Abelian covers of association schemes with applications to SIC-POVM

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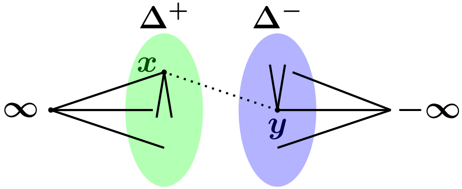
The graph extension construction (Shult, 1972)



Let A_1 be the adjacency matrix of Δ, let $A_2 = J - I - A_1$ be the adjacency matrix of the complement. Let

$$\tilde{A}_1 = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & A_1 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix},$$

Then the construction can be equivalently described by the adjacency matrix

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_2 & \tilde{A}_1 \end{bmatrix}.$$

Taylor graphs

A graph Δ is strongly regular with parameters (v, k, λ, μ) if

- Δ has v vertices and is k-regular,
- any pair of adjacent (non-adjacent) vertices have λ (μ) common neighbors.

Theorem

If Δ is a strongly regular graph with parameters (v, k, λ, μ) with $k = 2\mu$, then the graph extension gives a distance-regular antipodal 2-cover of K_{v+1}.

See Brouwer, Cohen and Neumaier, “Distance-Regular Graphs”, Section 1.5.

We call graphs arising in the above theorem a Taylor graph.

A directed graph Δ is a doubly regular tournament if

- Δ has $2k+1$ vertices and is k-regular, where k is odd,
- the adjacency matrix A_1 satisfies $A_1^2 = \frac{k-1}{2}A_1 + \frac{k+1}{2}A_2$, where $A_2 = A_1^\top = J - I - A_1$.

Theorem (Ikuta and M., 2023)

If Δ is a doubly regular tournament, then

$$\tilde{A}_1 = \begin{bmatrix} 0 & 1 \\ 0 & A_1 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 \\ 1^\top & A_2 \end{bmatrix},$$

$$B_0 = I, \quad B_1 = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_2 & \tilde{A}_1 \end{bmatrix}, B_2 = \begin{bmatrix} \tilde{A}_2 & \tilde{A}_1 \\ \tilde{A}_1 & \tilde{A}_2 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

gives a 3-class association scheme.

Theorem says the linear span of B_0, B_1, B_2, B_3 is closed under multiplication. This modified graph extension is essentially due to Babai and Cameron (2000).

Antipodal covers of complete graphs

Definition

Let $N \geq 3$ and $r \geq 2$. A graph Γ is called an antipodal r-cover of the complete graph K_N if

- $V(\Gamma)$ admits an equitable partition into N cocliques of size r,
- any pair of these cocliques induces a matching.

Taylor graphs are antipodal 2-covers.

1st Goal: A distance-regular antipodal 2^t-cover of K_{n+1} can be constructed from an amorphic pseudocyclic association scheme of 2^t classes on n points.

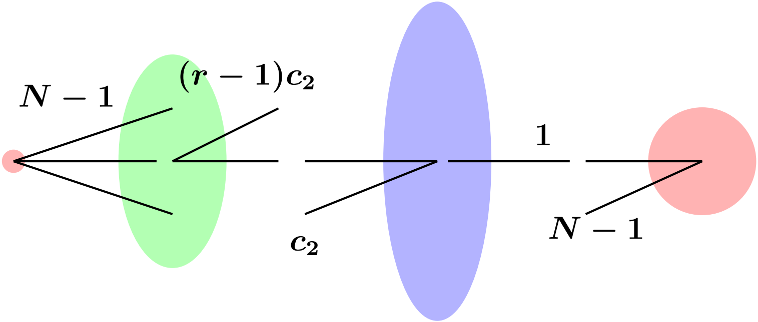
Example: Generalized Paley graphs $\mathrm{Cay}(\mathbb{F}_{49}, \zeta^i\langle\zeta^4\rangle)$, $i = 1, 2, 3, 4$, where $\mathbb{F}_{49}^\times = \langle\zeta\rangle$, give a 4-cover of K_{50}.

Definition

A graph of diameter 3 is called a distance-regular antipodal r-cover of K_N if there exists $c_2 > 0$ such that the distance partition with respect to a vertex is an equitable partition with quotient matrix

$$\begin{bmatrix} 0 & N-1 & 0 & 0 \\ 1 & N-2-(r-1)c_2 & (r-1)c_2 & 0 \\ 0 & c_2 & N-2-c_2 & 1 \\ 0 & 0 & N-1 & 0 \end{bmatrix}.$$

- "distance$= 3$" is an equivalence relation, and equivalence classes form an equitable partition into cocliques of size r,
- any pair of these cocliques induces a matching.


