PMATH 445/745 — Assignment 8 solutions

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The normalization of the Fourier transform used in this assignment is

$$\widehat{f}(\rho) = \sum_{g\in G} f(g)\rho(g), \tag{1}$$

whose inverse is given by

$$f(h) = \sum_{\rho\in\mathrm{Irr}(G)} \frac{\dim V_\rho}{\#G}\,\mathrm{Tr}(\widehat{f}(\rho)\rho(h^{-1})). \tag{2}$$

1. Given two functions $\varphi,\psi : G\to\mathbb{C}$, define the *convolution* $\varphi*\psi : G\to\mathbb{C}$ by

$$(\varphi*\psi)(h) \coloneqq \sum_{g\in G}\varphi(g)\psi(g^{-1}h).$$

Prove that

$$\widehat{\varphi*\psi} = \widehat{\varphi}\widehat{\psi}.$$

Solution: Associate to $\varphi,\psi : G\to\mathbb{C}$ elements of $\mathbb{C}[G]$ via

$$\varphi\mapsto\Phi \coloneqq \sum_{g\in G}\varphi(g)g, \qquad \psi\mapsto\Psi\coloneqq\sum_{g\in G}\psi(g)g. \tag{3}$$

Calculating the product in the algebra $\mathbb{C}[G]$,

$$\Phi\Psi = \left(\sum_{g\in G}\varphi(g)g\right)\left(\sum_{h\in G}\psi(h)h\right) = \sum_{g\in G}\sum_{h\in G}\varphi(g)\psi(h)gh.$$

For each g, in the inner sum make the change of variables $h=g^{-1}k$. Left multiplication is an automorphism of G, so in particular it's a bijection, and summing over $h\in G$ is the same as summing over $k\in G$. Continuing,

$$\Phi\Psi = \sum_{g\in G}\sum_{k\in G}\varphi(g)\psi(g^{-1}k)gg^{-1}k = \sum_{k\in G}\left(\sum_{g\in G}\varphi(g)\psi(g^{-1}k)\right)k = \sum_{k\in G}(\varphi*\psi)(k)\,k. \tag{4}$$

We see that $\Phi\Psi$ is the element of $\mathbb{C}[G]$ corresponding, under (3), to the function $\varphi*\psi : G\to\mathbb{C}$.

The representation $\rho : G\to\mathrm{GL}(V)$ is a group homomorphism, and, extending linearly, the map

$$\rho : \mathbb{C}[G]\to\mathrm{End}(V)$$
$$\rho\left(\sum_{g\in G}f(g)g\right) = \sum_{g\in G}f(g)\rho(g)$$

preserves $\mathbb{C}[G]$-algebra structure. The Fourier transform of φ is equal to $\rho(\Phi)$:

$$\widehat{\varphi}(\rho) = \sum_{g\in G}\varphi(g)\rho(g) = \rho\left(\sum_{g\in G}\varphi(g)g\right) = \rho(\Phi),$$

and the same for ψ. Since ρ is a homomorphism from $\mathbb{C}[G]$ to $\operatorname{End}(V)$, we have $\rho(\Phi\Psi) = \rho(\Phi)\rho(\Psi) = \widehat{\varphi}(\rho)\widehat{\psi}(\rho)$. On the other hand, our calculation (4) above shows that $\rho(\Phi\Psi) = \widehat{(\varphi * \psi)}(\rho)$. ♣

2. In this problem, choose the isomorphism class representatives of the irreducible representations $\rho \in \operatorname{Irr}(G)$ to be unitary, i.e. $\rho(h^{-1}) = \rho(h)^\dagger \coloneqq \overline{\rho(h)}^T$ for all $h \in G$. (Using $\dagger$ to denote the conjugate transpose is common in physics.) Define an inner product $\langle \cdot, \cdot \rangle_{\text{End}}$ on $\operatorname{End}(V_\rho)$ by $\langle A, B \rangle_{\text{End}} \coloneqq \operatorname{Tr}(AB^\dagger)$. Prove *Plancherel's theorem*

$$\#G \, \langle \varphi, \psi \rangle_G = \sum_{\rho \in \operatorname{Irr}(G)} \frac{\dim V_\rho}{\#G} \, \langle \widehat{\varphi}(\rho), \widehat{\psi}(\rho) \rangle_{\text{End}}.$$

