complementary** pairs, as below

	Primal Φ	**Complement** Ψ
Size	$d \times n$	$(n-d) \times n$
Short fat matrix	$\text{const} \cdot \Phi$ has ON rows	$\exists$ unitary $\begin{bmatrix} \text{const} \cdot \Phi \\ \text{const} \cdot \Psi \end{bmatrix}$
Gram matrix	G	$\text{const} \cdot (I - \text{const} \cdot G)$
Signature matrix	S	$-S$

Outline

Ex: Real ETF $\leftrightarrow$ special graph



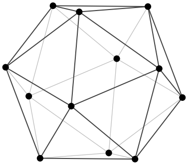
ETF

$$\begin{bmatrix} 0 & - & - & + & - & - \\ - & 0 & - & - & + & - \\ - & - & 0 & - & - & + \\ + & - & - & 0 & + & + \\ - & + & - & + & 0 & + \\ - & - & + & + & + & 0 \end{bmatrix}$$

signature matrix

$$\begin{bmatrix} 0&0&0&1&0&1&1&0&0&1&0&1 \\ 0&0&1&0&1&0&0&1&1&0&1&0 \\ 0&1&0&0&0&1&0&1&1&0&0&1 \\ 1&0&0&0&1&0&1&0&0&1&1&0 \\ 0&1&0&1&0&0&0&1&0&1&1&0 \\ 1&0&1&0&0&0&1&0&1&0&0&1 \\ 1&0&0&1&0&1&0&0&1&0&1&0 \\ 0&1&1&0&1&0&0&0&0&1&0&1 \\ 0&1&1&0&0&1&1&0&0&0&1&0 \\ 1&0&0&1&1&0&0&1&0&0&0&1 \\ 0&1&0&1&1&0&1&0&1&0&0&0 \\ 1&0&1&0&0&1&0&1&0&1&0&0 \end{bmatrix}$$

adjacency matrix



graph

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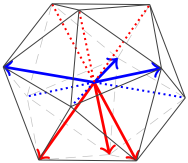


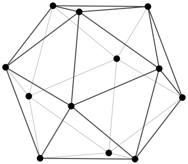
ETF

$$\begin{bmatrix} 0 & - & - & + & - & - \\ - & 0 & - & - & + & - \\ - & - & 0 & - & - & + \\ + & - & - & 0 & + & + \\ - & + & - & + & 0 & + \\ - & - & + & + & + & 0 \end{bmatrix}$$

signature matrix

$$\begin{bmatrix} 0&0&0&1&0&1&1&0&0&1&0&1\\ 0&0&1&0&1&0&0&1&1&0&1&0\\ 0&1&0&0&0&1&0&1&1&0&0&1\\ 1&0&0&0&1&0&1&0&0&1&1&0\\ 0&1&0&1&0&0&0&1&0&1&1&0\\ 1&0&1&0&0&0&1&0&1&0&0&1\\ 1&0&0&1&0&1&0&0&1&0&1&0\\ 0&1&1&0&1&0&0&0&0&1&0&1\\ 0&1&1&0&0&1&1&0&0&0&1&0\\ 1&0&0&1&1&0&0&1&0&0&0&1\\ 0&1&0&1&1&0&1&0&1&0&0&0\\ 1&0&1&0&0&1&0&1&0&1&0&0 \end{bmatrix}$$

adjacency matrix



graph

$$\begin{bmatrix} 0&0&0&1&0&1&1&0&0&1&0&1\\ 0&0&1&0&1&0&0&1&1&0&1&0\\ 0&1&0&0&0&1&0&1&1&0&0&1\\ 1&0&0&0&1&0&1&0&0&1&1&0\\ 0&1&0&1&0&0&0&1&0&1&1&0\\ 1&0&1&0&0&0&1&0&1&0&0&1\\ 1&0&0&1&0&1&0&0&1&0&1&0\\ 0&1&1&0&1&0&0&0&0&1&0&1\\ 0&1&1&0&0&1&1&0&0&0&1&0\\ 1&0&0&1&1&0&0&1&0&0&0&1\\ 0&1&0&1&1&0&1&0&1&0&0&0\\ 1&0&1&0&0&1&0&1&0&1&0&0 \end{bmatrix}^2 = \begin{bmatrix} 5&0&2&2&2&2&2&2&2&2&2&2\\ 0&5&2&2&2&2&2&2&2&2&2&2\\ 2&2&5&0&2&2&2&2&2&2&2&2\\ 2&2&0&5&2&2&2&2&2&2&2&2\\ 2&2&2&2&5&0&2&2&2&2&2&2\\ 2&2&2&2&0&5&2&2&2&2&2&2\\ 2&2&2&2&2&2&5&0&2&2&2&2\\ 2&2&2&2&2&2&0&5&2&2&2&2\\ 2&2&2&2&2&2&2&2&5&0&2&2\\ 2&2&2&2&2&2&2&2&0&5&2&2\\ 2&2&2&2&2&2&2&2&2&2&5&0\\ 2&2&2&2&2&2&2&2&2&2&0&5 \end{bmatrix}$$

Defn: Regular abelian DRACKN

Determined by:

- Γ, finite abelian group, order r
- $B \in \mathbb{C}[\Gamma]^{n\times n}$

Such that:

