THE KODAIRA DIMENSION OF MODULI SPACES OF CURVES WITH MARKED POINTS

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Abstract. The purpose of this paper is to generalize the methods and results due to Eisenbud, Harris, and Mumford on the moduli space of curves of genus g to the moduli space of curves of genus g with n marked points. The main result is the determination, for each g with $4 \leq g \leq 23$, of an n such that the moduli space is of general type.

1. Introduction. In the early 1980's, Eisenbud, Harris, and Mumford studied $\overline{\mathcal{M}}_g$, the moduli space of curves of a given genus, and succeeded in proving that if the genus is at least 24, the moduli space is of general type. This was contrary to many people's expectations, as it had been known for some time that moduli spaces of curves of small genus were rational or unirational.

Briefly, what is involved in their proof is the computation of the canonical divisor of $\overline{\mathcal{M}}_g$, the exhibition of certain effective divisors and the determination of their classes, and the verification that the canonical divisor is the sum of an ample and an effective divisor. In the other direction, it is known that $\overline{\mathcal{M}}_g$ has Kodaira dimension $-\infty$ when $g \leq 13$ or $g = 15$.

One can generalize the definition of $\mathcal{M}_g$ by asking for a parametrization of the curves of genus g together with n distinct points on them. This amounts to studying, instead of families of curves, families of curves with n disjoint sections. As before, it turns out that one obtains a noncompact moduli space, which is compactified by replacing "nonsingular curve" by "stable curve" in the definitions. In order to be considered stable, an n-pointed curve is required only to have finitely many automorphisms that fix all the marked points.

Definition 1.1. $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of genus g with n marked points.

In this paper, I will study divisor classes on $\overline{\mathcal{M}}_{g,n}$. Although $\operatorname{Pic} \overline{\mathcal{M}}_{g,n} \otimes \mathbf{Q}$ is known completely, the effective and ample cones are much less well understood. In fact, these still seem quite difficult even for $\overline{\mathcal{M}}_g$, and we will not come close to determining them. (Gibney, Keel, and Morrison have recently made notable progress on these problems [GKM].) Rather, I will compute the classes of a number of divisors on $\overline{\mathcal{M}}_{g,n}$, and apply these results to determine the Kodaira dimension of many of the varieties $\overline{\mathcal{M}}_{g,n}$. The methods are more suited to proving that they are of general type than to proving that they are not.

An effective divisor on $\overline{\mathcal{M}}_g$ can be thought of as a property of curves of genus g which is satisfied in codimension 1. Similarly, when $n = 1$, an effective divisor on $\overline{\mathcal{M}}_{g,n}$ is a property of points on curves which only finitely many points on a general curve will satisfy. To my knowledge, the first computation of the class of a divisor on $\overline{\mathcal{M}}_{g,1}$ was done by Cukierman [Cuk], who computed the class of the locus of Weierstrass points. We may also consider the property of being a ramification point of a map to $\mathbf{P}^1$ of minimal degree if g is even, the property of being a ramification point of appropriate order of a map to $\mathbf{P}^1$ of slightly higher degree (more generally), and so on. Computing the classes of these divisors often requires intersecting them with curves entirely contained in a boundary component of $\overline{\mathcal{M}}_{g,1}$, and therefore it is important to understand linear series and line bundles on singular curves.

Here we will compute the classes of some divisors on $\overline{\mathcal{M}}_{g,n}$ and use them to prove, in many cases, that $\overline{\mathcal{M}}_{g,n}$ is of general type. We now define the family of divisors that will be most useful:

Definition 1.2. Choose nonnegative integers $a_1, \dots, a_n$ with $\sum a_n = g$. Then we may define a divisor $d_{g;a_1,\dots,a_n}$ on an open subset of $\mathcal{M}_{g,n}$ as the locus of points corresponding to curves and points $(C, p_1, \dots, p_n)$ such that the divisor $\sum a_i p_i$ moves in a pencil. The union of the divisorial components of its closure in $\overline{\mathcal{M}}_{g,n}$ will be denoted $D_{g;a_1,\dots,a_n}$.

Using these divisors, it is possible to prove, for example, that $\overline{\mathcal{M}}_{5,17}$ and $\overline{\mathcal{M}}_{17,9}$ are of general type—we get a result of this sort for each $g \in [4, 22]$. This will follow from an expression of the canonical divisors of these moduli spaces as the sum of an ample and an effective divisor; the point is that such expressions are possible only once we know that the classes of the $D_{g;a_1,\dots,a_n}$ are effective. In the next few sections, we will compute the classes of the D in terms of a basis for $\operatorname{Pic} \overline{\mathcal{M}}_{g,n} \otimes \mathbf{Q}$ which we will introduce and use this to prove our general-type results. We will compare these results to classically-known unirational parametrizations of $\overline{\mathcal{M}}_{g,n}$; in small genus, our results are very sharp, but the gap between n for which $\overline{\mathcal{M}}_{g,n}$ is known to be of negative Kodaira dimension and n for which $\overline{\mathcal{M}}_{g,n}$ is known to be of general type increases with g. For a full statement of the main result, see Theorem 5.1 (the statement of the theorem uses no notation that has not yet been introduced).

By pointing out an error in [L1], Sean Keel has led me to realize that the result of theorem 5.1 can be strengthened significantly in some cases where $g+1$ is prime. I intend to communicate this improvement shortly.

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2. Basic Facts. Here we present some of the basic facts about $\overline{\mathcal{M}}_{g,n}$ that we will be using in the rest of the paper. For $\overline{\mathcal{M}}_g$, the best reference is [HM]; there is no general reference of the same sort for $\overline{\mathcal{M}}_{g,n}$, but [AC] is essentially sufficient for what is contained here. Henceforth, we will restrict attention to $g > 3$ to avoid all sorts of trouble.

To start with, we describe the Picard group of the moduli stack $\overline{\mathcal{M}}_{g,n}$. This is taken directly from [AC, Thm. 2]. We will also assume a certain familiarity with the Picard group of $\overline{\mathcal{M}}_g$, as discussed in [HM].

Definition 2.1. On $\overline{\mathcal{M}}_{g,n}$, let λ and δ_0 be the pullbacks from $\overline{\mathcal{M}}_g$ of the classes of the same name. Fix $0 \le i \le \lfloor g/2 \rfloor, S \subseteq \{1, 2, \dots, n\}$, where $\operatorname{card} S > 1$ if $i = 0$ and $1 \in S$ if $i = g/2$. We let $\delta_{i;S}$ be the divisor class on the moduli stack whose value on a family of curves with smooth general fiber is the number of curves in the family that have a disconnecting node whose removal creates a connected component of genus i with precisely the marked points corresponding to elements of S on it. Let ψ_i be the divisor class on the moduli stack which takes the value $-\pi_*(\sigma_i^2)$ on the family $\mathcal{X} \xrightarrow{\pi} B$ with the σ_i as sections. On $\overline{\mathcal{M}}_{g,1}$, let ω be the relative dualizing sheaf for the map $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$; on $\overline{\mathcal{M}}_{g,n}$, let $\omega_i = \pi^* \omega$, where π is the map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,1}$ that forgets all but the ith marked point. (The reader is cautioned that ω_i is not the relative dualizing sheaf for the map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ that forgets the ith marked point.)

We make the obvious convention that $\delta_{i;S} = \delta_{g-i;\bar{S}}$, where $\bar{S}$ means the complement of S in $\{1, 2, \dots, n\}$. For $n = 1$, we have $\psi_1 = \omega_1$, and we denote both by ω (or ψ). Also we write simply δ for $\sum_{i,S} \delta_{i;S} + \delta_0$, and, when $n = 1$, we write δ_i for $\delta_{i;\{1\}}$ and δ_{g-i} for $\delta_{i;\emptyset}$.

Theorem 2.2. *$\operatorname{Pic}_{fun} \overline{\mathcal{M}}_{g,n}$, the Picard group of the moduli stack $\overline{\mathcal{M}}_{g,n}$, is free on the* *following generators:* $\lambda, \delta_0, \psi_i\,(1 \leq i \leq n)$, *and* $\delta_{i;S}\,(0 \leq i \leq \lfloor g/2 \rfloor), S \subseteq \{1, 2, \ldots, n\})$. ∎

We define π_i to be the map from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g,n-1}$ given by forgetting the ith point. We will need to know the effect of pulling back divisors under this map. Again, the answer is given in [AC, p. 161]:

THEOREM 2.3. *The pullback map is given as follows:*

$$\begin{aligned}\pi_n^*\lambda &= \lambda,\\ \pi_n^*\delta_0 &= \delta_0,\\ \pi_n^*\omega_i &= \omega_i,\\ \pi_n^*\psi_i &= \psi_i - \delta_{0;\{i,n\}}\\ \pi_n^*\delta_{i;S} &= \delta_{i;S} + \delta_{i;S\cup\{n\}},\end{aligned}$$

except that

$$\pi_1^*\delta_{g/2;\emptyset} = \delta_{g/2;\emptyset}$$

for $n = 1$. ∎

For notational simplicity, we have only stated this for π_n, but the action of the symmetric group $\mathcal{S}_n$ on $\overline{\mathcal{M}}_{g,n}$ makes it easy to describe the effects of the other π. Also observe that, by induction starting from $\psi = \omega$ on $\overline{\mathcal{M}}_{g,1}$, we get

$$\psi_i = \omega_i + \sum_{i \in S \subseteq \{1,2,\ldots,n\}} \delta_{0;S}$$

on $\overline{\mathcal{M}}_{g,n}$.

Our main goal in this paper is to prove, whenever possible, that $\overline{\mathcal{M}}_{g,n}$ is of general type; to do this we will need to show that $K_{\overline{\mathcal{M}}_{g,n}}$ is the sum of an ample **Q**-divisor and an effective **Q**-divisor. As it happens, there are only finitely many cases to worry about (recall our assumption that $g > 3$).

THEOREM 2.4. *All but finitely many of the* $\mathcal{M}_{g,n}$ *with* $g > 3$ *are of general type. For* $g = 2, 3$, *all but finitely many of the* $\mathcal{M}_{g,n}$ *dominate a variety of general type.*

Proof. If a variety maps dominantly to a variety of general type with fibers of general type, the variety is itself of general type. The existence of a map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ whose general fiber is a curve of genus g, together with the Harris-Mumford-Eisenbud result mentioned above, then takes care of all cases with $g > 23$. (Actually, the general fiber of this map is not a curve of genus g in the case $g = 2, n = 0$, but this is of little importance.)