Solution: Substituting the definitions,

$$\sum_{\rho \in \operatorname{Irr}(G)} \frac{\dim V_\rho}{\#G} \, \langle \widehat{\varphi}, \widehat{\psi} \rangle_{\text{End}} = \sum_{\rho \in \operatorname{Irr}(G)} \frac{\dim V_\rho}{\#G} \operatorname{Tr}\left(\left(\sum_{g \in G} \varphi(g)\rho(g) \right) \left(\sum_{h \in G} \psi(h)\rho(h) \right)^\dagger \right).$$

Using linearity of $\dagger$ and Tr,

$$= \sum_{\rho \in \operatorname{Irr}(G)} \frac{\dim V_\rho}{\#G} \sum_{g \in G} \sum_{h \in G} \varphi(g)\overline{\psi(h)} \operatorname{Tr}\big(\rho(g)\rho(h)^\dagger\big).$$

Using the facts that ρ is unitary and ρ is a homomorphism,

$$\begin{aligned} &= \sum_{\rho \in \operatorname{Irr}(G)} \frac{\dim V_\rho}{\#G} \sum_{g \in G} \sum_{h \in G} \varphi(g)\overline{\psi(h)} \operatorname{Tr}\big(\rho(gh^{-1})\big) \\ &= \frac{1}{\#G} \sum_{g \in G} \sum_{h \in G} \varphi(g)\overline{\psi(h)} \sum_{\rho \in \operatorname{Irr}(G)} \dim V_\rho \, \chi_\rho(gh^{-1}). \end{aligned} \tag{5}$$

The regular representation $\mathbb{C}[G]$ factors as

$$\mathbb{C}[G] \cong \bigoplus_{\rho \in \operatorname{Irr}(G)} (\dim V_\rho) \, V_\rho,$$

so we recognize

$$\sum_{\rho \in \operatorname{Irr}(G)} \dim V_\rho \, \chi_\rho(gh^{-1}) = \chi_{\mathbb{C}[G]}(gh^{-1}) = \begin{cases} \#G & \text{if } g = h \\ 0 & \text{if } g \neq h. \end{cases}$$

Substituting into (5),

$$\begin{aligned} \sum_{\rho \in \operatorname{Irr}(G)} \frac{\dim V_\rho}{\#G} \, \langle \widehat{\varphi}, \widehat{\psi} \rangle_{\text{End}} &= \frac{1}{\#G} \sum_{g \in G} \sum_{h \in G} \varphi(g)\overline{\psi(h)} \cdot \#G \, \mathbb{1}\{g = h\} \\ &= \sum_{g \in G} \varphi(g)\overline{\psi(g)} \\ &= \#G \, \langle \varphi, \psi \rangle_G. \end{aligned}$$

This completes the proof.

A version of this identity, which also applies to cases in which the ρ's are not assumed to be unitary, is

$$\sum_{g \in G} \varphi(g)\psi(g^{-1}) = \sum_{\rho \in \operatorname{Irr}(G)} \frac{\dim V_\rho}{\#G} \operatorname{Tr}\Big(\widehat{\varphi}(\rho)\widehat{\psi}(\rho)\Big).$$

The fastest way I know how to prove this is to observe that the left hand side is $(\varphi * \psi)(e)$, and then apply the Fourier inversion formula (2) along with the result of problem 1. ♣

3. "Nice" functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ have Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e(nx) \tag{6}$$

with

$$\widehat{f}(n) = \langle f, e(n\cdot)\rangle = \int_{\mathbb{R}/\mathbb{Z}} f(x)\overline{e(nx)}\,dx.$$

(Here $e(nx) \coloneqq \exp(2\pi i n x)$.)