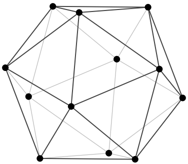
- each $B_{ii} = 0$
- $B_{ij} \in \Gamma$ when $i \neq j$
- $B_{ji} = B_{ij}^{-1}$ when $i \neq j$
- apply regular rep of Γ to B entrywise to get $A \in (\mathbb{C}^{r\times r})^{n\times n}$, then

$$A^2 = (n - rc - 2)A + (n-1)I + c(J - I) \otimes J$$

Parameters: DRACKN(n, r, c)

$$B = \begin{bmatrix} 0 & \delta_- & \delta_- & \delta_+ & \delta_- & \delta_- \\ \delta_- & 0 & \delta_- & \delta_- & \delta_+ & \delta_- \\ \delta_- & \delta_- & 0 & \delta_- & \delta_- & \delta_+ \\ \delta_+ & \delta_- & \delta_- & 0 & \delta_+ & \delta_+ \\ \delta_- & \delta_+ & \delta_- & \delta_+ & 0 & \delta_+ \\ \delta_- & \delta_- & \delta_+ & \delta_+ & \delta_+ & 0 \end{bmatrix}$$

$$A = \left[\begin{array}{cc|cc|cc|cc|cc|cc} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$



Example: $\Gamma = \mathbb{Z}_3$ and $(n, r, c) = (9, 3, 3)$

$$B = \begin{bmatrix} 0 & \delta_0 & \delta_0 & \delta_0 & \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 \\ \delta_0 & 0 & \delta_0 & \delta_2 & \delta_0 & \delta_1 & \delta_1 & \delta_0 & \delta_2 \\ \delta_0 & \delta_0 & 0 & \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 & \delta_0 \\ \delta_0 & \delta_1 & \delta_2 & 0 & \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 \\ \delta_2 & \delta_0 & \delta_1 & \delta_1 & 0 & \delta_2 & \delta_0 & \delta_0 & \delta_0 \\ \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 & 0 & \delta_0 & \delta_0 & \delta_0 \\ \delta_0 & \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 & 0 & \delta_1 & \delta_2 \\ \delta_1 & \delta_0 & \delta_2 & \delta_0 & \delta_0 & \delta_0 & \delta_2 & 0 & \delta_1 \\ \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 & \delta_0 & \delta_1 & \delta_2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix}
0&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&1&0 \\
0&0&0&0&1&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&1&0&0&0&0&1 \\
0&0&0&0&0&1&0&0&1&0&0&1&1&0&0&0&1&0&0&0&1&0&1&0&1&0&0 \\
1&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&1&0&1&0&0&0&0&1 \\
0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1&0&0 \\
0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&1&0&0&1&0&0&0&0&1&0&1&0 \\
1&0&0&1&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&1&0&1&0&0 \\
0&1&0&0&1&0&0&0&0&0&0&1&1&0&0&0&1&0&1&0&0&0&0&1&0&1&0 \\
0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&0&1 \\
1&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&1&0&1&0&0&1&0&0&1&0&0 \\
0&1&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&1&0&0&1&0 \\
0&0&1&1&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1 \\
0&0&1&1&0&0&0&1&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&1&0&0 \\
1&0&0&0&1&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&1&0&0&1&0 \\
0&1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1 \\
0&1&0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0 \\
0&0&1&1&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0 \\
1&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&1 \\
1&0&0&0&0&1&0&1&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&1 \\
0&1&0&1&0&0&0&0&1&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&1&0&0 \\
0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0 \\
0&1&0&1&0&0&0&0&1&1&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&1&0 \\
0&0&1&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&0&0&0&0&1 \\
1&0&0&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0 \\
0&0&1&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\
1&0&0&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0 \\
0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&1&1&0&0&0&1&0&0&0&0
\end{bmatrix}$$