Since $\overline{\mathcal{M}}_{g,n}$ is birational to the n-fold fiber product of $\overline{\mathcal{M}}_{g,1}$ with itself over $\overline{\mathcal{M}}_g$ for any $g > 3$ and any n, we can apply the Caporaso-Harris-Mazur correlation theorem [CHM, Thm. 1.3], which states that a sufficiently high fiber power of any map of varieties with "maximal variation of moduli" whose general fiber is a curve of genus 2 or greater will be of general type. (The interested reader is referred to [CHM] for a definition of maximal variation of moduli; the condition is fairly trivially satisfied here.) When $g = 2, 3$, we can only assert that $\mathcal{M}_{g,k+1} \times_{\mathcal{M}_{g,k}} \mathcal{M}_{g,k+1}$ is birational to $\mathcal{M}_{g,k+2}$ starting from some $k > 0$, and so we do not have a family with maximal variation of moduli. Thus we can only conclude that $\mathcal{M}_{g,n}$ will eventually admit a dominant map to a variety of general type. In contrast to this, it can easily be shown that $\overline{\mathcal{M}}_{g,n}$ has nonpositive Kodaira dimension for $g = 0, 1$. ∎

This method requires $\overline{\mathcal{M}}_{g,n}$ to have canonical singularities (otherwise conclusions about $K_{\overline{\mathcal{M}}_{g,n}}$ do not allow us to bound the plurigenera of its desingularizations from below). More precisely, let $\overline{\mathcal{M}}_{g,n}^0$ be the subvariety of $\overline{\mathcal{M}}_{g,n}$ consisting of pointed curves without nontrivial automorphisms. If $\overline{\mathcal{M}}_{g,n}$ has canonical singularities, then any global section of $mK_{\overline{\mathcal{M}}_{g,n}^0}$ pulls back to a global section of $mK_{\tilde{\mathcal{M}}_{g,n}}$, where $\tilde{\mathcal{M}}_{g,n}$ is any desingularization of $\overline{\mathcal{M}}_{g,n}$. Therefore, if we can prove that the pluricanonical sheaves on $\overline{\mathcal{M}}_{g,n}$ have lots of global sections, the same will follow for the desingularizations, which is what we need in order to conclude that $\overline{\mathcal{M}}_{g,n}$ is (for instance) of general type. Fortunately, the singularities are canonical in genus at least 4. To prove this, we first remark that, by the well-known local structure of $\overline{\mathcal{M}}_{g,n}$—a point corresponding to a pointed curve C has an analytic neighborhood isomorphic to $\mathbf{C}^{3g-3+n}/\mathrm{Aut}\, C$—all of the singularities of $\overline{\mathcal{M}}_{g,n}$ are finite quotient singularities. (It follows that every Weil divisor on $\overline{\mathcal{M}}_{g,n}$ is $\mathbf{Q}$-Cartier, also a fact which we will need.)

Theorem 2.5. *Let $g \geq 4, n \geq 0$. Then $\overline{\mathcal{M}}_{g,n}$ has only canonical singularities.*

Proof. We will apply the Reid–Tai–Shepherd-Barron criterion [HMu, p. 27]. Suppose that $\overline{\mathcal{M}}_{g,n}$ has a noncanonical singularity at P, and let $(C, p_1, \ldots, p_n)$ be the stable n-pointed curve which corresponds to P. The deformation space of $(C, p_1, \ldots, p_n)$ is

$$H^1(C, T_C(-\sum p_i)),$$

according to [HM, ex. 1, p. 94], and Serre duality identifies this with

$$H^0(C, \Omega^1_C \otimes \omega_C \otimes \mathcal{O}(\sum p_i)).$$

To say that P violates the criterion is to say that for some $k > 0$ and some choice ζ_k of primitive kth root of 1, there is an automorphism σ of C fixing all the p_i whose eigenvalues, with multiplicity, are $\{\{\zeta_k^{r_i}\}\}$ with $\sum r_i < k$.

Consider σ as an automorphism of C. According to [HMu, §1, thm. 2], σ satisfies the Reid–Tai–Shepherd-Barron criterion unless it acts nontrivially on a single component whose genus is 1 and which is attached to the rest of the curve at only one point. Except in this case, therefore, σ as an automorphism of the pointed curve still satisfies the criterion, because $H^0(C, \Omega^1_C \otimes \omega_C)$, the deformation space of the curve, embeds into the deformation space of the pointed curve. Thus all eigenvalues on the unpointed curve appear with at least as high a multiplicity on the pointed curve. The only thing left to mention is that contracting $\mathbf{P}^1$-components of C (as we may need to do for C to remain stable when we remove the marked points) does not alter the action of σ on $H^0(C, \Omega^1_C \otimes \omega_C)$. This can be seen by considering the three cases of removing a $\mathbf{P}^1$ attached at one point, attached at two points on the same component, and attached at two points on different components separately.

Suppose now that we are in the exceptional case described above in which σ violates the criterion. If none of the p_i is on the component of genus 1, then by arguing as in [HMu, pp. 40-44] we conclude that no singularity appears. If at least one of the p_i is on this component, the singularity will be canonical; for the restriction of the sheaf $\Omega^1_C \otimes \omega_C \otimes \mathcal{O}(\sum p_i)$ to this elliptic curve is the same as it would be if another component were attached at this p_i. Thus the computations done in that situation in [HMu, p. 36] give the desired conclusion. ∎

It is convenient to determine $K_{\overline{\mathcal{M}}_{g,n}}$ at this point.

THEOREM 2.6. *The canonical divisor of* $\overline{\mathcal{M}}_{g,n}$ *is*

$$13\lambda - 2\delta_0 + \sum_1^n \omega_i + \sum_{S\subseteq\{1,2,\ldots,n\}} (\operatorname{card} S - 2)\delta_{0;S} - 3\sum_S \delta_{1;S} - 2\sum_{i=2}^{\lfloor g/2\rfloor}\sum_S \delta_{i;S},$$

where if g *is even the* S *in the last sum with* $i = g/2$ *runs only over subsets containing* n.

I thank Sean Keel for supplying the nontrivial parts of the proof that follows. The statement must be known, but I have not been able to find it in the literature.

Proof. For technical reasons it is essential to know that $\overline{\mathcal{M}}_{g,n}$ is Cohen-Macaulay. The only singularities are finite quotient singularities, and the local rings there are generated over the base field by monomials including $x_i^{m_i}$ for each local coordinate x_i. It is easily verified that $x_1^{m_1}, \ldots, x_{3g-3+n}^{m_{3g-3+n}}$ is a regular sequence in the local rings, and the claim follows.

We proceed by induction on n. The base case

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\sum_{i=2}^{\lfloor g/2\rfloor}\delta_i$$

is known [HM, eqn. 3.113]. Essentially we would like to use the fact that $K_{\overline{\mathcal{M}}_{g,n}} = \pi_n^* K_{\overline{\mathcal{M}}_{g,n-1}} + \omega_{\pi_n}$, where π_n is the nth projection map. It is necessary to compare ω_{π_n} with ω_n.

Now, we know that relative differentials are compatible with base change [Ha, prop. 8.10], and that the dualizing sheaf can be computed from the relative differentials [KM, prop. 5.75], so we would like to understand the map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1} \otimes_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,1}$ obtained from the universal property of a fiber product. (Informally, this map takes a curve of genus g with n marked points to the same curve without the last point and the same curve with only the last point.) Manifestly, such a map is surjective. If two points of $\overline{\mathcal{M}}_{g,n}$ have the same image, it can only mean that the last marked point is on a component of genus 0 which would disappear when all the points are unmarked. The only codimension-1 loci where this happens are the $\delta_{0;S}$ with $n \in S, \operatorname{card} S \geq 3$. It follows that the relative dualizing sheaf ω_{π_n} differs from ω_n only in a linear combination of their classes. This is now enough to prove the theorem except as regards the coefficients of the $\delta_{0;S}$, because $\pi_n^*\omega_i = \omega_i, \pi_n^* \sum_S \delta_{i;S} = \pi_n^* \sum_S \delta_{i;S}$, and likewise λ and δ_0 pull back to classes of the same name.

Because of the action of $\mathcal{S}_n$ on $\overline{\mathcal{M}}_{g,n}$, the coefficient of $\delta_{0;S}$ depends only on $\operatorname{card} S$. Also, if $n \notin S$, the coefficient of $\delta_{0;S}$ in $K_{\overline{\mathcal{M}}_{g,n}}$ is equal to its coefficient in $K_{\overline{\mathcal{M}}_{g,n-1}}$, because $\delta_{0;S}$ pulls back to $\delta_{0;S} + \delta_{0;S\cup\{n\}}$ and $\delta_{0;S}$ does not appear in the pullback otherwise. Therefore, by induction, it is only necessary to show that the coefficient a_n of $\delta_{0;\{1,2,\ldots,n\}}$ in $K_{\overline{\mathcal{M}}_{g,n}}$ is $n-2$. By the above discussion, it is guaranteed for $n = 2$.

Let D be the subvariety $\delta_{0;\{1,2,\ldots,n\}}$ of $\overline{\mathcal{M}}_{g,n}$. Its points are curves of genus g attached at a point to an n-pointed curve of genus 0; thus $D \cong \overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{0,n+1}$ as an abstract variety. Let C be a curve in $\overline{\mathcal{M}}_{g,n}$ obtained by fixing a curve of genus g and a point on it and attaching an n-pointed $\mathbf{P}^1$ at a point that varies on the $\mathbf{P}^1$. By the adjunction formula (whose use is justified here by [KM, prop. 5.73]) we have

$$(K_{\overline{\mathcal{M}}_{g,n}} + D)|_D = K_D.$$

We intersect both sides with C. We find that C misses λ and all boundary components except $\delta_{0;S}$ with $\operatorname{card} S = n-1$, which C intersects once, and $\delta_{0;\{1,2,\ldots,n\}}$, which C intersects with multiplicity $2-n$. (This is the self-intersection of the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$ with n points blown up.) Recalling that $\omega_i = \psi_i - \sum_{S \ni i} \delta_{0;S}$ and noticing that the self-intersection of each section to the family is -1, we get $\omega_i \cdot C = 1-(n-1)+(2-n) = 0$. Therefore

$$(K_{\overline{\mathcal{M}}_{g,n}} + D)|_D \cdot C = na_{n-1} - (n-2)(a_n+1).$$

I claim that $K_D \cdot C = -2$. Indeed,

$$K_D = \nu_1^* K_{\overline{\mathcal{M}}_{g,1}} \times \overline{\mathcal{M}}_{0,n+1} + \overline{\mathcal{M}}_{g,1} \times \nu_2^* K_{\overline{\mathcal{M}}_{0,n+1}}$$

(where ν_1, ν_2 are the maps that express D as a product). Since C is constant in $\overline{\mathcal{M}}_{g,1}$, we conclude that $K_D \cdot C = K_{\overline{\mathcal{M}}_{0,n+1}} \cdot \nu_{2*}C$. But C is a fiber of $\pi_{n+1} : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$; the result follows, since the fibers are all curves of genus 0 and thus the canonical meets each of them -2 times.

In summary, we have shown that the sequence of a_n, the coefficients of $\delta_{0;S}$ with card $S = n$, satisfies the recurrence $na_{n-1} - (n-2)(a_n+1) = -2$ with $a_2 = 0$, whence yet another induction rapidly demonstrates that $a_n = n-2$ and completes the proof of the theorem. ∎

Corollary 2.7. *The class of the relative dualizing sheaf of the map $\pi_n : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ is*

$$\omega_n + \sum_{\substack{n \in S,\ |S| \geq 3 \\ S \subseteq \{1,2,\ldots,n\}}} \delta_{0;S}.$$

Proof. $K_{\overline{\mathcal{M}}_{g,n}} = \pi_n^* K_{\overline{\mathcal{M}}_{g,n-1}} + \omega_{\pi_n}$.