$N-1$
$(r-1)c_2$
1
c_2
$N-1$

Definition (Godsil–Hensel, 1992)

A symmetric arc function is a function

$$f\colon \{(x,y) \mid x,y \in X,\ x \neq y\} \to G,$$

where G is an abelian group, satisfying

$$f(x,y) = f(y,x)^{-1} \quad (x,y \in X,\ x \neq y).$$

A symmetric arc function gives rise to an antipodal r-cover, denoted Γ_f, of K_N, where $r = |G|$ and $N = |X|$.

- $V(\Gamma_f) = X \times G$,
- $(x,g) \sim (y,h) \iff f(x,y)g = h$.

1st Goal: $G = (\mathbb{Z}_2)^t$, $X = \{\infty\} \cup Y$, where Y carries an amorphic pseudocyclic association scheme.

SRG of (negative) Latin square type

Definition

Let (Y, R_i) $(i = 1, \ldots, r)$ be a collection of edge-disjoint strongly regular graphs with parameters

$$((rg \mp 1)^2, g(rg \mp 2), g^2 - 1 \pm (r - 3), g(g \mp 1)).$$

Let $R_0 = \{(y, y) \mid y \in Y\}$. We say that $(Y, \{R_i\}_{i=0}^r)$ is an amorphic pseudocyclic association scheme,

provided

- the linear span of $\mathfrak{A}$ the adjacency matrices A_i of R_i $(i = 0, \ldots, r)$ is closed under multiplication,
- the multiplication in $\mathfrak{A}$ is commutative,
- $\ldots$

but unnecessary, by Van Dam (2003)

Theorem (Lansdown–M., 2024+)

Let $(Y, \{R_i\}_{i=0}^r)$ be an amorphic pseudocyclic association scheme with adjacency matrices $\{A_i\}_{i=0}^r$, where $r = 2^t$. Fix a bijection $\iota\colon [r] \to (\mathbb{Z}_2)^t$. Define

$$\tilde{A}_i = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1^\top & A_i \end{bmatrix} & \text{if } \iota(i) = 0 \in (\mathbb{Z}_2)^t, \\ \begin{bmatrix} 0 & 0 \\ 0 & A_i \end{bmatrix} & \text{otherwise.} \end{cases}$$

Let $\rho\colon (\mathbb{Z}_2)^t \to GL(r, \mathbb{Z})$ be the regular representation. Then

$$B = \sum_{i \in [r]} \tilde{A}_i \otimes \rho(\iota(i))$$

is the adjacency matrix of a distance-regular antipodal r-cover of $K_{|Y|+1}$.

By Godsil–Hensel (1995), our construction results in graphs equivalent to certain group divisible designs. In fact, our graphs are divisible design graphs.

In the expression

$$\boldsymbol{B} = \sum_{i \in [r]} \tilde{\boldsymbol{A}}_i \otimes \boldsymbol{\rho}(\boldsymbol{\iota}(i)),$$

where the Kronecker product $\otimes$ is used to describe the adjacency matrix, the matrix algebra over the group algebra of the abelian group was used in Godsil–Hensel (1995), but also more recently Coutinho–Godsil–Shirazi–Zhan (2016).

Theorem (Lansdown–M., 2024+)

Let $(Y, \{R_i\}_{i=0}^r)$ be an amorphic pseudocyclic association scheme with adjacency matrices $\{A_i\}_{i=0}^r$, where $r = 2^t$. Fix a bijection $\iota\colon [r] \to (\mathbb{Z}_2)^t$. Define

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$$B = \sum_{i \in [r]} \tilde{A}_i \otimes \rho(\iota(i))$$

is the adjacency matrix of a distance-regular antipodal r-cover of $K_{|Y|+1}$.

- Let $(Y, \{R_i\}_{i=0}^r)$ be a commutative association scheme with adjacency matrices $\{A_i\}_{i=0}^r$.
- Let G be a finite abelian group of order r.
- Let $\iota\colon [r] \to G$ be a bijection such that, if $R_i^\top = R_{i'}$ then $\iota(i)^{-1} = \iota(i')$.