$$\begin{bmatrix}
0&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&1&0 \\
0&0&0&0&1&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&1&0&0&0&0&1 \\
0&0&0&0&0&1&0&0&1&0&0&1&1&0&0&0&1&0&0&0&1&0&1&0&1&0&0 \\
1&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&1&0&1&0&0&0&0&1 \\
0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1&0&0 \\
0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&1&0&0&1&0&0&0&0&1&0&1&0 \\
1&0&0&1&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&1&0&1&0&0 \\
0&1&0&0&1&0&0&0&0&0&0&1&1&0&0&0&1&0&1&0&0&0&0&1&0&1&0 \\
0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&0&1 \\
1&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&1&0&1&0&0&1&0&0&1&0&0 \\
0&1&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&1&0&0&1&0 \\
0&0&1&1&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1 \\
0&0&1&1&0&0&0&1&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&1&0&0 \\
1&0&0&0&1&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&1&0&0&1&0 \\
0&1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1 \\
0&1&0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0 \\
0&0&1&1&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0 \\
1&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&1 \\
1&0&0&0&0&1&0&1&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&1 \\
0&1&0&1&0&0&0&0&1&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&1&0&0 \\
0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0 \\
0&1&0&1&0&0&0&0&1&1&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&1&0 \\
0&0&1&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&0&0&0&0&1 \\
1&0&0&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0 \\
0&0&1&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\
1&0&0&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0 \\
0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&1&1&0&0&0&1&0&0&0&0
\end{bmatrix}^2 = \begin{bmatrix}
8&0&0&1&3&3&1&3&3&1&3&3&3&1&3&3&3&1&1&3&3&3&3&1&3&1&3 \\
0&8&0&3&1&3&3&1&3&3&1&3&3&3&1&1&3&3&3&1&3&1&3&3&3&3&1 \\
0&0&8&3&3&1&3&3&1&3&3&1&1&3&3&3&1&3&3&3&1&3&1&3&1&3&3 \\
1&3&3&8&0&0&1&3&3&3&3&1&1&3&3&3&1&3&3&1&3&1&3&3&3&3&1 \\
3&1&3&0&8&0&3&1&3&1&3&3&3&1&3&3&3&1&3&3&1&3&1&3&1&3&3 \\
3&3&1&0&0&8&3&3&1&3&1&3&3&3&1&1&3&3&1&3&3&3&3&1&3&1&3 \\
1&3&3&1&3&3&8&0&0&3&1&3&3&3&1&1&3&3&3&3&1&3&1&3&1&3&3 \\
3&1&3&3&1&3&0&8&0&3&3&1&1&3&3&3&1&3&1&3&3&3&3&1&3&1&3 \\
3&3&1&3&3&1&0&0&8&1&3&3&3&1&3&3&3&1&3&1&3&1&3&3&3&3&1 \\
1&3&3&3&1&3&3&3&1&8&0&0&3&3&1&3&1&3&1&3&3&1&3&3&1&3&3 \\
3&1&3&3&3&1&1&3&3&0&8&0&1&3&3&3&3&1&3&1&3&3&1&3&3&1&3 \\
3&3&1&1&3&3&3&1&3&0&0&8&3&1&3&1&3&3&3&3&1&3&3&1&3&3&1 \\
3&3&1&1&3&3&3&1&3&3&1&3&8&0&0&3&3&1&1&3&3&1&3&3&1&3&3 \\
1&3&3&3&1&3&3&3&1&3&3&1&0&8&0&1&3&3&3&1&3&3&1&3&3&1&3 \\
3&1&3&3&3&1&1&3&3&1&3&3&0&0&8&3&1&3&3&3&1&3&3&1&3&3&1 \\
3&1&3&3&3&1&1&3&3&3&3&1&3&1&3&8&0&0&1&3&3&1&3&3&1&3&3 \\
3&3&1&1&3&3&3&1&3&1&3&3&3&3&1&0&8&0&3&1&3&3&1&3&3&1&3 \\
1&3&3&3&1&3&3&3&1&3&1&3&1&3&3&0&0&8&3&3&1&3&3&1&3&3&1 \\
1&3&3&3&3&1&3&1&3&1&3&3&1&3&3&1&3&3&8&0&0&3&1&3&3&3&1 \\
3&1&3&1&3&3&3&3&1&3&1&3&3&1&3&3&1&3&0&8&0&3&3&1&1&3&3 \\
3&3&1&3&1&3&1&3&3&3&3&1&3&3&1&3&3&1&0&0&8&1&3&3&3&1&3 \\
3&1&3&1&3&3&3&3&1&1&3&3&1&3&3&1&3&3&3&3&1&8&0&0&3&1&3 \\
3&3&1&3&1&3&1&3&3&3&1&3&3&1&3&3&1&3&1&3&3&0&8&0&3&3&1 \\
1&3&3&3&3&1&3&1&3&3&3&1&3&3&1&3&3&1&3&1&3&0&0&8&1&3&3 \\
3&3&1&3&1&3&1&3&3&1&3&3&1&3&3&1&3&3&3&1&3&3&3&1&8&0&0 \\
1&3&3&3&3&1&3&1&3&3&1&3&3&1&3&3&1&3&3&3&1&1&3&3&0&8&0 \\
3&1&3&1&3&3&3&3&1&3&3&1&3&3&1&3&3&1&1&3&3&3&1&3&0&0&8
\end{bmatrix}$$

$$A^2 = (n - rc - 2)A + (n - 1)I + c(J - I) \otimes J$$

Theorem (Godsil & Hensel)

Given $B \in \mathbb{C}[\Gamma]^{n \times n}$, a regular abelian DRACKN(n, r, c).
Apply **any** character $1 \neq \alpha \in \hat{\Gamma}$ entrywise to find $\hat{\alpha}(B) \in \mathbb{C}^{n \times n}$.
Then

- $\hat{\alpha}(B) =$ signature matrix of $d \times n$ ETF,
- $-\hat{\alpha}(B) =$ signature matrix of $(n-d) \times n$ ETF,

where $d = f(n, r, c)$.

$$B = \begin{bmatrix} 0 & \delta_0 & \delta_0 & \delta_0 & \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 \\ \delta_0 & 0 & \delta_0 & \delta_2 & \delta_0 & \delta_1 & \delta_1 & \delta_0 & \delta_2 \\ \delta_0 & \delta_0 & 0 & \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 & \delta_0 \\ \delta_0 & \delta_1 & \delta_2 & 0 & \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 \\ \delta_2 & \delta_0 & \delta_1 & \delta_1 & 0 & \delta_2 & \delta_0 & \delta_0 & \delta_0 \\ \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 & 0 & \delta_0 & \delta_0 & \delta_0 \\ \delta_0 & \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 & 0 & \delta_1 & \delta_2 \\ \delta_1 & \delta_0 & \delta_2 & \delta_0 & \delta_0 & \delta_0 & \delta_2 & 0 & \delta_1 \\ \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 & \delta_0 & \delta_1 & \delta_2 & 0 \end{bmatrix} \qquad \hat{\alpha}(B) = \begin{bmatrix} 0 & 1 & 1 & 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ 1 & 0 & 1 & \omega^2 & 1 & \omega & \omega & 1 & \omega^2 \\ 1 & 1 & 0 & \omega & \omega^2 & 1 & \omega^2 & \omega & 1 \\ 1 & \omega & \omega^2 & 0 & \omega^2 & \omega & 1 & 1 & 1 \\ \omega^2 & 1 & \omega & \omega & 0 & \omega^2 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega^2 & \omega & 0 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & 1 & 1 & 1 & 0 & \omega & \omega^2 \\ \omega & 1 & \omega^2 & 1 & 1 & 1 & \omega^2 & 0 & \omega \\ \omega^2 & \omega & 1 & 1 & 1 & 1 & \omega & \omega^2 & 0 \end{bmatrix}$$