For later use, we will describe the pushforwards of intersections of divisors on $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g,n-1}$. The main tool for this is the push-pull formula, but other techniques are needed as well.

Theorem 2.8. *Let D_1, D_2 be divisor classes which are among the standard list of generators given above. Then the table below gives $\pi_{n*}(D_1 \cdot D_2)$. (In the table, $n \notin S$, $j = g-i$, and $r = 2g-2$, while an open statement followed by a quantity, such as $(i \in S)\delta_{i;S}$, means the specified quantity if the statement is true and 0 if it is false.)*

	ω_1	ω_2	$\ldots$	ω_n	λ	δ_0	$\delta_{0;\{1,n\}}$	$\delta_{0;\{2,n\}}$	$\delta_{i;S}$	$\delta_{i;S\cup\{n\}}$	$\ldots$
ω_1	0	0	$\ldots$	$r\omega_1$	0	0	ω_1	ω_1	0	0	$\ldots$
ω_2	0	0	$\ldots$	$r\omega_2$	0	0	ω_2	ω_2	0	0	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
ω_n	$r\omega_1$	$r\omega_2$	$\ldots$	see below	$r\lambda$	$r\delta_0$	ω_1	ω_2	$(2j-1)\delta_{i;S}$	$(2i-1)\delta_{i;S}$	$\ldots$
λ	0	0	$\ldots$	$r\lambda$	0	0	λ	λ	0	0	$\ldots$
δ_0	0	0	$\ldots$	$r\delta_0$	0	0	δ_0	δ_0	0	0	$\ldots$
$\delta_{0;\{1,n\}}$	ω_1	ω_2	$\ldots$	ω_1	λ	δ_0	$-\psi_1$	0	$(1 \notin S)\delta_{i;S}$	$(1 \in S)\delta_{i;S}$	$\ldots$
$\delta_{0;\{2,n\}}$	ω_1	ω_2	$\ldots$	ω_2	λ	δ_0	0	$-\psi_2$	$(2 \notin S)\delta_{i;S}$	$(2 \in S)\delta_{i;S}$	$\ldots$
$\delta_{i;S}$	0	0	$\ldots$	$(2j-1)\delta_{i;S}$	0	0	$(1 \notin S)\delta_{i;S}$	$(2 \notin S)\delta_{i;S}$	$-\delta_{i;S}$	$\delta_{i;S}$	0
$\delta_{i;S\cup\{n\}}$	0	0	$\ldots$	$(2i-1)\delta_{i;S}$	0	0	$(1 \in S)\delta_{i;S}$	$(2 \in S)\delta_{i;S}$	$\delta_{i;S}$	$-\delta_{i;S}$	0
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	0	0	$\ldots$

Proof. Different entries in this table require different techniques, many of which will be seen repeatedly in the arguments to come. Thus we will indicate (and justify, if appropriate) each method as it arises, grouping together the entries that are computed in the same way.

According to Theorem 2.3, the divisor classes $\omega_1, \ldots, \omega_{n-1}, \lambda$, and δ_0 are pullbacks of classes from $\overline{\mathcal{M}}_{g,n-1}$ with the same name. The push-pull formula states that $f_*(f^*x \cdot y) = x \cdot f_*y$, and therefore $\pi_{n*}(D \cdot D_2) = (\deg D_2)D$ when D is any of these classes, where deg means the degree on a fiber of π_n. It is clear, now, that $\delta_{\{i,n\}}$ has degree 1 on a fiber, while λ, the $\omega_i (i < n)$, and all the other boundary components have degree 0 on a fiber, since they are supported in the inverse image by π_n of a codimension-1 subvariety. Finally, ω_n has degree $2g-2$ on a fiber, since up to divisors of form $\delta_{0;S}$ with card $S \geq 3$ it is the relative dualizing sheaf of a morphism whose fibers are curves of genus g, and such divisors have degree 0 on a fiber. This enables us to fill in the rows for λ, δ_0, and the ω_i $(i < n)$.

We have $\pi_n^*\delta_{i;S} = \delta_{i;S} + \delta_{i;S\cup\{n\}}$, so the $\delta_{i;S}$ are not pulled back from the base. But like divisors that are pulled back from the base, their support is entirely contained in a set of fibers of codimension 1, except in the case $i = 0, S = \{j, n\}$, and it is uniquely determined by i and S. Therefore $\pi_{n*}(\delta_{i;S} \cdot \delta_{j;S'})$ must be a divisor supported in codimension 2 unless $i = j, S \cap \{1, 2, \ldots, n-1\} = S' \cap \{1, 2, \ldots, n-1\}$. The only such divisor, of course, is the trivial one.

More generally, if the support of D is contained in the union of the fibers over an irreducible locus L of codimension 1, then $\pi_{n*}D \cdot D_2$ must be a multiple of D_L, the reduced divisor supported on L. We will be able to determine the coefficient of D_L by intersecting both sides with a single curve. Since, for $n \notin S$, the class $\delta_{i,S} + \delta_{i;S\cup\{n\}}$ is $\pi_n^*\delta_{i,S}$, we will not have to do this for both S and $S \cup \{n\}$, as we can use the push-pull formula as before.

Our first application of this technique will compute $\pi_{n*}(\delta_{i;S}^2)$. Without loss of generality, we assume $n \notin S$ (we can do this because $\delta_{i;S} = \delta_{g-i,\bar{S}}$). I claim that

$$\pi_{n*}(\delta_{i;S}^2) = -\delta_{i;S}.$$

Since

$$\pi_{n*}(\delta_{i;S} \cdot (\delta_{i;S} + \delta_{i;S\cup\{n\}})) = 0,$$

it suffices to prove that

$$\pi_{n*}(\delta_{i;S} \cdot \delta_{i;S\cup\{n\}}) = \delta_{i;S}.$$

It must be a multiple of $\delta_{i;S}$ by the reasoning above; we only need to find the coefficient. We will use the curve C described as follows: fix $k \in S$, if possible, and fix general curves C_{g-i} and C_i of genus $g-i$ and i respectively and a general point on each at which they will be attached. For each $j \neq k \in \{1, 2, \ldots, n\}$, fix a position for p_j on C_{g-i} if $j \in S$, and on C_i if not. Then consider the curve C in $\overline{\mathcal{M}}_{g,n}$ whose base locus is isomorphic to C_{g-i}, and such that the point p of C corresponds to the stable curve with C_{g-i} and C_i attached at the given points, the kth marked point is p_k, and all the others are in their fixed assigned positions. Plainly, C has intersection number 1 with $\delta_{i;S}$ on $\overline{\mathcal{M}}_{g,n-1}$; when we pull back to $\overline{\mathcal{M}}_{g,n}$, the only point on the resulting surface contained in both $\delta_{i;S}$ and $\delta_{i;S\cup\{n\}}$ is that at which p_k is at the attachment point (so that it is blown up to live on a $\mathbf{P}^1$) and p_n is at the attachment point of that $\mathbf{P}^1$ to the component of genus $g-i$. Therefore the coefficient is 1, as claimed. (This doesn't work if S is empty, but in that case we instead use the test curve whose base is the given curve of genus i and whose fiber at the point P consists of the given curve of genus i, with $p_1 = P$, attached to a general curve of genus $g-i$ with the other points in fixed positions. Again this meets $\delta_{i;S}$ exactly once.)

Now we compute the pushforward of $\omega_n \cdot \delta_{i;S}$, except in the case $i = 0, \operatorname{card} S = 2, n \in S$ which we will deal with soon. Again, we may and will assume $n \notin S$, and again the answer is some multiple of $\delta_{i;S}$, the problem being to find the coefficient. This we do as follows: again, choose $k \in S$, fix curves of genus $g - i$ and i, and let p_k vary on the curve of genus $g - i$ while all the other points are fixed, and precisely those in S are on the genus-i component. (We have assumed S to be nonempty, but if it isn't we can proceed as in the previous paragraph.) This curve in $\overline{\mathcal{M}}_{g,n-1}$, of course, meets $\delta_{i;S}$ once. When we take the inverse image of this curve in $\overline{\mathcal{M}}_{g,n}$ and intersect with $\delta_{i;S}$, we get the curve whose base is the curve of genus $g - i$ and on which p_n varies on this curve, p_k is at the blown-up point of attachment, and all the other points are in their usual fixed positions. The intersection of this curve with ω_n being $2(g-i)-1$, we conclude that

$$\pi_{n*}(\omega_n \cdot \delta_{i;S}) = (2(g-i)-1)\delta_{i;S}$$

when $n \notin S$. Observe that

$$\pi_{n*}(\omega_n \cdot (\delta_{i;S} + \delta_{i;S\cup\{n\}})) = (2g-2)\delta_{i;S};$$

this is as it should be, because ω_n has degree $2g-2$ on fibers and $\delta_{i;S} + \delta_{i;S\cup\{n\}} = \pi_n^*\delta_{i;S}$.

It is clear that if $\operatorname{card}(\{i,n\} \cap S) = 1$, then $\delta_{0;\{i,n\}}$ cannot intersect $\delta_{0;S}$ at all, so the pushforward in this case is 0. From this, it follows that if $i, n \notin S$, then $\pi_{n*}(\delta_{0;\{i,n\}} \cdot \delta_{i;S}) = \delta_{i;S}$, while if $i, n \in S, S \neq \{i,n\}$, the pushforward is $\delta_{i;S-\{n\}}$. These both follow from the push-pull formula, in view of the facts that $\pi_n^*\delta_{i;S} = \delta_{i;S} + \delta_{i;S\cup\{n\}}$ and that $\delta_{0;\{i,n\}}$ has degree 1 on a fiber of π_n.

There are still a few entries left. The next one we will treat is $\pi_{n*}(\delta_{0;\{i,n\}}^2)$, which we show to be equal to $-\psi_i$ by proving that both have the same intersection with any curve in $\overline{\mathcal{M}}_{g,n-1}$. (All nonzero classes in $\operatorname{Pic}_{\overline{\mathcal{M}}_{g,n}}$ have nonzero intersection with some curve, so this is enough.) Take a curve in $\overline{\mathcal{M}}_{g,n-1}$, then, and pull it back to $\overline{\mathcal{M}}_{g,n}$. Intersecting once with $\delta_{0;\{i,n\}}$ gives us the curve in $\overline{\mathcal{M}}_{g,n}$ which projects to the curve we started with in $\overline{\mathcal{M}}_{g,n-1}$ and which has points i and n together on their own $\mathbf{P}^1$. Intersecting again produces the sum of the self-intersection of a constant section to a $\mathbf{P}^1$ and the ith section on the curve in $\overline{\mathcal{M}}_{g,n-1}$; that is, it gives us $-\psi_i \cdot C$. Thus

$$\pi_{n*}(\delta_{0;\{i,n\}}^2) = -\psi_i.$$

For $\pi_{n*}(\delta_{0;\{i,n\}} \cdot \omega_n)$, note that when restricted to $\delta_{0;\{i,n\}}$, the classes ω_i and ω_n become equal. Thus the pushforward in question is ω_i. This completes the verification of the accuracy of the table.