3.1. Figure out what the analogue of the result from problem 1 is for $\mathbb{R}/\mathbb{Z}$, and prove it.

Solution: Convolving f and g and substituting (6) for each of $f(x)$ and $g(x)$ gives

$$\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}/\mathbb{Z}} f(x-y)g(y)\,dy \\
&= \int_{\mathbb{R}/\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \widehat{f}(n)e(n(x-y))\right)\left(\sum_{m\in\mathbb{Z}} \widehat{g}(m)e(my)\right)\,dy \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \widehat{f}(n)\widehat{g}(m)e(nx)\int_{\mathbb{R}/\mathbb{Z}} e((m-n)y)\,dy \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \widehat{f}(n)\widehat{g}(m)e(nx)\,\mathbb{1}\{m=n\} \\
&= \sum_{n\in\mathbb{Z}} \widehat{f}(n)\widehat{g}(n)e(nx),
\end{aligned}$$

from which we read off that $\widehat{(f * g)}(n) = \widehat{f}(n)\widehat{g}(n)$.

On the other side of the Fourier transform, the conversion from multiplication to convolution looks like

$$\begin{aligned}
f(x)g(x) &= \left(\sum_{n\in\mathbb{Z}} \widehat{f}(n)e(nx)\right)\left(\sum_{m\in\mathbb{Z}} \widehat{g}(m)e(mx)\right) \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \widehat{f}(n)\widehat{g}(m)\,e((n+m)x) \\
&= \sum_{\ell\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \widehat{f}(n)\widehat{g}(\ell-n)\right) e(\ell x) \\
&= \sum_{\ell\in\mathbb{Z}} (\widehat{f} * \widehat{g})(\ell)\, e(\ell x).
\end{aligned}$$

♣

3.2. Figure out what the analogue of the result from problem 2 is for $\mathbb{R}/\mathbb{Z}$, and prove it.

Solution: Insert the Fourier transforms of f and g into the inner product:

$$\begin{aligned}\langle f, g\rangle_{\mathbb{R}/\mathbb{Z}} &= \int_{\mathbb{R}/\mathbb{Z}} f(x)\overline{g(x)}\,dx \\ &= \int_{\mathbb{R}/\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \widehat{f}(n)e(nx)\right)\overline{\left(\sum_{m\in\mathbb{Z}} \widehat{g}(m)e(mx)\right)}dx \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(m)} \int_{\mathbb{R}/\mathbb{Z}} e(nx)\overline{e(mx)}\,dx \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(m)}\langle e(n\,\cdot), e(m\,\cdot)\rangle_{\mathbb{R}/\mathbb{Z}} \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(m)}\,\mathbb{1}\{m=n\} \\ &= \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}.\end{aligned}$$

♣

4. The Fourier transform is an isomorphism

$$\mathbb{C}[G] \xrightarrow{\sim} \bigoplus_{\rho\in\mathrm{Irr}(G)} \mathrm{End}(V_\rho) \cong \bigoplus_{\rho\in\mathrm{Irr}(G)} \mathrm{Mat}_{\dim V_\rho\times\dim V_\rho}(\mathbb{C}).$$

Take $G = S_3$. In this problem, fix the representative of the isomorphism class of the 2-dimensional irreducible representation of S_3 to be

$$\begin{aligned}\rho_{2D}(()) &= \begin{bmatrix}1&0\\0&1\end{bmatrix} & \rho_{2D}((123)) &= \begin{bmatrix}0&-1\\1&-1\end{bmatrix} & \rho_{2D}((132)) &= \begin{bmatrix}-1&1\\-1&0\end{bmatrix}\\ \rho_{2D}((12)) &= \begin{bmatrix}0&1\\1&0\end{bmatrix} & \rho_{2D}((23)) &= \begin{bmatrix}1&-1\\0&-1\end{bmatrix} & \rho_{2D}((31)) &= \begin{bmatrix}-1&0\\-1&1\end{bmatrix}.\end{aligned}$$

Let

$$A := \left(37, -1009, \begin{bmatrix}3&i\\0&\pi\end{bmatrix}\right) \in \bigoplus_{\rho\in\mathrm{Irr}(S_3)} \mathrm{Mat}_{\dim V_\rho\times\dim V_\rho}(\mathbb{C}).$$

(Ordered (trivial, sign, 2D).) Find the element $f \in \mathbb{C}[S_3]$ whose Fourier transform is A.