Define

$$\tilde{A}_i = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1^\top & A_i \end{bmatrix} & \text{if } \iota(i) = 1_G, \\ \begin{bmatrix} 0 & 0 \\ 0 & A_i \end{bmatrix} & \text{otherwise.} \end{cases}$$

Let $\rho\colon G \to GL(r, \mathbb{Z})$ be the regular representation. For $g \in G$, define

$$B_g = \sum_{i \in [r]} \tilde{A}_i \otimes \rho(g\iota(i)), \quad C_g = I \otimes \rho(g).$$

Then $\{B_g, C_g \mid g \in G\}$ forms the set of adjacency matrices of an association scheme, provided... (conditions to be described)

To ease notation, let us identify $[r]$ with G, so that $(Y, \{R_g\}_{g\in\{0\}\cup G})$ is a commutative association scheme with adjacency matrices $\{I\} \cup \{A_g\}_{g\in G}$, $A_g^\top = A_{g^{-1}}$ for $g \in G$. Then

$$B_g = \sum_{h\in G} \tilde{A}_h \otimes \rho(gh).$$

where

$$\tilde{A}_h = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1^\top & A_h \end{bmatrix} & \text{if } h = 1_G, \\ \begin{bmatrix} 0 & 0 \\ 0 & A_h \end{bmatrix} & \text{otherwise.} \end{cases}$$

Let $(Y, \{R_g\}_{g\in\{0\}\cup G})$ be a commutative association scheme with adjacency matrices $\{I\} \cup \{A_g\}_{g\in G}$, $A_g^\top = A_{g^{-1}}$ for $g \in G$.

Lemma

TFAE

1. $\{B_g, C_g \mid g \in G\}$ *forms the set of adjacency matrices of an association scheme,*
2. B_1^2 *belongs to the linear span of* $\{B_g, C_g \mid g \in G\}$,
3. $\sum_{g\in G} A_g^\top A_{gh} = \sum_{g\in G} k_g A_{gh} + \delta_{h,1}(|Y|I - J)$ *for* $h \in G$.

The last condition can be expressed easily in terms of intersection numbers or eigenvalues of the association scheme $(Y, \{R_i\}_{i\in\{0\}\cup G})$.

The resulting association scheme has $2|G| - 1$ classes (note that $C_1 = I$), and is symmetric if and only if $G \cong (\mathbb{Z}_2^t)$ (since $C_g^\top = C_{g^{-1}}$). But it admits fusion if $G \cong (\mathbb{Z}_2^t)$.

Let $\mathcal{Y} = (Y, \{R_g\}_{g\in\{0\}\cup G})$ be a commutative association scheme with adjacency matrices $\{I\} \cup \{A_g\}_{g\in G}$, $A_g^\top = A_{g^{-1}}$ for $g \in G$.

Lemma

TFAE

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3. $\sum_{g\in G} A_g^\top A_{gh} = \sum_{g\in G} k_g A_{gh} + \delta_{h,1}(|Y|I - J)$ *for* $h \in G$.

If $\mathcal{Y}$ is an amorphic pseudocyclic association scheme with $r = 2^t$, then $B_1^2 \in \langle I, B_1, B_2\rangle$, where

$$B_2 = J - B_1 - I \otimes J = J - B_1 - I - B_3,$$
$$B_3 = \sum_{g\in G\setminus\{1\}} C_g.$$

Let $G = \mathbb{Z}_4$, and let $\mathcal{Y} = (Y, \{R_g\}_{g \in \{0\} \cup G})$ be a commutative association scheme with adjacency matrices $\{I\} \cup \{A_g\}_{g \in G}$, $A_g^\top = A_{g^{-1}}$ for $g \in G$. If

$$\sum_{g \in G} A_g^\top A_{gh} = \sum_{g \in G} k_g A_{gh} + \delta_{h,1}(|Y|I - J) \quad (h \in G),$$

then we have a 7-class association scheme with adjacency matrices $\{B_g, C_g \mid g \in G\}$. The condition is satisfied by the association scheme with first eigenmatrix

$$P = \begin{bmatrix} 1 & 6 & 16 & 24 & 16 \\ 1 & 3 & -2 & 0 & -2 \\ 1 & -1 & 2 & -4 & 2 \\ 1 & -3 & -2+6i & 6 & -2-6i \\ 1 & -3 & -2-6i & 6 & -2+6i \end{bmatrix}$$

Its symmetrization is one of the dual pair of $GH(2,2)$.

To compute the first eigenmatrix of the resulting association scheme, we first compute the intersection matrices in general.