Finally, we consider $\pi_{n*}(\omega_n)^2$. Letting Ω be the relative dualizing sheaf, we have $\pi_{n*}(\Omega^2) = \kappa_1 = 12\lambda - \delta$. (cf. [HM, Section 2.D]). But as seen in Corollary 2.7, we have

$$\Omega = \omega_n + \sum_{\substack{n \in S,\ |S| \geq 3 \\ S \subseteq \{1,2,\ldots,n\}}} \delta_{0;S}.$$

It follows that

$$\pi_{n*}(\omega_n)^2 = 12\lambda - \delta - 2\sum \pi_{n*}(\omega_n \cdot \delta_{0;S}) - \pi_{n*}(\sum \delta_{0;S})^2.$$

Here, $\sum \pi_{n*}(\omega_n \cdot \delta_{0;S})$ will give $-\sum \delta_{0;S}$. All the cross terms in $\pi_{n*}(\sum \delta_{0;S})^2$ will vanish, leaving $-\sum \delta_{0;S}$. In conclusion, then, we get $\pi_{n*}(\omega_n)^2 = 12\lambda - \delta + 3\sum_S \delta_{0;S}$. $\square$

Now we turn to the problem of finding ample $\mathbf{Q}$-divisors:

THEOREM 2.9. *Let the a_i be positive, and let $b > 11$. Then the divisor*

$$\sum a_i\psi_i + b\lambda - \delta_0 - \sum_{i>0,S} \delta_{i;S}$$

is ample on $\overline{\mathcal{M}}_{g,n}$.

Proof. It suffices to show that the intersection of this divisor with any curve is bounded below by a fixed positive number. Recall ([HM, thm. 6.40]) that $b\lambda - \delta$ is ample on $\overline{\mathcal{M}}_g$. Therefore, by push-pull, its pullback takes a value greater than some fixed positive quantity q on any curve in $\overline{\mathcal{M}}_{g,n}$ which remains a family after forgetting the sections, and 0 on any other curve in $\overline{\mathcal{M}}_{g,n}$. On the other hand, it is well-known that ψ_i is nonnegative on any curve in $\overline{\mathcal{M}}_{g,n}$, and is 0 only on curves for which σ_i is a constant section to a constant curve. It is not possible for all of the σ_i to be constant sections to constant curves while the curves in the family do not vary (if so, we would not have a curve in $\overline{\mathcal{M}}_{g,n}$, but a point). The theorem follows from this, with the bound being $\min\{q, a_1, \ldots, a_n\}$. ▯

3. Auxiliary Computations. Here we collect a few auxiliary enumerative computations that don't seem to fit in anywhere else in particular. Some of these, at least, are surely known already, but we understand that in some cases the method of computation may be new, and in any event we do not know of an appropriate reference. This section relies heavily on [ACGH].

Let us start with an easy example. Recall that the dimension of the space of g^1_d's on a general curve of genus g is $2d - g - 2$.

THEOREM 3.1. *Let C be a general curve of genus g, let P_1, P_2 be general points on it, and let $B(g, m, n) = B'(g, d, n)$ be the number of g^1_d's on C which are ramified at P_1 to order $m - 1$ and at P_2 to order $n - 1$, where $2d = g + m + n$ so that $B(g, m, n)$ is positive and finite. Then $B'(g, d, n) = \binom{g}{d-n} - \binom{g}{d}$.*

Before proving this formula, we remark that special cases are well-known: for example, if $m = n = 1, d = (g + 2)/2$, we recover the fact, which goes back to Castelnuovo, that a general curve of genus g carries $c_{g/2} = g!/(g/2)!(g/2 + 1)!$ $g^1_{(g+2)/2}$'s. Also, it is clear from the definition that $B(g, m, n) = B(g, n, m)$ and $B'(g, d, n) = B'(g, d, 2d - g - n)$. It is easily verified that the formula asserted is consistent with this.

To prove the theorem, we use the Excess Intersection Formula on $\mathrm{Pic}^d(C)$. Let V_1 be the locus on $\mathrm{Pic}^d(C)$ consisting of divisors of the form

$$mP_1 + \sum_1^{d-m} x_i,$$

and let V_2 be the locus of divisors of the form

$$nP_2 + \sum_1^{d-n} y_j.$$

A point contained in the intersection of these loci must correspond to a g^1_d of the type we are looking for, unless it corresponds simply to a divisor of the form

$$mP_1 + nP_2 + \sum_1^{d-m-n} z_k.$$

By Poincaré's formula [ACGH I.§5], we have loci whose classes are $\theta^{g-d+m}/(d-m)!$ and $\theta^{g-d+n}/(d-n)!$; their intersection number is $\binom{g}{d-m} = \binom{g}{d-n}$. But there is an excess intersection of $\theta^d/(g-d)!$, whose contribution we must determine and account for. Let C_i be the ith symmetric product of C, and W_i its image in $\operatorname{Pic} C$. (We are only computing topologically rather than in the Chow ring of one of the components of the Picard variety, so it is unnecessary to specify the degree.)

According to Excess Intersection, now, we may define a map

$$E = \mathcal{O}_{C_{g-d}} + \mathcal{N}_{C_{g-d}/\operatorname{Pic} C} \to F = \mathcal{N}_{C_{d-m}/\operatorname{Pic} C}\big|_{C_{g-d}} + \mathcal{N}_{C_{d-n}/\operatorname{Pic} C}\big|_{C_{g-d}},$$

where C_i means the ith symmetric product of C. The degeneracy locus of this map, according to Porteous' formula, is the degree of the $g-d$-th Chern class of $F-E$. This degree will be the contribution to the intersection number $\binom{g}{d-m}$ that we must remove to get the correct answer to our enumerative problem.

The computation is rather easier than it might be, because the tangent sheaf to an abelian variety is free, and so

$$c_t(\mathcal{N}_{C_k/\operatorname{Pic} C})c_t(C_k) = 1.$$

According to [ACGH, VII.5.4],

$$c_t(C_k) = (1+tx)^{k-g+1}e^{-t\theta/(1+tx)},$$

where θ is the pullback of the Θ-divisor from J and x is the class of C_{k-1} in C_k. We can afford to be careless about which symmetric product these live in, as under natural inclusions $C_k \to C_{k+l}$, θ pulls back to θ and x to x.

Since of course $c_t(\mathcal{O}_{C_{g-d}}) = 1$, this means that

$$\begin{aligned} c_t(F-E) &= (1+tx)^{1-d+(g-d+m-1)+(g-d+n+1)}e^{(1+1-1)t\theta/(1+tx)} \\ &= (1+tx)^{g-d-1}e^{t\theta/(1+tx)}. \end{aligned}$$

Expanding the exponential in a power series, we get

$$\sum_{l=0}^{\infty}(1+tx)^{g-d-1-l}\frac{(t\theta)^l}{l!}.$$

But actually the sum may be cut off at $l = g-d$, because these computations are taking place in the Chow ring of C_{g-d}. We are only looking for the $g-d$-th Chern class, which is the coefficient of t^{g-d}. Except for the last term $l = g-d$, all of the others are polynomials in t of degree $g-d-1$, so they do not contribute. When $l = g-d$, though, we find $\theta^{g-d}/(g-d)!$, multiplied by the constant coefficient of $(1+tx)^{-1}$, which is 1. So we must subtract off $\theta^{g-d}/(g-d)!$, where here θ is the pullback of the θ-divisor to C_{g-d}. Applying push-pull for the map $C_{g-d} \to J$, and letting θ now denote the usual θ-divisor on the Jacobian, we find that

$$\begin{aligned} &\frac{\theta^{g-d}}{(g-d)!} \text{ on } C_{g-d} \\ =&\frac{\theta^{g-d}}{(g-d)!}\cdot\frac{\theta^d}{d!} \text{ on } \operatorname{Jac} C \\ =&\binom{g}{d}. \end{aligned}$$

This gives us our claimed answer of $\binom{g}{d-n} - \binom{g}{d}$. □

A similar, but somewhat more complicated, problem is to determine the function $A(g,m,n)$ which counts g^1_d's on a general curve of genus g vanishing to order m at a specified point and order n at an unspecified point, where now $2d = g+m+n-1$. For notational consistency, let $A'(g,d,n) = A(g,m,n)$. This could be approached using a similar Excess Intersection setup, but now instead of one excess intersection locus there are two—one as before, and one where the unspecified point coincides with the specified one—and these excess intersection loci intersect each other. This seems excessively complicated, and we therefore seek another way to proceed. The Schubert calculus suggests itself.

THEOREM 3.2. *Let $A(g,m,n) = A'(g,d,n)$ be the number of g^1_d's on a general curve of genus g ramified to order $m-1$ at a specified point and order $n-1$ at an unspecified point, where $2d = g+m+n-1$. Then*

$$A'(g,d,n) = g!(n^2-1) \sum_{j=\max(0,m+n-d-1)}^{\min(m-1,n-1,d)} \frac{(m+n-2j-1)}{((d-m-n+j+1)!(d-j)!)}.$$

Again we observe that this formula is well-known in special cases before giving the proof. For example, $A'(g,g,g) = A(g,1,g)$ counts the Weierstrass points on a curve of genus g, and the formula gives

$$g!(g^2-1)g/(0!g!) = g^3 - g,$$

as it must. If g is even, then $A(g,1,2)$ should be the number of $g^1_{(g+2)/2}$'s on a curve of genus g multiplied by the number of ramification points of each, so $6\binom{g}{(g-2)/2}$. One readily confirms that this is predicted by the formula.

Instead of using a general smooth curve to prove the theorem, we will use a *comb curve*, which is a stable curve consisting of one component (the "spine" of the comb) of genus 0, with g elliptic tails (the "teeth") attached at general points. The specified point will be on the spine. It is fairly easy to compute on such curves, and they turn out to be general with respect to most of the properties that we are interested in. As these curves are reducible, we must use the theory of limit linear series. For an introduction to this theory, see [HM, Chapter 5].

If we consider the aspect of the g^1_d on the spine, we see that it must have a ramification point at each of the g points of attachment, so if it is to vanish to order m at another point of the spine and order n at another, there are $g+m+n-2 = 2d-1$ ramification points on the spine, which is more than are allowed. Therefore the unspecified point must be on a tooth. We claim that it must differ from the point of attachment by n-torsion. Indeed, if it does not, then the vanishing sequence at the point of attachment on the elliptic tail is at most $\langle d-n-1, d\rangle$, so it is at least $\langle 0, n+1\rangle$ on the spine, again giving us $2d-1$ ramification points there. Therefore, the vanishing sequence on the elliptic tail must in fact be $\langle d-n, d\rangle$, and it is $\langle 0, n\rangle$ on the spine, while any g^1_d with the right ramifications on the spine together with an n-torsion point on the appropriate tail must correspond to a unique limit g^1_d on the curve as a whole. Since we can choose any of g tails and any of n^2-1 points of order n, we must compute the number of g^1_d's on a curve of genus 0 which vanish to order m at one point, vanish to order n at another, and ramify simply at other prescribed points, and then multiply the answer we get by $g(n^2-1)$.