Solution: The f we are looking for is the inverse Fourier transform (2) of A. I'll start by doing calculations for the general element $(x, y, \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))$. By (2), for any $h \in G$,

$$\begin{aligned}f(h) &= \frac{1}{6}\mathrm{Tr}(x\cdot 1) + \frac{1}{6}\mathrm{Tr}(y\cdot\rho_{\mathrm{sgn}}(h^{-1})) + \frac{1}{3}\mathrm{Tr}\big(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\rho_{\mathrm{2D}}(h^{-1})\big)\\ &= \frac{x}{6} + \frac{y}{6}\mathrm{sgn}(h) + \frac{1}{3}\mathrm{Tr}\big(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\rho_{\mathrm{2D}}(h^{-1})\big),\end{aligned} \tag{7}$$

where in the second line, $\mathrm{sgn}(h)$ means the sign as a permutation. Clearly most of the work is in calculating the last term. Here is a table with the values

h	h^{-1}	$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\rho_{2\mathrm{D}}(h^{-1})$	$\mathrm{Tr}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\rho_{2\mathrm{D}}(h^{-1})\right)$
$()$	$()$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$a+d$
(12)	(12)	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ c & d \end{bmatrix}$	$b+c$
(23)	(23)	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -a-b \\ c & -c-d \end{bmatrix}$	$a-c-d$
(31)	(31)	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -a-b & b \\ -c-d & d \end{bmatrix}$	$-a-b+d$
(123)	(132)	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -a-b & a \\ -c-d & c \end{bmatrix}$	$-a-b+c$
(132)	(123)	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} b & -a-b \\ d & -c-d \end{bmatrix}$	$b-c-d$

For $x = 37$, $y = -1009$, $a = 3$, $b = i$, $c = 0$, and $d = \pi$, substituting into (7) gives

h	$f(h)$
$()$	$\frac{37}{6} - \frac{1009}{6} + \frac{3+\pi}{3}$
(12)	$\frac{37}{6} + \frac{1009}{6} + \frac{i}{3}$
(23)	$\frac{37}{6} + \frac{1009}{6} + \frac{3-\pi}{3}$
(31)	$\frac{37}{6} + \frac{1009}{6} + \frac{-3-i+\pi}{3}$
(123)	$\frac{37}{6} - \frac{1009}{6} + \frac{-3-i}{3}$
(132)	$\frac{37}{6} - \frac{1009}{6} + \frac{i-\pi}{3}$

♣

5. Let $\sigma_1, \ldots, \sigma_n$ be 3-cycles in A_4 chosen independently and uniformly at random. Using Fourier analysis, prove that

$$\mathrm{Prob}(\sigma_1 \cdots \sigma_n = h) = \frac{1}{12}\left(1 + \frac{1}{(-2)^n}\begin{cases} -1 & \text{if } h \text{ is a 3-cycle} \\ 2 & \text{if } h \text{ is not a 3-cycle} \end{cases}\right).$$

Solution: The first step here is to relate the problem statement to convolutions. Let φ and ψ be probability distributions on A_4, i.e. $0 \leqslant \varphi(\sigma) \leqslant 1$ for each $\sigma \in A_4$ and $\sum_{\sigma \in A_4} \varphi(\sigma) = 1$, and idem for ψ. Suppose you generate a random element σ_1, with $\mathrm{Prob}(\sigma_1 = \sigma) = \varphi(\sigma)$ for all σ, and then another random element σ_2, drawn from ψ instead of φ. For any prescribed $\tau \in A_4$, the probability that $\sigma_1\sigma_2 = \tau$ is

$$\begin{aligned} &\mathrm{Prob}\big(\sigma_1 = ()\big)\,\mathrm{Prob}\big(\sigma_2 = \tau\big) \;+\; \mathrm{Prob}\big(\sigma_1 = (123)\big)\,\mathrm{Prob}\big(\sigma_2 = (123)^{-1}\tau\big) \;+\; \ldots \\ &\qquad = \sum_{g \in A_4} \mathrm{Prob}\big(\sigma_1 = g\big)\,\mathrm{Prob}\big(\sigma_2 = g^{-1}\tau\big) \\ &\qquad = \sum_{g \in A_4} \varphi(g)\psi(g^{-1}\tau) \\ &\qquad = (\varphi * \psi)(\tau). \end{aligned}$$