Recall that the intersection matrices are nothing but the matrix representation of the left multiplication of the adjacency matrices, expressed in terms of the basis $\mathcal{B} = ((C_g)_{g\in G}; (B_g)_{g\in G})$ of adjacency matrices:

$$C_h \mathcal{B} = \mathcal{B} \begin{bmatrix} \rho(h) & 0 \\ 0 & \rho(h) \end{bmatrix},$$

$$B_h \mathcal{B} = \mathcal{B} \begin{bmatrix} 0 & |Y|\rho(h) \\ \rho(h) & \sum_{g\in G} k_g \rho(gh) \end{bmatrix}.$$

The first eigenmatrix is the table of eigenvalues of the intersection matrices. This can be computed using the ideas from Coutinho–Godsil–Shirazi–Zhan (2016) or the concept of polyphase matrices in Fickus–Jasper–Mixon–Peterson–Watson (2019).

Since G is an abelian group, the intersection matrices can be block-diagonalized as

$$\begin{bmatrix} \rho(h) & 0 \\ 0 & \rho(h) \end{bmatrix} \sim \left(\begin{bmatrix} \chi(h) & 0 \\ 0 & \chi(h) \end{bmatrix} \right)_{\chi \in \mathrm{Irr}(G)}$$

$$\begin{bmatrix} 0 & n\rho(h) \\ \rho(h) & \sum_{\ell \in G} k_\ell \rho(h\ell h) \end{bmatrix} \sim \left(\begin{bmatrix} 0 & |Y|\chi(h) \\ \chi(h) & \sum_{g \in G} k_g \chi(gh) \end{bmatrix} \right)_{\chi \in \mathrm{Irr}(G)}$$

Computing the eigenvalues for the example $G = \mathbb{Z}_4$, Y: fission of $GH(2,2)$, we have

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 63 & 63 & 63 & 63 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & i & -1 & -i & 3 & 3i & -3 & -3i \\ 1 & i & -1 & -i & -21 & -21i & 21 & 21i \\ 1 & -1 & 1 & -1 & 7 & -7 & 7 & -7 \\ 1 & -1 & 1 & -1 & -9 & 9 & -9 & 9 \\ 1 & -i & -1 & i & 3 & -3i & -3 & 3i \\ 1 & -i & -1 & i & -21 & 21i & 21 & -21i \end{bmatrix}$$

The second eigenmatrix has column (written as a row vector)

$$\begin{array}{cccccccc} C_1 & C_i & C_{-1} & C_{-i} & B_1 & B_i & B_{-1} & B_{-i} \\ 8 & -8i & -8 & 8i & -\frac{8}{3} & \frac{8}{3}i & \frac{8}{3} & -\frac{8}{3}i \end{array}$$

This implies that the association scheme can be realized as a subset consisting of unit vectors in $\mathbb{C}^8$ invariant under $\langle i \rangle$, in such a way that the absolute values of the inner products are 1 and $\frac{1}{3}$ only. Since $|Y| = 63$, the number of vectors is

$$(1 + |Y|)|\mathbb{Z}_4| = 256,$$

and the $256/4 = 64$ vectors up to $\langle i \rangle$ form a SIC-POVM in $\mathbb{C}^8$.

Definition (SIC-POVM)

$X \subseteq \{x \in \mathbb{C}^n \mid \|x\| = 1\}$, $|X| = n^2$,

$$|(x, y)| = \frac{1}{\sqrt{n+1}} \quad (x, y \in X,\ x \neq y)$$

SIC-POVM in $\mathbb{C}^3$

From a non-symmetric 3-class association scheme on 8 points, together with $\mathbb{Z}_3$, one can similarly construct a 5-class association scheme on

$$(1+8)3 = 27$$

points.

This association scheme can be realized as a subset consisting of unit vectors in $\mathbb{C}^3$ invariant under $\langle\omega\rangle$, in such a way that the absolute values of the inner products are 1 and $\frac{1}{2}$ only.

The $27/3 = 9$ vectors up to $\langle\omega\rangle$ form a SIC-POVM in $\mathbb{C}^3$.

Remark

The data of the association scheme on $\mathbf{256}$ points representing SIC-POVM in $\mathbb{C}^{\mathbf{8}}$ was provided to us by Sho Suda who, together with Alexander Gavrilyuk, showed the uniqueness and triple regularity of the association scheme.

Because of the triple regularity, the neighborhood of a point with respect to a relation carries an association scheme. So it is natural to expect that the bigger association scheme can be reconstructed from the neighborhood by the construction

$$\tilde{\boldsymbol{A}}_{\mathbf{1}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1}^{\top} & \boldsymbol{A}_{\mathbf{1}} \end{bmatrix}.$$

Thank you very much for your attention!