This can be done by standard methods. If we embed the rational curve as a rational normal curve in $\mathbf{P}^d$, a g^1_d is given by fixing a $d-2$-plane and considering the intersections with the curve of $d-1$-planes containing it. The condition that the g^1_d ramify to order r at a point is precisely the condition that one of the $d-1$-planes contain a given r-plane, in other words, that the fixed $d-2$-plane intersect a given r-plane in an $r-1$-plane. This is the Schubert cycle

$$\sigma_{\underbrace{1,1,\ldots,1}_{m}}.$$

For convenience, we will compute the intersections, not of these cycles in $\mathbf{G}(d-2,d)$, but of their duals in $\mathbf{G}(1,d)$. The dual of the cycle

$$\sigma_{\underbrace{1,1,\ldots,1}_{m}}$$

is σ_m.

We have ramification of order m once, order n once, and order 1 a number of other times, so when we dualize we get the Schubert cycles $(d-m-1,d)=\sigma_m, (d-n-1,d)=\sigma_n$, and copies of σ_1. Thus we must find the degree of the Schubert cycle $\sigma_m \cdot \sigma_n$ on $\mathbf{G}(1,d)$. Our answer can be extracted from Example 14.7.11 (ii) of [F].

Without loss of generality, let $m \geq n$. Then

$$\begin{aligned}\sigma_m \cdot \sigma_n &= \sum_{0}^{n} \sigma_{m+j,n-j} \\ &= \sum (d-m-j-1, d-n+j).\end{aligned}$$

According to Fulton's formula,

$$\deg(a_0,a_1) = (a_0+a_1-1)!\,(a_1-a_0)/(a_0!\,a_1!).$$

Translating this formula into our notation and cutting off the values of j that do not produce actual Schubert cycles gives our result. Note that $a_0+a_1-1 = 2d-m-n-2 = g-1$, so $g(a_0+a_1-1)! = g!$. ∎

The following proposition will be very important in section 5.

PROPOSITION 3.3. *Suppose that $\sum_{i=1}^{n} a_i = g$, and let p_1 through p_{n-1} be fixed general points on a general curve C of genus g. Then the number of points p_n such that the divisor $\sum a_i p_i$ moves on C is $a_n(ga_n-1)$.*

Proof. This is a calculation in intersection theory. We need to find the number of divisors of the form $\sum a_i p_i$ which move, where p_n is allowed to vary and the other p_i are not. A first approach to this number is obtained by choosing q to be a point different from all the p_i, and intersecting the curve given by $\sum a_i p_i$ with the codimension-1 subvariety of the Jacobian consisting of $q+\sum_{j=1}^{g-1} r_j$. (A divisor is linearly equivalent—thus, equal in the Jacobian—to some divisor containing any fixed point iff it moves in a pencil.) The class of the first curve is $\Theta^{g-1}a_n^2/(g-1)!$, while the divisor is just the Θ-divisor. (As we are only concerned with the number of intersections, it is unnecessary to worry about how we identify the Picard varieties in different degrees.) Since $\Theta^g = g!$, the intersection number is $a_n^2 g$. However, the choice $p_n = q$ contributes a_n to this number, and the actual answer is therefore $a_n(ga_n-1)$. We remark that an essentially identical proof has been given by Diaz [D, proof of lemma 6.2]. ∎

For completeness (and for its rather smaller importance later) we state and prove a generalization of this formula.

PROPOSITION 3.4. *Let C be a general curve of genus g, let p_1 be a general point on it, and let $a_1 + a_2 = g - 2h$. The number of points on C such that some divisor of the form $\sum a_i p_i + \sum_{j=1}^{h} q_j$ moves in a g^1_{g-h} on C is given by the following recursive formula:*

$$d(g, a_1, 1) = \binom{g}{h+1} - \binom{g}{h},$$
$$d(g, g-a_2, a_2) = a_2(a_2 g - 1),$$

and

$$d(g, a_1, a_2) = a_2^2\left(\binom{g-1}{h} - \binom{g-1}{h-1}\right) + d(g-1, a_1+1, a_2) + \begin{cases} A(g-1, 1, a_2), & \text{if } a_1 = 1; \\ d(g-1, a_1-1, a_2), & \text{otherwise.} \end{cases}$$

Proof. The first base case is a specialization of Theorem 3.1 above, and the second is Proposition 3.3. For the recursion, consider a degeneration of the curve of genus g to a reducible curve consisting of two component: E, a curve of genus 1 on which p_1 lies, and a curve of genus $g - 1$. No more than one of the points p_2, q_j may lie on E, because then they could move in a 1-parameter family, while a dimension count would show in any event that there is no possibility for the aspect on the genus-$g - 1$ component. Thus, there are three possibilities, corresponding precisely to the three terms in the formula above:

1. p_2 goes to E, the q_j to the component of genus $g-1$. In this case, the vanishing sequence of the aspect on the genus-1 component can be at most $\langle g-h-a_1-a_2, g-h\rangle$, with equality for a_2^2 choices of p_2. In this case the vanishing sequence on the genus-$g-1$ component must be $\langle 0, a_1 + a_2\rangle$, so we must find a g^1_{g-h} on a curve of genus $g - 1$ containing one fixed divisor of degree h and another fixed disjoint divisor of degree $g - 2h$. By Theorem 3.1, there are $\binom{g-1}{h} - \binom{g-1}{h-1}$ of these.
2. q_1 is on E, and all the other points on the component of genus $g - 1$. In this case we must again have $a_1 p_1 + q_1 = (a_1 + 1)n$ on E, where n is the point of attachment, and this uniquely determines q_1. Then the problem is to find a g^1_{g-h} on the component of genus $g-1$ which vanishes to order a_1+1 at the point of attachment and a_2 somewhere else (at p_2). The number of these is $d(g-1, a_1+1, a_2)$.
3. p_2 and all the q_i are on the component of genus $g - 1$. The vanishing sequence on E is at most $\langle g-h-a_1, g-h-1\rangle$ (since this aspect must contain a divisor of form $a_1 p_1 + (g - h - a_1)n$), so on the component of genus $g - 1$ it is at least $\langle 1, a_1\rangle$, and this is uniquely achieved if $a_1 > 1$. In this case, then, we need a g^1_{g-h-1} on the component of genus $g - 1$ which includes a divisor containing $(a_1 - 1)p_1 + a_2 p_2$, and by definition there are $d(g-1, a_1-1, a_2)$ of these. If $a_1 = 1$, though, the vanishing sequence on the component of genus $g - 1$ will have to be $\langle 1, 2\rangle$, so we are looking for a g^1_{g-h-1} there which vanishes to order a_2 at some unspecified point. The number of these is $A(g-1, 1, a_2)$. ∎

Finally, we remark that because an intersection-theoretic calculation like the one in the previous theorem can be set up, the number would be the same if the divisor $a_1 p_1$ were replaced by any fixed effective divisor of the same degree.

4. The First Case. In this section we will explain our attempts to prove that $\overline{\mathcal{M}}_{g,1}$ is of general type for suitable g. The contents of this section are essential for our subsequent work on $\overline{\mathcal{M}}_{g,n}$.

In order to show that $\overline{\mathcal{M}}_{g,1}$ is of general type, we must show that the canonical divisor is the sum of an effective and an ample divisor. We have seen that the canonical class on $\overline{\mathcal{M}}_{g,1}$ is

$$\omega + 13\lambda - 2\sum_{i=0}^{g-1} \delta_i - \delta_1 - \delta_{g-1}.$$

One way to get an effective divisor on $\overline{\mathcal{M}}_{g,1}$ is simply to pull one back from $\overline{\mathcal{M}}_g$. Of course, this is unsatisfactory, since such divisors will only allow us to prove that $\overline{\mathcal{M}}_{g,1}$ is of general type when $\overline{\mathcal{M}}_g$ is. We will, however, need the best known effective divisor classes on $\overline{\mathcal{M}}_g$:

Definition 4.1. Let BN be the divisor class

$$(g+3)\lambda - \frac{(g+1)}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i$$

on $\overline{\mathcal{M}}_g$.

If $g+1$ is not prime, then BN is a positive multiple of the class of an effective divisor, namely that of the union of the codimension-1 components of the locus of curves admitting a g^r_d, where $(r+1)(g-d+r) = g+1$ [EH, Thm. 1] (the condition on r, d, and g ensures that there are such components). We remark on the striking fact that the truth of this statement is independent of r and d, as long as they satisfy the condition. (For a more general version of this result, see [L1] or [L2].) For $g+1$ is prime, it is not known whether this divisor class is effective, but we will still use the notation BN in this case. The divisor class BN is best in the sense that it minimizes the ratio of $-l/d_0$, called the *slope*, over known effective divisors with negative coefficient of δ_0. Here l (resp. d_0) is the coefficient of λ (resp. δ_0). This is what we need because we know that δ_0 is effective, so we can always increase this ratio as needed, but we must find divisors of small slope in order to express the canonical divisor usefully. For further discussion of this point, see [HM, Sect. 6.F], but note that their statement of the Slope Conjecture is slightly incorrect, as it ignores the possibility of adding multiples of δ_0 to obtain divisors of negative slope.

It is also necessary, though, to use divisors with positive degree on a fiber. Perhaps the most obvious such divisor is the Weierstrass divisor, that is, the locus of Weierstrass points. Its class was computed by Cukierman [Cuk, Thm. 2.0.12 and following remark]. His result is as follows:

Theorem 4.2. *Let $g \geq 2$. The class on $\overline{\mathcal{M}}_{g,1}$ of the Weierstrass divisor $\mathcal{W}$, which is the closure of the divisor on $\mathcal{M}_{g,1}$ given by Weierstrass points of smooth curves, is*

$$\frac{g(g+1)}{2}\omega - \lambda - \sum_{i=1}^{g-1} \frac{(g-i)(g-i+1)}{2}\delta_i.$$

Proof. See [Cuk]. Alternatively, this can be done by the method of test curves. ∎

Using this theorem, it is straightforward to prove that $\overline{\mathcal{M}}_{23,1}$ is of general type. Indeed, since 24 is not prime, the Brill-Noether class

$$26\lambda - 4\delta_0 - \sum_i i(24-i)\delta_i$$

is effective. Consider

$$\begin{aligned} K &- \frac{1}{2}BN - \frac{2}{g(g+1)-\varepsilon}\mathcal{W} \\ &= \frac{2}{g(g+1)} - \varepsilon\lambda + \frac{2\varepsilon}{g(g+1)}\omega + \text{boundary components}, \end{aligned}$$

where ε is a small positive number. According to Theorem 2.9, this is the sum of an ample and an effective divisor, and the result follows.