Consequently, if $\sigma_1, \dots, \sigma_n$ are all drawn according to φ, then

$$\operatorname{Prob}(\sigma_1 \cdots \sigma_n = h) = (\underbrace{\varphi * \varphi * \cdots * \varphi}_{n \text{ times}})(h).$$

Taking an n-fold convolution seems daunting; it's n nested sums over the group. Thankfully, we have problem 1 to save us. Let $\mathcal{F}\{f\} = \widehat{f}$ denote the Fourier transform (1), and let $\mathcal{F}^{-1}\{A\}$ denote the inverse Fourier transform (2). Then

$$\begin{aligned}(\underbrace{\varphi * \varphi * \cdots * \varphi}_{n \text{ times}})(h) &= \mathcal{F}^{-1}\big\{\mathcal{F}\{\underbrace{\varphi * \varphi * \cdots * \varphi}_{n \text{ times}}\}\big\}(h) \\ &= \mathcal{F}^{-1}\big\{\widehat{\varphi}^{\,n}\big\}(h). \end{aligned} \tag{8}$$

Raising a function to the n^{th} power is something we can do. Let's start by calculating $\widehat{\varphi}$ for our given distribution, which is $\varphi(\sigma) = \frac{1}{8}$ if σ is a 3-cycle, and $\varphi(\sigma) = 0$ otherwise.

For any irreducible representation ρ of A_4,

$$\widehat{\varphi}(\rho) = \sum_{g \in A_4} \varphi(g)\rho(g)$$

is an endomorphism of V_ρ. Moreover, by a fantastic stroke of luck, our given φ is constant on conjugacy classes, so, for any $h \in A_4$,

$$\rho(h)\widehat{\varphi}(\rho)\rho(h^{-1}) = \sum_{g \in A_4} \varphi(g)\rho(hgh^{-1}) = \sum_{g \in A_4} \varphi(h^{-1}gh)\rho(g) = \sum_{g \in A_4} \varphi(g)\rho(g) = \widehat{\varphi}(\rho),$$

i.e. $\widehat{\varphi}$ is A_4-equivariant. Hence, by Schur's lemma, $\widehat{\varphi}(\rho)$ is a scalar multiple λI of the identity. We calculate the scalar λ by taking a trace:

$$\operatorname{Tr}(\widehat{\varphi}(\rho)) = \sum_{g \in A_4} \varphi(g)\chi_\rho(g) = \# A_4 \, \langle \varphi, \chi_\rho^* \rangle.$$

(Here $\langle \cdot\,, \cdot \rangle$ denotes the inner product for functions on A_4.) Comparing with $\operatorname{Tr}(\lambda I) = \lambda \dim V_\rho$ gives

$$\widehat{\varphi}(\rho) = \frac{\# A_4 \, \langle \varphi, \chi_\rho^* \rangle}{\dim V_\rho} I,$$

a result we've seen before. Taking the n^{th} power in (8) is consequently really easy:

$$\widehat{\varphi}(\rho)^n = \left(\frac{\# A_4 \, \langle \varphi, \chi_\rho^* \rangle}{\dim V_\rho} \right)^n I.$$

Finally, we must take the inverse Fourier transform. Substituting into (2),

$$\begin{aligned}\mathcal{F}^{-1}\big\{\widehat{\varphi}^{\,n}\big\}(h) &= \sum_{\rho \in \operatorname{Irr}(A_4)} \frac{\dim V_\rho}{\# A_4} \operatorname{Tr}(\widehat{\varphi}(\rho)\rho(h^{-1})) \\ &= \sum_{\rho \in \operatorname{Irr}(A_4)} \frac{\dim V_\rho}{\# A_4} \operatorname{Tr}\left(\left(\frac{\# A_4 \, \langle \varphi, \chi_\rho^* \rangle}{\dim V_\rho} \right)^n I \, \rho(h^{-1}) \right) \\ &= \sum_{\rho \in \operatorname{Irr}(A_4)} \left(\frac{\# A_4}{\dim V_\rho} \right)^{n-1} \langle \varphi, \overline{\chi}_\rho \rangle^n \, \overline{\chi}_\rho(h). \end{aligned} \tag{9}$$