$$\alpha : \mathbb{Z}_3 \to \mathbb{T} : \delta_1 \mapsto e^{\frac{2\pi i}{3}} = \omega$$

Godsil & Hensel, J. Combin. Theory Ser. B, 1992
Coutinho, Godsil, Shirazi & Zhan, Linear Algebra Appl., 2016

$$B = \begin{bmatrix} 0 & \delta_0 & \delta_0 & \delta_0 & \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 \\ \delta_0 & 0 & \delta_0 & \delta_2 & \delta_0 & \delta_1 & \delta_1 & \delta_0 & \delta_2 \\ \delta_0 & \delta_0 & 0 & \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 & \delta_0 \\ \delta_0 & \delta_1 & \delta_2 & 0 & \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 \\ \delta_2 & \delta_0 & \delta_1 & \delta_1 & 0 & \delta_2 & \delta_0 & \delta_0 & \delta_0 \\ \delta_1 & \delta_2 & \delta_0 & \delta_2 & \delta_1 & 0 & \delta_0 & \delta_0 & \delta_0 \\ \delta_0 & \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 & 0 & \delta_1 & \delta_2 \\ \delta_1 & \delta_0 & \delta_2 & \delta_0 & \delta_0 & \delta_0 & \delta_2 & 0 & \delta_1 \\ \delta_2 & \delta_1 & \delta_0 & \delta_0 & \delta_0 & \delta_0 & \delta_1 & \delta_2 & 0 \end{bmatrix} \qquad \hat{\alpha}(B) = \begin{bmatrix} 0 & 1 & 1 & 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ 1 & 0 & 1 & \omega^2 & 1 & \omega & \omega & 1 & \omega^2 \\ 1 & 1 & 0 & \omega & \omega^2 & 1 & \omega^2 & \omega & 1 \\ 1 & \omega & \omega^2 & 0 & \omega^2 & \omega & 1 & 1 & 1 \\ \omega^2 & 1 & \omega & \omega & 0 & \omega^2 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega^2 & \omega & 0 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & 1 & 1 & 1 & 0 & \omega & \omega^2 \\ \omega & 1 & \omega^2 & 1 & 1 & 1 & \omega^2 & 0 & \omega \\ \omega^2 & \omega & 1 & 1 & 1 & 1 & \omega & \omega^2 & 0 \end{bmatrix}$$

$$\alpha : \mathbb{Z}_3 \to \mathbb{T} : \delta_1 \mapsto e^{\frac{2\pi i}{3}} = \omega$$

$$-\hat{\alpha}(B) = \begin{bmatrix} 0 & -1 & -1 & -1 & -\omega & -\omega^2 & -1 & -\omega^2 & -\omega \\ -1 & 0 & -1 & -\omega^2 & -1 & -\omega & -\omega & -1 & -\omega^2 \\ -1 & -1 & 0 & -\omega & -\omega^2 & -1 & -\omega^2 & -\omega & -1 \\ -1 & -\omega & -\omega^2 & 0 & -\omega^2 & -\omega & -1 & -1 & -1 \\ -\omega^2 & -1 & -\omega & -\omega & 0 & -\omega^2 & -1 & -1 & -1 \\ -\omega & -\omega^2 & -1 & -\omega^2 & -\omega & 0 & -1 & -1 & -1 \\ -1 & -\omega^2 & -\omega & -1 & -1 & -1 & 0 & -\omega & -\omega^2 \\ -\omega & -1 & -\omega^2 & -1 & -1 & -1 & -\omega^2 & 0 & -\omega \\ -\omega^2 & -\omega & -1 & -1 & -1 & -1 & -\omega & -\omega^2 & 0 \end{bmatrix}$$

$$\Phi = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 1 & 0 & -\omega & \omega^2 & 0 & -\omega^2 & \omega \\ 1 & 0 & -1 & \omega^2 & 0 & -\omega & \omega & 0 & -\omega^2 \\ -1 & 1 & 0 & -\omega & \omega^2 & 0 & -\omega^2 & \omega & 0 \end{bmatrix}$$

3×9 ETF

Example: Symplectic DRACKNs

Given a prime power q and $m \geq 1$.

Consider $V = \mathbb{F}_q^{2m}$ with symplectic form $[\cdot, \cdot]$.

Then $B \in \mathbb{C}[\mathbb{F}_q]^{V \times V}$ with $B_{uu} = 0$ and $B_{uv} = \delta_{[u,v]}$ for $u \neq v$

is a regular abelian DRACKN with $(n, r, c) = (q^{2m}, q, q^{2m-1})$.