Lamentably, this exhausts the utility of $\mathcal{W}$ for the present purposes:

PROPOSITION 4.3. *Let $g < 23$. Then the canonical divisor of $\overline{\mathcal{M}}_{g,1}$ is not a positive linear combination of $BN, \mathcal{W}$, boundary components, and divisors shown to be ample by Theorem 2.9.*

Proof. It suffices, in fact, to consider coefficients of λ, ω, and δ_0—equivalently, to consider $\mathrm{Pic}(\mathcal{M}_{g,1} \cup \delta_0) \otimes \mathbf{Q}$. The space spanned by these is also spanned by $BN, \mathcal{W}$, and δ_0, and it is easily checked that

$$13\lambda + \omega - 2\delta_0 = \frac{13g(g+1)+2}{g(g+1)(g+3)}BN + \frac{2}{g(g+1)}\mathcal{W} + (\frac{13g^2+13g+2}{6g^2+18g} - 2)\delta_0.$$

If $g < 23$, then $(13g^2 + 13g + 2)/(6g^2 + 18g) < 2$, which is bad news. Obviously the other boundary coefficients are irrelevant, and divisors shown to be ample by Theorem 2.9 cannot help either, as on this subvariety they are of the form $a\lambda - \delta_0$ for $a > 11$, and are therefore positive linear combinations of BN and δ_0. ∎

In fact I do not know how to prove that $\overline{\mathcal{M}}_{g,1}$ is of general type for any $g < 23$. The following results are proved mainly to be used in the next section.

DEFINITION 4.4. $D_{g;g-2h} = D_{g;a}$ is the divisor of points on curves of genus g at which some g^1_{g-h} vanishes to order $g - 2h\,(> 1)$.

THEOREM 4.5.

$$D_{g;a} = \frac{(a^3-a)}{(g^3-g)}\binom{g}{(g-a)/2}\mathcal{W} + \frac{6a}{(g+1)(g-2)}\binom{g-2}{(g-a-2)/2}BN.$$

Proof. That $D_{g;a}$ is a linear combination of $\mathcal{W}$ and BN is a very special case of the results of [L2]. Let $D_{g;a} = x\mathcal{W} + yBN$. We start by determining x. The degree of $D_{g;a}$ on a fiber is $A(g,1,a)$, while that of $\mathcal{W}$ is $g^3 - g$. Thus the coefficient of $\mathcal{W}$ is $A(g,1,a)/(g^3 - g)$. To confirm that

$$A(g,1,a) = g!\frac{a^3-a}{((g-a)/2)!((g+a)/2)!}$$

is a simple matter of expanding the expression in Theorem 3.2.

Next we consider the coefficient of BN. Consider a family of curves C whose base is a fixed general curve of genus g, and whose fiber at a point p is that curve attached at the point p to an elliptic curve, with the marked point at a fixed point of the elliptic curve not differing by torsion from the point of attachment. This curve meets δ_1 with multiplicity $4 - 2g$ and none of the other standard generators. Since the coefficient of δ_1 in $\mathcal{W}$ is $-g(g-1)/2$, and in BN is $-(g-1)$, the intersection with the right side is $xg(g-1)(g-2) + y(g-1)(2g-4)$. We apply the theory of limit linear series to find its intersections with the left side.

Consider a limit g^1_{g-h} on a fiber of C which vanishes to order a at p_1. Its vanishing sequence at the point of attachment on the elliptic curve must, by a dimension count, be either $(g-a)/2, (g+a-2)/2$ or $(g-a-2)/2, (g+a)/2$, giving respectively $1, a$ or $0, a+1$ on the component of genus $g-1$. Thus the intersection $C \cdot D_{g;a}$ is equal to

$$\begin{aligned} &A(g-1,1,a-1) + A(g-1,1,a+1) \\ &= (a^3 - 3a^2 + 2a)\binom{g-1}{(g-a)/2} + (a^3 + 3a^2 + 2a)\binom{g-1}{(g-a-2)/2}. \end{aligned}$$

Solving this equation for y, we get

$$y = \frac{\frac{a(a-1)(a-2)(g-1)!}{((g-a)/2)!((g+a-2)/2)!} + \frac{a(a+1)(a+2)(g-1)!}{((g-a-2)/2)!((g+a)/2)!} - \frac{a(a+1)(a-1)(g-2)g!}{(g+1)((g-a)/2)!((g+a)/2)!}}{2(g-2)(g-1)},$$

which simplifies to the asserted formula. ☐

5. The Main Theorem. The purpose of this section is to describe and prove our main results about $\overline{\mathcal{M}}_{g,n}$, which are indicated in the table and theorem below. See the next section for slight improvements.

Theorem 5.1. *$\overline{\mathcal{M}}_{g,n}$ is of general type (resp. positive Kodaira dimension, resp. nonnegative Kodaira dimension) as indicated by this table:*

Genus	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$n : \kappa \geq 0$	16*	15*	16*	14*	14	13	14*	11*	13	11	11	10*	12	9	11	8	7	5	8
$\overline{\mathcal{M}}_{g,n}$ gen. type	17	16	17	15	14	13	15	12	13	11	11	11	12	9	11	8	7	6	8

(Note that an asterisk indicates Kodaira dimension ≥ 1 for the relevant variety, but is of course omitted when the next line shows that it is of general type.)

The proof of this theorem will unfold gradually over the next few pages. We start by defining some divisors.

Definition 5.2. Let $g > 1$ as usual, and let $a_1, a_2, \ldots, a_n$ be nonnegative integers such that

$$\sum_{i=1}^{n} a_i = g$$

The divisor $D_{g;a_1,a_2,\ldots,a_n}$ is defined on a suitable open subset as the locus of curves C with n points $p_1, \ldots, p_n$ such that there is a g^1_g on C containing some divisor of the form

$$\sum_{i=1}^{n} a_i p_i,$$

where the q_i are unspecified points on the curve. It is then defined on all of $\overline{\mathcal{M}}_{g,n}$ by taking the union of the codimension-1 components in the closure.

This definition generalizes certain familiar ones. For example, $D_{g;g}$ is the Weierstrass divisor on $\overline{\mathcal{M}}_{g,1}$. The divisor $D_{3;1,1,1}$ is the divisor which when restricted to the locus of plane quartics in $\overline{\mathcal{M}}_3$ picks out the sets of 3 collinear points.

PROPOSITION 5.3. $\pi_{n*}D_{g;a_1,a_2,\ldots,a_n}\cdot\delta_{0;\{n-1,n\}} = D_{g;a_1,a_2,\ldots,a_{n-2},a_{n-1}+a_n}$.

As usual, the restriction to $\delta_{0;\{n-1,n\}}$ is made for notational simplicity only; it is straightforward to use some other $\delta_{0;\{i,j\}}$, in view of the $\mathcal{S}_n$-action on $\overline{\mathcal{M}}_{g,n}$.

Proof. Consider a reducible curve with one component of genus 0, containing marked points p_{n-1} and p_n, and attached to a curve of genus g which contains all the others. Now consider a limit g^1_g on the reducible curve which has the given property. Recall that this means that there is a limit g^1_g on this curve whose aspect on the $\mathbf{P}^1$ includes a divisor containing $a_{n-1}p_{n-1}+a_np_n$, whose aspect on the curve of genus g includes a divisor containing $\sum_{i=1}^{n-2}a_ip_i$, and such that the sum of the multiplicities of these divisors at the point of attachment is at least g.

If the aspect on the $\mathbf{P}^1$ includes the divisor mentioned above, its vanishing sequence at the attachment point cannot exceed $\langle g-a_{n-1}-a_n, g\rangle$. Therefore, the vanishing sequence at this point on the other aspect must be at least $\langle 0, a_{n-1}+a_n\rangle$. A dimension count shows that, in general, the vanishing sequence cannot exceed this either. Therefore, in a general situation, the aspect on the $\mathbf{P}^1$ can only be the unique linear series containing the divisors

$$gp^0_a$$

and

$$a_{n-1}p_{n-1}+a_np_n+(g-a_{n-1}-a_n)p^0_a,$$

where p^0_a is the point of attachment, and the linear series on the genus-g part must be one which contains a divisor

$$\sum_{i=1}^{n-2}a_ip_i+(a_{n-1}+a_np^g_a),$$

where p^g_a is the point of attachment on this component. Conversely, it is very easy to go in the other direction. Thus the intersections of the two divisors named in the theorem with any curve must be equal, and therefore the divisors themselves must be. (By abuse of language, we are allowing ourselves to talk about the "aspect" on a possibly-reducible "component"; we mean, of course, the limit linear series on that union of components.) ▯

It is now possible to determine the most important coefficients of the $D_{g;a_1,\ldots,a_n}$.

THEOREM 5.4. *With respect to the standard basis of* $\mathrm{Pic}_{fun}\overline{\mathcal{M}}_{g,n}$, *the coefficient of* ω_i *in* $D_{g;a_1,\ldots,a_n}$ *is* $a_i(a_i+1)/2$, *and the coefficient of* $\delta_{0;\{i,j\}}$ *is* $-a_ia_j$. *Also, the coefficient of* λ *is* -1, *and that of* δ_0 *is* 0.

Before beginning the proof, we observe that if $a_i=0$, then the coefficients of ω_i and $\delta_{0;\{i,j\}}$ are all 0. This is as it should be, because in this case D is the pullback by π_i of the divisor

$$D_{g;a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n}.$$

Proof. Induction on n. The base case $n=1$ is part of Cukierman's theorem [Cuk] (our Theorem 4.2). For the inductive step, we show that if this formula is correct for some $D_{g;a_1,\ldots,a_{n-1}+a_n}$, then it gives the correct coefficients for ω_{n-1},ω_n, and $\delta_{0;\{n-1,n\}}$. Of course this is enough, by virtue of the action of $\mathcal{S}_n$, which acts on the D's by permuting the a_i's.

Now, for $n=2$, we have three equations for the three coefficients of ω_1,ω_2, and $\delta_{0;\{1,2\}}$, obtained by considering the pushforward of

$$D_{g;a_1,a_2}\cdot\delta_{0;\{1,2\}}$$

to $\overline{\mathcal{M}}_{g,1}$ and the degrees on the fibers with respect to π_1 and π_2. It is a simple matter to solve them to obtain the result claimed in this case.

For the general case, again consider the effect of pushing forward the intersection of the divisor $D_{g;a_1,\ldots,a_n}$ with $\delta_{0;\{n-1,n\}}$ by π_n. This preserves the coefficient of ω_1, which proves our claim for ω_1, and, by symmetry, for all the ω. As for the $\delta_{0;\{i,j\}}$, I assert first that the coefficient $-a_ia_j$ is consistent with the degree on a fiber claimed. To see this, note that the degree on a fiber of π_i is $2g-2$ times the coefficient of ω_i plus the sum of the coefficients of $\delta_{0;\{i,j\}}$, which would thus be

$$\begin{aligned}(g-1)(a_i+1)(a_i) - \sum_{j\neq i} a_ia_j &= a_i(ga_i + g - 1 - \sum a_i)\\ &= a_i(ga_i - 1)\end{aligned}$$

as desired.

This does not prove that these are the correct coefficients. However, it is now easy for us to do so in the case $n = 3$, for if we modified them by adding $d_{1,2}, d_{1,3}$, and $d_{2,3}$ to them, in order to get the correct degrees we would need $d_{i,j} = -d_{k,l}$ whenever the sets $\{i,j\}$ and $\{k,l\}$ were not equal. Plainly this requires all the d's to be 0. Then if $n > 3$, the result follows, because the coefficient of $\delta_{0;\{1,2\}}$ in $D_{g;a_1,\ldots,a_{n-1}+a_n}$ is equal to that in $\pi_{n*}D_{g;a_1,\ldots,a_n}\cdot\delta_{0;\{n-1,n\}}$, which is $-a_1a_2$ by induction. Once again, the full claim follows by symmetry.