We have an explicit φ, so let's calculate the values of $\langle \varphi, \overline{\chi}_\rho \rangle$. Fetching the character table for A_4 from assignment 4 or the midterm,

	$\{()\}$	$\{(12)(34),(13)(24),(14)(23)\}$	$\{(123),(134),(142),(243)\}$	$\{(124),(132),(143),(234)\}$
χ_{triv}	1	1	1	1
χ_ω	1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
$\chi_{\overline{\omega}}$	1	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
$\chi_{3\text{D}}$	3	-1	0	0

I'll write $\omega := e^{\frac{2\pi i}{3}}$ for brevity. For an arbitrary class function f, with

$$f(()) = a, \quad f((12)(34)) = b, \quad f((123)) = c, \quad f((124)) = d,$$

we have

$$\begin{aligned}
\langle f, \overline{\chi}_{\text{triv}} \rangle &= \frac{1}{12}(a + 3b + 4c + 4d) \\
\langle f, \overline{\chi}_{\omega} \rangle &= \frac{1}{12}(a + 3b + 4c\omega + 4d\overline{\omega}) \\
\langle f, \overline{\chi}_{\overline{\omega}} \rangle &= \frac{1}{12}(a + 3b + 4c\overline{\omega} + 4d\omega) \\
\langle f, \overline{\chi}_{3\text{D}} \rangle &= \frac{1}{12}(3a - 3b).
\end{aligned}$$

Our distribution φ corresponds to $a = b = 0$ and $c = d = \frac{1}{8}$, giving

$$\begin{aligned}
\langle \varphi, \overline{\chi}_{\text{triv}} \rangle &= \frac{1}{\#A_4} \\
\langle \varphi, \overline{\chi}_{\omega} \rangle &= \frac{\omega + \overline{\omega}}{2\#A_4} = -\frac{1}{2\,\#A_4} \\
\langle \varphi, \overline{\chi}_{\overline{\omega}} \rangle &= -\frac{1}{2\,\#A_4} \\
\langle \varphi, \overline{\chi}_{3\text{D}} \rangle &= 0.
\end{aligned}$$

Substituting into the right hand side of (9),

$$\begin{aligned}
&\sum_{\rho \in \operatorname{Irr}(A_4)} \left(\frac{\#A_4}{\dim V_\rho} \right)^{n-1} \langle \varphi, \overline{\chi}_\rho \rangle^n \, \overline{\chi}_\rho(h) \\
&\quad = (\#A_4)^{n-1} \left(\frac{1}{\#A_4} \right)^n \; + \; (\#A_4)^{n-1} \left(-\frac{1}{2\,\#A_4} \right)^n \overline{\chi}_\omega(h) \; + \; (\#A_4)^{n-1} \left(-\frac{1}{2\,\#A_4} \right)^n \overline{\chi}_{\overline{\omega}}(h) \; + \; 0 \\
&\quad = \frac{1}{12} \left(1 + \frac{\overline{\chi}_\omega(h) + \overline{\chi}_{\overline{\omega}}(h)}{(-2)^n} \right).
\end{aligned}$$

From the table above or otherwise,

$$\overline{\chi}_\omega(h) + \overline{\chi}_{\overline{\omega}}(h) = \begin{cases} -1 & \text{if } h \text{ is a 3-cycle} \\ 2 & \text{if } h \text{ is not a 3-cycle.} \end{cases}$$

Note that the same approach would have worked for any φ. The calculation was simpler because φ was a class function, but, as demonstrated in problem 4, you can take the inverse Fourier transform of anything. Raising a matrix to an n^{th} power is straightforward: put it into Jordan normal form. ♣