ETF sizes: $\frac{q^m(q^m+1)}{2} \times q^{2m}$, $\frac{q^m(q^m-1)}{2} \times q^{2m}$

$q = 3$, $m = 1$: 6×9, 3×9

Thas, J. Comb. Theory, Ser. B, 1977

Somma, Rend. Mat. Appl., 1983

Coutinho, Godsil, Shirazi & Zhan, Linear Algebra Appl., 2016

JWI & Mixon, J. Combin. Theory, Ser. A, 2022

Example: Unitary DRACKNs

Given a prime power q, put $n = q^3 + 1$.

Consider $V = \mathbb{F}_{q^2}^3$ with Hermitian form $(\cdot, \cdot)$.

Choose reps $t_1, \ldots, t_n \in V$ for the isotropic lines: $(t_i, t_i) = 0$.

Finally, let $\Gamma \leq \mathbb{F}_{q^2}^\times$ have order $q + 1$.

Then $B \in \mathbb{C}[\Gamma]^{n \times n}$ with $B_{ii} = 0$ and $B_{ij} = \delta_{-(t_i, t_j)^{q-1}}$ for $i \neq j$

is a regular abelian DRACKN.

DRACKN parameters: $(n, r, c) = (q^3 + 1, q + 1, q^2 - 1)$

ETF sizes: $(q^3 - q^2 + q) \times (q^3 + 1)$, $(q^2 - q + 1) \times (q^3 + 1)$

$q = 2$: 6×9, 3×9

Godsil, Australas. J. Combin., 1992

Fickus et al, Appl. Comput. Harmon. Anal., 2019

JWI & Mixon, Algebr. Comb., 2024

$r = 2$

Real ETFs $\quad\leftrightarrow\quad$ DRACKNs with $r = 2$

$$\begin{bmatrix} 0 & - & - & + & - & - \\ - & 0 & - & - & + & - \\ - & - & 0 & - & - & + \\ + & - & - & 0 & + & + \\ - & + & - & + & 0 & + \\ - & - & + & + & + & 0 \end{bmatrix} \quad\leftrightarrow\quad \left[\begin{array}{cc|cc|cc|cc|cc|cc} 0&0&0&1&0&1&1&0&0&1&0&1 \\ 0&0&1&0&1&0&0&1&1&0&1&0 \\ \hline 0&1&0&0&0&1&0&1&1&0&0&1 \\ 1&0&0&0&1&0&1&0&0&1&1&0 \\ \hline 0&1&0&1&0&0&0&1&0&1&1&0 \\ 1&0&1&0&0&0&1&0&1&0&0&1 \\ \hline 1&0&0&1&0&1&0&0&1&0&1&0 \\ 0&1&1&0&1&0&0&0&0&1&0&1 \\ \hline 0&1&1&0&0&1&1&0&0&0&1&0 \\ 1&0&0&1&1&0&0&1&0&0&0&1 \\ \hline 0&1&0&1&1&0&1&0&1&0&0&0 \\ 1&0&1&0&0&1&0&1&0&1&0&0 \end{array}\right]$$

Also equivalent to:

- Regular two-graphs
- Strongly regular graphs with $k = 2\mu$

Outline

Problem

Does there exist a regular abelian DRACKN(n, r, c) with

$$n = (t^2 - 1)^2, \qquad rc = (t-1)^2(t^2 + t - 1)$$

for some integer $t \neq 2$?

If so: $\exists\ d \times d^2$ ETF for $d = t^2 - 1$

Feasible:

t	d	n	r	c
6	35	1 225	5	205
8	63	3 969	7	497
11	120	14 400	5	2 620
12	143	20 449	11	1 705
14	195	38 025	13	2 717
15	224	50 176	7	6 692
16	255	65 025	5	12 195
18	323	104 329	17	5 797
20	399	159 201	19	7 961
21	440	193 600	5	36 880

Spoiler: NO

Coutinho, Godsil, Shirazi & Zhan, Linear Algebra Appl., 2016

Theorem (Gerzon's "absolute" bound)

If there are n equiangular lines in $\mathbb{C}^d$, then $n \leq d^2$.

If equality holds, then unit norm reps form an ETF.

Michael Gerzon

Lemmens and Seidel, J. Algebra, 1973

Theorem (Gerzon's "absolute" bound)

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Michael Gerzon

Sketch of proof for "$\leq$"

- Pick unit norm reps $\varphi_1, \ldots, \varphi_n$
- Define $P_j = \varphi_j \varphi_j^* = P_j*$
- $P_1, \ldots, P_n$ are linearly independent:

$$\left[\; \langle P_i, P_j \rangle_F \; \right]_{i,j} = \left[\; |\langle \varphi_i, \varphi_j \rangle|^2 \; \right]_{i,j} = \mu^2 J + (1 - \mu^2) I$$

- $n \leq \dim\{\text{Hermitian } d \times d \text{ matrices}\}$

Lemmens and Seidel, J. Algebra, 1973

Zauner's conjecture

For every d, there exists a $d \times d^2$ ETF

Gerhard Zauner

- Projections are basis for Hermitian mats (quantum info thy)
- Numerical solns: all $d \leq 53$, others
- Exact solns: all $d \leq 193$, others (finitely many)
- Recent news: implied by Stark conjectures+ (num thy)
- Related to Hilbert's 12th problem?