Finally, the claim about the coefficients of δ_0 and λ is immediate by induction, since when we cut with $\delta_{0;\{i,i+1\}}$ and push forward, the only sources for coefficients of δ_0 and λ in the new divisor are these coefficients in the old one, and $\mathcal{W}$ does have coefficient -1 of λ, and 0 of δ_0. ∎

In addition, we need to know that the coefficients of the other boundary divisors with respect to the basis of $\mathrm{Pic}_{fun}\overline{\mathcal{M}}_{g,n}\otimes\mathbf{Q}$ that uses the ψ in place of the ω in these divisors are nonpositive. We introduce the notation D_g^1 for $D_{g;1,1,\ldots,1}$ (with g ones). This divisor being $\mathcal{S}_n$-invariant, its coefficient of $\delta_{i;S}$ depends only on i and $\operatorname{card} S$. We will call this coefficient $d_{i,c}^g$, where $c = \operatorname{card} S$. When appropriate, the g will be suppressed.

THEOREM 5.5. $d_{0,c} = -c(c-1)/2$.

Proof. For $c = 2$, this is proved just above. In general, we proceed by induction on c. Rather abusively, we let $p(D) = \pi_{1*}D\cdot\delta_{0;\{1,2\}}$ when D is a divisor on any $\overline{\mathcal{M}}_{g,n}(n\geq 2)$. We will determine the $\delta_{0;\{1,2\}}$-coefficients of $D, p(D), p(p(D)), \ldots$ in terms of the coefficients of D. Considering the table of pushforwards (Theorem 2.8), it is clear that the only important coefficients of D are those of $\omega_1, \delta_{0;\{1,2\}}, \ldots, \delta_{0;\{1,2,\ldots,n\}}$. Let these be $w_1, d_2, \ldots, d_n$ respectively, and let the coefficient of $\delta_{0;\{1,2\}}$ in $p^k(D)$ be d_2^p, so that, for instance, $d_2^1 = d_2$. Then, from the table of pushforwards, it follows that the coefficients of $\omega_1, \delta_{0;\{1,2\}}, \ldots, \delta_{0;\{1,2,\ldots,n\}}$ in $p(D)$ are $w_1 - d_2, d_3 - d_2, d_4 - d_2, \ldots, -d_2$. In general, we see by induction that the coefficients in $p^k(D)$ are $w_1 - d_{k-1}, d_{k+2} - d_{k+1}, d_{k+3} - d_{k+1}, \ldots$, at least up to the coefficient of $\delta_{0;\{1,2,\ldots,n-k\}}$, and thus that up to $p^{n-2}(D)$ the coefficient of $\delta_{0;\{1,2\}}$ is $d_{k+2} - d_{k+1}$.

Apply this result to the divisor D_g^1 on $\overline{\mathcal{M}}_{g,g}$. We know that

$$p^k(D) = D_{g;k+1,1,\ldots,1}$$

by Proposition 5.3, and therefore that its coefficient of $\delta_{0;\{1,2\}}$ is $k+1$ (using Theorem 5.4). This tells us that $d_{0,c+1} - d_{0,c} = c$ for $c = 2, 3, \ldots, n-1$. The theorem now follows. ∎

Now we determine the values of the other $d_{i,c}$ below. Note in particular that they are all nonpositive.

PROPOSITION 5.6. *Let* $i \in \{1, 2, \ldots, \lfloor g/2 \rfloor\}$. *Then* $d_{i,c} = |c-i|\,(|c-i|+1)/2$.

Proof. We start by proving that $d^{2i}_{i,c} = |c-i|\,(|c-i|+1)/2$ for all i, c. Of course $d^{2i}_{i,c} = d^{2i}_{i,2i-c}$, because these are the coefficients of one and the same divisor, so we only need to prove the claim for $c \geq i$. To start with $c = 2i$, consider the test curve C whose base locus is a general curve of genus i, and whose fiber at a point consists of a constant general $2i$-pointed curve of genus i attached to a curve isomorphic to the base at the given point. In view of the generality of the $2i$-pointed curve, a g^1_{2i} on it containing the divisor which is the sum of the marked points can vanish to order at most i, and only one such g^1_{2i} vanishes to order i. Thus limit g^1_{2i}'s on the whole curve with this as an aspect correspond to g^1_i's on the component isomorphic to the base which are totally ramified at the point of attachment. Of course, there are $i^3 - i$ of these. But the curve in question meets $\delta_{i,\{1,2,\ldots,n\}}$ $2-2i$ times, and none of the other generators of $\mathrm{Pic}\,\overline{\mathcal{M}}_{g,n} \otimes \mathbf{Q}$. This shows that $d^{2i}_{i,2i} = i(i+1)/2$, in accordance with the formula above.

Now we proceed by descending induction on c. Fix general c-pointed and $2i-c$-pointed curves of genus i, and consider a test curve whose base is the second of these and whose fiber at a point attaches the first at a general point to the second at that point. This curve meets $2i-c$ of the ω's once each, $2i-c$ of the $\delta_{i,c+1}$'s, and intersects a $\delta_{i,c}$ with degree $4i-c-2$. I claim that it intersects D^1_g in $(c-i)(i(c-i)-1)$ points. To see this, consider the aspect of a g^1_{2i} on the c-pointed component. It must include $\sum p_j + (2i-c)n$, where p_j are the marked points and n is the point of intersection. As the p_j are general points, the maximal possible order of vanishing at n is i, and this is only possible with one g^1_{2i}. (Actually the situation is even worse when $i = c$, but this is bad enough.) Thus we need a g^1_{2i} on the other side whose vanishing sequence is at least i, c and which includes a divisor of the form $\sum q_j + cn$, where now n is the point of attachment on this side and the q_j are the marked points. Removing the i base points, we see that we have reduced to an already solved problem: given $2i-c$ points on a curve of genus i, how many points are there such that $(i-c)$ times that point plus the sum of the others moves? The answer, as shown in Proposition 3.3, is $(c-i)(i(c-i)-1)$.

Combining all of these facts, we see that

$$2i - c - \frac{(2i-c)(c-i+1)(c-i+2)}{2} + (4i-c-2)d_{i,c} = (c-i)(i(c-i)-1).$$

One checks that $d_{i,c} = (c-i)(c-i+1)/2$, as claimed.

This handles the case $g = 2i$; perhaps surprisingly, this is essentially the only one. To be precise, I claim that $d^g_{i,c} = d^{2i}_{i,c}$ whenever $g \leq 2i$. This takes care of all of the d, because $d_{i,c} = d_{g-i,g-c}$.

The proof of this fact is not difficult. All that we need to do is consider the test curves whose base is a fixed curve of genus i and whose fiber at p consists of a fixed curve of genus $g-i$ attached to a curve isomorphic to the base at p. The marked points are fixed general points, distributed in all the possible ways on the two curves. One sees easily that these test curves have the same intersection numbers with all the boundary components of the same names, and the same intersection numbers with D^1_g. This completes the proof of this fact and of the theorem as a whole. ∎

The reader may be wondering about the discrepancy between this proposition and the previous theorem, since, after all, $d_{0,c}$ should equal $d_{g,g-c}$. This difference results from taking the ω_i as basis elements in place of the ψ_i. If the ψ_i were performing this function, we would have $d_{0,c} = -c(c+1)/2$ for $c > 1$.

COROLLARY 5.7. *All of the* $D_{g;a_1,\ldots,a_n}$ *have nonpositive coefficients of all of the higher boundary components.*

Proof. This follows from the previous theorem, the construction of the $D_{g;a_1,\dots,a_n}$ from D_g^1 by repeated intersection with a $d_{0;\{i,j\}}$ and pushing forward, and the table of pushforwards given in Theorem 2.8, in which the higher boundary components appear as pushforwards in only two ways: from higher boundary components with positive coefficients, and from ω's with negative coefficients. Since the higher boundary components start by appearing with negative coefficients, and since the ω's always appear with nonnegative coefficients, the coefficients of the higher boundary components remain negative. □

We are now in a position to prove the validity of the table which opened the section. We start by verifying the easiest of the entries ($g = 11$) to illustrate the method. Consider the divisor D_{11}^1 on $\overline{\mathcal{M}}_{11,11}$. By the theorems of this section, it equals

$$\sum_1^{11} \omega_i - \lambda - \sum_{i,j} \delta_{0,\{i,j\}} - \text{higher boundary components.}$$

Since $11 + 1 = 12$ is not prime, the Brill-Noether class is effective; it is

$$14\lambda - 2\delta_0 - \sum_{i>0,S} i(11-i)\delta_{i,S}.$$

The canonical divisor minus the sum of these, then, is a sum of boundary components with positive coefficients (recall that

$$K = \sum \omega_i + 13\lambda - 2\delta_0 - \sum_S 3\delta_{1,S} - \sum_{i>1,S} 2\delta_{i,S} + \sum_S (\operatorname{card} S - 2)\delta_{0;S}).$$

Therefore K is effective and the Kodaira dimension is nonnegative. We show that it is positive at the end of this section.

The method for proving that $\overline{\mathcal{M}}_{11,12}$ is of general type is very similar. Start with the divisor D_{11}^1 as before, and pull it back to $\overline{\mathcal{M}}_{11,12}$ by the 12 different projection maps. If we average these 12 divisors, add the Brill-Noether divisor, and subtract from K, we have a divisor which is ample by Theorem 2.9. More precisely, consider

$$\begin{aligned} D_{av} &= \frac{1}{12}\sum_{i=1}^{12} \pi_i^* D_{11}^1 \\ &= \sum \frac{11}{12}\omega_i - \lambda + \dots. \end{aligned}$$

Consider the divisor

$$K - D_{av} - (1-\varepsilon)BN = \sum \frac{\omega_i}{12} + \varepsilon\lambda + \text{boundary components.}$$

We must show that it is the sum of an ample divisor and some boundary components, thereby proving that K is the sum of an ample and an effective divisor. $\overline{\mathcal{M}}_{11,12}$ will thus be of general type.

Replace ω_i by $\psi_i - \sum_{i\in S\subseteq\{1,2,\dots,n\}} \delta_{0;S}$. We must show that the coefficients of the $\delta_{0;S}$ in the resulting expression are nonnegative. If $\operatorname{card} S = 2$, we get $-2/12$ from the change from ω to ψ, and $10/12$ from the coefficient of $\delta_{0;S}$ expressed with ω's, total $2/3$. If $\operatorname{card} S = j > 2$, we get $-j/12$ from the change of basis, and 1 from the coefficient as it was before, so the total is nonnegative.

At this point the procedure for proving the main theorem of this section should be clear, and we state it in a simple proposition.

PROPOSITION 5.8. *Suppose that we have an effective* $\mathbf{Q}$*-divisor* D *on* $\overline{\mathcal{M}}_g$ *in which the coefficient of* λ *is* $13+l$*, the coefficient of* δ_0 *is* -2*, and all the coefficients of the higher boundary components are* -3 *or less. If* gl *is an integer, then* $\overline{\mathcal{M}}_{g,gl}$ *is of nonnegative Kodaira dimension, and if* $n > gl$*, then* $\overline{\mathcal{M}}_{g,n}$ *is of general type.*

Proof. The example of genus 11 shows the way to go. For the $\overline{\mathcal{M}}_{g,gl}$ case, all that we need to do is take the average of all of the different pullbacks of D_g^1 to $\overline{\mathcal{M}}_{g,n}$ in all the different ways, add D, and add some higher boundary components to show that K is effective. In the more general situation, we again pull back, average, and add a small multiple of an ample divisor. The only potential problem is with the coefficients of the $\delta_{0;S}$. But actually this problem can easily be taken care of, for we will have coefficients at most 1 for ω in the optimal cases. The coefficient of a $\delta_{0;S}$ in the difference before changing ω_i to ψ_i will always be at least $2\,\mathrm{card}\,S-2$ when $\mathrm{card}\,S > 2$, and so at most $\mathrm{card}\,S$ will be subtracted, leaving a positive coefficient. The case $\mathrm{card}\,S = 2$ is slightly more troublesome.

First we consider the cases in which $l > 1$. In this case, when we sum the n different pullbacks of D_1^g to $\overline{\mathcal{M}}_{g,n}$ and multiply by l/n, we get

$$\sum \frac{gl}{n}\omega_i - \sum \frac{g(g-1)l}{n(n-1)}\delta_{0;\{i,j\}} - l\lambda.$$

Adding $(13+l)\lambda - 2\delta_0$ and subtracting from K leaves

$$\sum(1-\frac{gl}{n})\omega_i + \sum \frac{g(g-1)l}{n(n-1)}\delta_{0;\{i,j\}},$$

and what we need to know is that replacing ω_i by ψ_i does not lead to negative coefficients. That means we want $g(g-1)l/(n(n-1)) \geq 2 - 2gl/n$, an inequality easily checked in all the cases shown in the table.

On the other hand, when $l < 1$, we start again with the divisor $D_{g;a_1,\ldots,a_n}$. Let d_{av} be the average of the coefficients of $\delta_{0;\{i,j\}}$ and let w_{av} be the average of the coefficients of ω_i. Now take the average of D over the permutations of $\{1,\ldots,n\}$, and multiply by l, obtaining an effective divisor of the form $\sum lw_{av}\omega_i + \sum ld_{av}\delta_{0;\{i,j\}} - l\lambda$, and what matters is that $ld_{av} \geq 2(1-lw_{av})$, which again is easily checked in each case.

When $g+1$ is not prime, applying this proposition with BN as D gives the main theorem of the section for $g < 10$ and $g = 11$. When $g+1$ is prime, on the other hand, we must use the less efficient divisor E_d^1, because BN is not known to be effective. For definitions, see [EH, introduction]; here the relevant fact is simply that $E_d^1 = (6d^2+d-6)\lambda - d(d-1)\delta_0 -$ higher boundary coefficients, where $d = g/2+1$.

In fact, in all cases where gl is an integer, except $g = 11$, we get that the Kodaira dimension of $\overline{\mathcal{M}}_{g,gl}$ is positive. Again, we consider it sufficient to illustrate this with a single example: $g = 4, l = 4$. In this case, we choose two different partitions of $\{1,\ldots,16\}$ into subsets of 4 elements. We pull back the divisor D_4^1 in the ways specified by these partitions and add. Thus we get two distinct expressions of the canonical divisor of $\overline{\mathcal{M}}_{4,16}$ as effective divisors, and the supports are plainly distinct. Thus the Kodaira dimension is positive, and a similar approach works in some of the other cases, namely $g = 5, 6, 7, 10$.

In the other cases, it is necessary to use divisors other than the D^1_g. To start with, note that it is obviously best to use divisors $D_{g;a_1,\ldots,a_n}$ in which the a_i are as close to equal as possible, because we are trying to minimize the average of the coefficients of the ω for fixed n. Thus we should never have $|a_i - a_j| > 1$; the only thing to choose is n. Then, by averaging and adding E^1_d or BN as appropriate, we prove the main theorem. If we are proving that $\overline{\mathcal{M}}_{g,n}$ is of general type for some $n \geq g$, we use the divisor D^1_g on $\overline{\mathcal{M}}_{g,g}$ pulled back in all the possible ways as indicated; if we are able to show that $\overline{\mathcal{M}}_{g,n}$ is of general type for some $n < g$, it is by using the divisors $D_{g;a_1,\ldots,a_n}$ on $\overline{\mathcal{M}}_{g,n}$, where $a_i \in (g/n - 1, g/n + 1)$ and $\sum a_i = g$. That is, we choose two disjoint cyclic subgroups of order n in $\mathcal{S}_n$, and average $\sigma(D_{g;a_1,\ldots,a_n})$ over each cyclic subgroup to get two different effective divisors with the same coefficients of ω, and each of them, when subtracted from K, is the sum of an ample and an effective divisor.

When $g = 15$, we are using the divisor

$$D = D_{15;2,2,2,2,2,1,1,1,1,1};$$

we can use $D + D^\sigma$ to produce an effective canonical divisor, where

$$\sigma = (16)(27)(38)(49)(5,10) \in \mathcal{S}_{10}.$$

Alternatively, we could take $D^\tau + D^{\tau\sigma}$, where, say, $\tau = (16)$, thereby proving that $\kappa_{\overline{\mathcal{M}}_{15,10}} > 0$. This sort of trick does not work when $g = 21$, so I cannot prove that $\overline{\mathcal{M}}_{21,5}$ has positive Kodaira dimension.

Sean Keel points out that in the case $g = 11$, we could obtain two effective canonical divisors with distinct support if it could be proved that the different Brill-Noether divisors on $\overline{\mathcal{M}}_{11}$ had distinct support. Since these divisors are known by classical results to be irreducible, it suffices to find a curve of genus 11 with a g^2_9 but no g^1_6.

PROPOSITION 5.9. *There are stable curves of genus* 11 *with a* g^2_9 *but no* g^1_6. *(This was previously proved by Gavril Farkas (p.c., by arguments similar to but simpler than those used in [Fa]; the following argument is due to Joe Harris.)*

Proof. Let C be a stable curve with two components: a general curve of genus 2 attached at a general point p and a general curve of genus 9 attached at an unspecified point q. We show that there are more choices of p such that C has a g^2_9 than choices such that C has a g^1_6.

If C has a g^1_6, the vanishing sequences must be $(3,6)$ on the curve of genus 2 and $(0,3)$ on the curve of genus 9, or else $(4,5)$ and $(1,2)$. The latter is impossible, because general curves of genus 9 do not have a g^1_5. In the former case, the aspect on the genus-2 component is uniquely determined. Thus the number of points, counted with multiplicity, is the degree on a fiber of $D_{9;3}$, which is 2016 by Theorem 4.5.

To find a g^2_9, we might look for a vanishing sequence of $(5,7,8)$ on the curve of genus 2 and $(1,2,4)$ on the curve of genus 9. The former can be managed by taking the aspect on the component of genus 2 to be $5p + |3p + p'|$, where p' is the hyperelliptic conjugate of p. Thus we must count the number of ramification points—which correspond to flexes and unibranch singularities of the plane model—of g^2_8's on a curve of genus 9. There are 42 g^2_8's on such a curve, by Castelnuovo's formula [ACGH V, formula 1.2], each with 72 ramification points (Plücker's formula, [HM, Lemma 5.21]), making 3024. That these points are in general distinct follows from the fact that a general plane curve of given degree and genus has only simple flexes, a special case of the main result of [CH]. ▯

COROLLARY 5.10. $\overline{\mathcal{M}}_{11,11}$ *has positive Kodaira dimension.* ▯

6. Rationality. In the previous sections, we have described and applied a technique for demonstrating that $\overline{\mathcal{M}}_{g,n}$ is of general type. We would like to know how close these results are to being optimal: in other words, for which g and n can we prove that $\overline{\mathcal{M}}_{g,n}$ is not of general type, or of negative Kodaira dimension? I have no new results on this question; everything in this section was communicated to me by J. Harris and S. Keel. As will be seen, our results are essentially optimal in genus $4, 6$, and 11, but somewhat less good in other genera. We decline to state an opinion on the nature of the $\overline{\mathcal{M}}_{g,n}$ not covered by either of the theorems.

THEOREM 6.1. *For $g = 2, \ldots, 9$, $\overline{\mathcal{M}}_{g,n}$ is unirational for $n \leq f(g)$, where $f(g)$ is as described in the table below. (For the reader's convenience, we have repeated the $h(g)$ such that we have proved that $\overline{\mathcal{M}}_{g,h(g)}$ is not rational in the third line.)*

g	2	3	4	5	6	7	8	9	11
$f(g)$	11	14	15	12	15	11	5	8	10
$h(g)$			16	15	16	14	14	13	11

Proof. For illustration, we will give a detailed proof of the result for genus 5. A general curve of genus 5 may be embedded into $\mathbf{P}^4$ by the canonical series as the intersection of three quadrics, and it is so embedded in only one way. Quadrics in $\mathbf{P}^4$ form a $\mathbf{P}^{14}$. Thus, if we fix 12 general points in $\mathbf{P}^4$, there is a unique net (that is, two-dimensional linear series) of quadrics vanishing on all of them. The intersection of all the quadrics in this net then cuts out the curve.

We consider the map from $(\mathbf{P}^4)^{12}$ to $\overline{\mathcal{M}}_{5,12}$ given as follows: given 12 general points in $\mathbf{P}^4$, take the unique canonical curve of genus 5 containing them. This gives a family of curves of genus 4 over an open subset of $(\mathbf{P}^4)^{12}$ with 12 sections, and an open subset of the space of 12-pointed curves of genus 5 is swept out—in other words, the family gives a dominant map $(\mathbf{P}^4)^{12} \to \overline{\mathcal{M}}_{5,12}$. This proves that $\overline{\mathcal{M}}_{5,12}$ is unirational.

The other entries are proved by similar methods, but starting with different families. The reader is invited to fill in the details; we give the families here. For curves of genus 2, use curves of type $(2,3)$ on a quadric surface. In genus 3 and 4, use the canonical model. For genus 6, take the curve to lie on a del Pezzo surface of degree 5. In genus 7, let it be a plane curve of degree 7 with 8 nodes. In genus 8 or 9, let it be a plane curve of degree 8 with 13 or 12 nodes, respectively. The result in genus 11 comes by representing such a curve as a hyperplane section of a $K3$ surface. One must check that all curves of the given genus are represented in this form, and that the marked points impose linear conditions on these linear parameter spaces. ∎

In addition, it is known that $\overline{\mathcal{M}}_g$ is of negative Kodaira dimension for $g = 10, 12, 13, 15$ (see the references in [HM, sect. 6.F]), but I have not tried to extend these results to show that $\overline{\mathcal{M}}_{g,n}$ is of negative Kodaira dimension for g in this range, even with $n = 1$.

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