Zauner, PhD Thesis, U Vienna, 1999

Renes, Blume-Kohout, Scott & Caves, J. Math. Phys., 2004

Appleby, Flammia & Kopp, `arXiv:2501.03970`

Recall: Welch bound

For n unit vectors $\Phi = \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix}$ in $\mathbb{C}^d$,

$$\max_{i\neq j} |\langle \varphi_i, \varphi_j \rangle| \geq \sqrt{\frac{n-d}{d(n-1)}},$$

with equality $\iff$ Φ is an ETF.

Corollary (Waldron)

If S is the signature matrix of a $d \times d^2$ ETF, then $S^{\circ 2}$ is the signature matrix of a $\frac{d(d+1)}{2} \times d^2$ ETF.

Waldron, IEEE Trans. Inf. Theory, 2017

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Proof:

- Consider outer products $\varphi_1 \varphi_1^\top, \ldots, \varphi_{d^2} \varphi_{d^2}^\top \in \mathbb{C}^{d \times d}_{\mathrm{sym}}$
- Welch $\Leftarrow$: $|\langle \varphi_i \varphi_i^\top, \varphi_j \varphi_j^\top \rangle_F| = |\langle \varphi_i, \varphi_j \rangle^2| = \frac{1}{d+1}$
- $\dim \mathbb{C}^{d \times d}_{\mathrm{sym}} = \frac{d(d+1)}{2}$, sig $= S^{\circ 2}$
- Welch $\Rightarrow$: outer products are an ETF in $\mathbb{C}^{d \times d}_{\mathrm{sym}}$

Waldron, IEEE Trans. Inf. Theory, 2017

Corollary: Here be no DRACKNs

If a regular abelian DRACKN exists to produce a $d \times d^2$ ETF, then $d \in \{1, 3\}$.

JWI & Mixon, J. Combin. Theory, Ser. A, 2022

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Recall:

- $\hat{\alpha}(B)$ applies $\alpha \in \hat{\Gamma}$ entrywise to $B \in \mathbb{C}[\Gamma]^{n \times n}$
- same d for **any** $1 \neq \alpha \in \hat{\Gamma}$

Proof:

- Let $S = \pm\hat{\alpha}(B)$ be the sig mat
- Waldron: $S^{\circ 2} = \widehat{\alpha^2}(B)$ is the sig mat of a $\frac{d(d+1)}{2} \times d^2$ ETF

JWI & Mixon, J. Combin. Theory, Ser. A, 2022

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- same d for **any** $1 \neq \alpha \in \hat{\Gamma}$

Proof:

- Let $S = \pm\hat{\alpha}(B)$ be the sig mat
- Waldron: $S^{\circ 2} = \widehat{\alpha^2}(B)$ is the sig mat of a $\frac{d(d+1)}{2} \times d^2$ ETF
- If $\alpha^2 = 1$, then $\widehat{\alpha^2}(B) = J - I$ implies $\frac{d(d+1)}{2} = 1$, so $d = 1$
- If $\alpha^2 \neq 1$, then $\frac{d(d+1)}{2} \in \{d, d^2 - d\}$, so $d \in \{1, 3\}$

JWI & Mixon, J. Combin. Theory, Ser. A, 2022

Feasible DRACKN parameters, $n \leq 500$, r odd prime

d	n	r	c	∃?
6	9	3	3	Y
15	25	5	5	Y
11	33	3	9	
21	36	3	12	Y
12	45	3	12	Y
33	45	5	10	
28	49	7	7	Y
34	51	3	18	
22	55	5	10	Y
52	65	5	15	Y
45	81	3	27	Y
65	91	7	14	
76	96	3	36	
33	99	3	30	
55	100	5	20	Y
14	105	3	27	
40	105	7	14	Y
65	105	3	36	
35	120	3	36	
66	121	11	11	Y
105	126	3	48	Y
86	129	3	45	
78	144	3	48	Y
29	145	5	25	
46	161	7	21	
91	169	13	13	Y

d	n	r	c	∃?
30	175	5	30	
145	175	7	28	
133	190	5	40	
105	196	7	28	Y
67	201	3	63	
77	210	5	40	
133	210	3	72	
186	217	7	35	
120	225	3	75	
120	225	5	45	
175	225	3	81	
70	231	3	72	
161	231	11	22	
162	243	3	84	
41	246	3	72	
92	253	11	22	Y
52	273	7	35	
221	273	3	99	
217	280	5	60	
42	288	3	84	
153	289	17	17	Y
177	295	5	60	
129	301	7	42	
88	320	5	60	
171	324	3	108	Y
225	325	13	26	

d	n	r	c	∃?
260	325	5	70	
113	339	3	108	
78	351	3	108	
126	351	13	26	Y
225	351	3	120	
190	361	19	19	Y
117	378	3	120	
261	378	7	56	
33	385	5	65	
55	385	7	49	
105	385	11	33	
154	385	5	75	
262	393	3	135	
210	400	5	80	Y
145	406	7	56	
56	441	7	56	
231	441	3	147	
231	441	7	63	
385	441	3	162	
369	451	11	44	
391	460	5	100	
370	481	13	39	
253	484	11	44	Y
97	485	5	90	
209	495	3	162	
286	495	5	100	

Outline

DRACKNs: another view

Lemma (Godsil & Hensel)

Given $B \in \mathbb{C}[\Gamma]^{n \times n}$ such that

- each $B_{ii} = 0$,
- $B_{ij} \in \Gamma$ when $i \neq j$,
- $B_{ji} = B_{ij}^{-1}$ when $i \neq j$,

then B is a DRACKN(n, r, c) iff

- $B^2 = (n-1)I + [n - c(r-1) - 2]\delta_{\text{id}} B + \sum_{g \neq \text{id}} c\delta_g B.$

$$\begin{bmatrix} 0 & \delta_- & \delta_- & \delta_+ & \delta_- & \delta_- \\ \delta_- & 0 & \delta_- & \delta_- & \delta_+ & \delta_- \\ \delta_- & \delta_- & 0 & \delta_- & \delta_- & \delta_+ \\ \delta_+ & \delta_- & \delta_- & 0 & \delta_+ & \delta_+ \\ \delta_- & \delta_+ & \delta_- & \delta_+ & 0 & \delta_+ \\ \delta_- & \delta_- & \delta_+ & \delta_+ & \delta_+ & 0 \end{bmatrix}^2 = \begin{bmatrix} 5\delta_0 & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- \\ 2\delta_+ + 2\delta_- & 5\delta_0 & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- \\ 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 5\delta_0 & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- \\ 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 5\delta_0 & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- \\ 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 5\delta_0 & 2\delta_+ + 2\delta_- \\ 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 2\delta_+ + 2\delta_- & 5\delta_0 \end{bmatrix}$$

$$B^2 = 5I + 2\delta_+ B + 2\delta_- B = (n-1)I + \sum_{g \in \Gamma} c_g \delta_g B$$

Godsil & Hensel, J. Combin. Theory Ser. B, 1992

Defn: “Roux”

Determined by:

- Γ, finite abelian group, order r
- $B \in \mathbb{C}[\Gamma]^{n\times n}$

Such that:

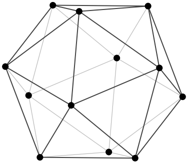
- each $B_{ii} = 0$
- $B_{ij} \in \Gamma$ when $i \neq j$
- $B_{ji} = B_{ij}^{-1}$ when $i \neq j$
- $B^2 = (n-1)I + \sum_{g\in\Gamma} c_g \delta_g B$

Parameters: $\text{roux}(n, r, \{c_g\}_{g\in\Gamma})$

$$\text{DRACKN} \iff c_g = \text{const for } g \neq \text{id}$$

$$B = \begin{bmatrix} 0 & \delta_- & \delta_- & \delta_+ & \delta_- & \delta_- \\ \delta_- & 0 & \delta_- & \delta_- & \delta_+ & \delta_- \\ \delta_- & \delta_- & 0 & \delta_- & \delta_- & \delta_+ \\ \delta_+ & \delta_- & \delta_- & 0 & \delta_+ & \delta_+ \\ \delta_- & \delta_+ & \delta_- & \delta_+ & 0 & \delta_+ \\ \delta_- & \delta_- & \delta_+ & \delta_+ & \delta_+ & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$



JWI & Mixon, J. Combin. Theory, Ser. A, 2022

Example: $\Gamma = \mathbb{Z}_4$ and $(n, r, \{c_g\}_{g=0}^{3}) = (4, 4, \{0, 1, 0, 1\})$

$$B = \begin{bmatrix} 0 & \delta_1 & \delta_1 & \delta_1 \\ \delta_3 & 0 & \delta_1 & \delta_3 \\ \delta_3 & \delta_3 & 0 & \delta_1 \\ \delta_3 & \delta_1 & \delta_3 & 0 \end{bmatrix} \qquad A = \left[\begin{array}{cccc|cccc|cccc|cccc} 0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0 \\ \hline 0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0 \\ 0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0 \\ \hline 0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0 \\ 1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0 \\ \hline 0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0 \end{array}\right]$$

$$\begin{bmatrix} 0 & \delta_1 & \delta_1 & \delta_1 \\ \delta_3 & 0 & \delta_1 & \delta_3 \\ \delta_3 & \delta_3 & 0 & \delta_1 \\ \delta_3 & \delta_1 & \delta_3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3\delta_0 & \delta_2 + \delta_0 & \delta_2 + \delta_0 & \delta_2 + \delta_0 \\ \delta_2 + \delta_0 & 3\delta_0 & \delta_2 + \delta_0 & \delta_2 + \delta_0 \\ \delta_2 + \delta_0 & \delta_2 + \delta_0 & 3\delta_0 & \delta_2 + \delta_0 \\ \delta_2 + \delta_0 & \delta_2 + \delta_0 & \delta_2 + \delta_0 & 3\delta_0 \end{bmatrix}$$

$$B^2 = 3I + \delta_1 B + \delta_3 B = (n-1)I + \sum_{g \in \Gamma} c_g \delta_g B$$

Properties of roux graphs

Given a $\mathrm{roux}(n, r, \{c_g\}_{g\in\Gamma})$:

- cover of K_n
- connected $\iff \langle g : c_g \neq 0\rangle = \Gamma$
- every $c_g \neq 0 \iff$ diam 3 $\implies$ regular antipodal cover of K_n
- graph spectrum determined by Fourier spectrum $\{\hat{c}_\alpha\}_{\alpha\in\hat{\Gamma}}$

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Quotients of roux graphs

Given a roux$(n, r, \{c_g\}_{g\in\Gamma})$, $B \in \mathbb{C}[\Gamma]^{n\times n}$:

- apply any group hom $\psi\colon \Gamma \to \Lambda$ entrywise to get a roux for Λ, params $c_\lambda = \sum_{g\in\psi^{-1}(\lambda)} c_g$ for $\lambda \in \Lambda$
- connected roux graph, $\Lambda = \mathbb{Z}_p$ for prime $p \mid r$
  $\implies$ quotient is a DRACKN(n, p, c)

Theorem (JWI & Mixon)

Given $B \in \mathbb{C}[\Gamma]^{n\times n}$ such that

- each $B_{ii} = 0$,
- $B_{ij} \in \Gamma$ when $i \neq j$,
- $B_{ji} = B_{ij}^{-1}$ when $i \neq j$,

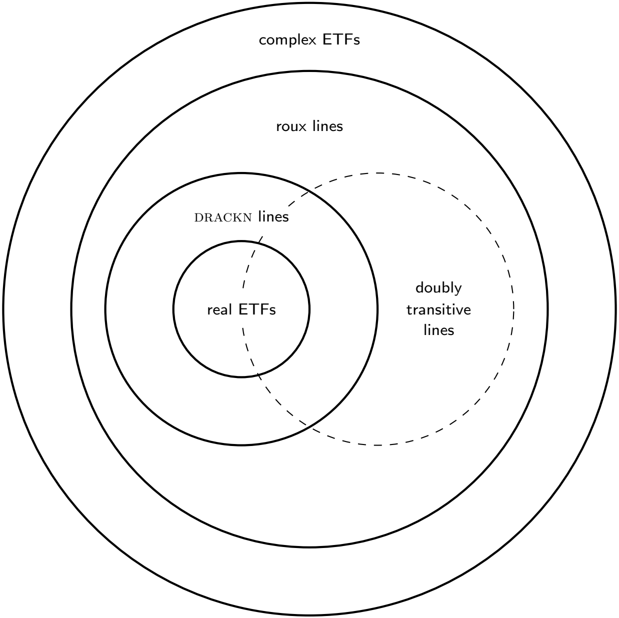
then B is a roux **if and only if**

- $\forall \alpha \in \hat{\Gamma}$, $\hat{\alpha}(B) =$ signature matrix of $d_\alpha \times n$ ETF.

In that case, $d_\alpha = f(n, r, \hat{c}_\alpha)$, where $\{c_g\}_g$ are roux params.

$$B = \begin{bmatrix} 0 & \delta_1 & \delta_1 & \delta_1 \\ \delta_3 & 0 & \delta_1 & \delta_3 \\ \delta_3 & \delta_3 & 0 & \delta_1 \\ \delta_3 & \delta_1 & \delta_3 & 0 \end{bmatrix} \qquad \hat{\alpha}(B) = \begin{bmatrix} 0 & i & i & i \\ -i & 0 & i & -i \\ -i & -i & 0 & i \\ -i & i & -i & 0 \end{bmatrix}$$

$\alpha\colon \mathbb{Z}_4 \to \mathbb{T}\colon \delta_1 \mapsto i$ — sig mat of 2×4 ETF


complex ETFs
roux lines
DRACKN lines
real ETFs
doubly
transitive
lines

Theorem (JWI & Mixon)

Given an ETF Φ, change line reps so sig mat S of ΦD has all 1's in first row/col. Then the ETF vectors span roux lines iff, for some r:

- $\forall i \neq j$, $S_{ij}^r = 1$, and
- $\forall 1 < k < r$, $S^{\circ k}$ is the sig mat of some ETF.

In that case, the roux can be taken over $\Gamma = \mathbb{Z}_r$.

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Example: Hoggar's lines

- $\exists$ 8×64 ETF with $S_{ij} \in \{\pm i, \pm 1\}$ $\forall i \neq j$, so $r = 4$
- Waldron: $S^{\circ 2}$ is the sig mat of a (real) 36×64 ETF
- $S^{\circ 3} = \overline{S}$ is the sig mat of another 8×64 ETF
- thus, S comes from a roux over $\Gamma = \mathbb{Z}_4$
- roux params: $(n, r, \{c_g\}_{g=0}^{3}) = (64, 4, \{24, 16, 6, 16\})$

Problem

Does there exist a roux over $\Gamma = \mathbb{Z}_4$ with params $n = 16t^2(t+1)^2$,

$$c_0 = 4t^4 + 12t^3 + 10t^2 - 2, \qquad c_1 = c_3 = 4t^4 + 8t^3 + 4t^2,$$

$$c_2 = 4t^4 + 4t^3 - 2t^2$$

for some integer $t > 1$?

If so: $\exists d \times d^2$ ETF with $d = 4t(t+1)$

$t = 1$: Hoggar's lines

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All $t \geq 1$ feasible:

- Only known integrality constraint satisfied
- Dimension of $S^{\circ 2} = \widehat{\alpha^2}(B)$ satisfies Waldron constraint
- $S^{\circ 2}$ makes a real ETF of a size that exists
  if a Hadamard of order $d = 4t(t+1)$ exists

Questions?

Picture generated by Microsoft Copilot

Doubly transitive lines I: Higman pairs and roux
JW Iverson and DG Mixon,
J. Combin. Theory Ser. A, 2022

Doubly transitive lines II: Almost simple symmetries
JW Iverson and DG Mixon,
Algebr. Comb., 2024