

# PMATH 445/745 — Assignment 8

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The normalization of the Fourier transform used in this assignment is

$$\widehat{f}(\rho) = \sum_{g \in G} f(g)\rho(g),$$

whose inverse is given by

$$f(h) = \sum_{\rho \in \text{Irr}(G)} \frac{\dim V_\rho}{\#G} \text{Tr}(\widehat{f}(\rho)\rho(h^{-1})).$$

1. Given two functions  $\varphi, \psi : G \rightarrow \mathbb{C}$ , define the *convolution*  $\varphi * \psi : G \rightarrow \mathbb{C}$  by

$$(\varphi * \psi)(h) := \sum_{g \in G} \varphi(g)\psi(g^{-1}h).$$

Prove that

$$\widehat{\varphi * \psi} = \widehat{\varphi}\widehat{\psi}.$$

2. In this problem, choose the isomorphism class representatives of the irreducible representations  $\rho \in \text{Irr}(G)$  to be unitary, i.e.  $\rho(h^{-1}) = \rho(h)^\dagger := \overline{\rho(h)}^T$  for all  $h \in G$ . (Using  $\dagger$  to denote the conjugate transpose is common in physics.) Define an inner product  $\langle \cdot, \cdot \rangle_{\text{End}}$  on  $\text{End}(V_\rho)$  by  $\langle A, B \rangle_{\text{End}} := \text{Tr}(AB^\dagger)$ . Prove *Plancherel's theorem*

$$\#G \langle \varphi, \psi \rangle_G = \sum_{\rho \in \text{Irr}(G)} \dim V_\rho \langle \widehat{\varphi}, \widehat{\psi} \rangle_{\text{End}}.$$

3. “Nice” functions  $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$  have Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)e(nx)$$

with

$$\widehat{f}(n) = \langle f, e(n \cdot) \rangle = \int_{\mathbb{R}/\mathbb{Z}} f(x)\overline{e(nx)} dx.$$

(Here  $e(nx) := \exp(2\pi inx)$ .)

3.1. Figure out what the analogue of the result from problem 1 is for  $\mathbb{R}/\mathbb{Z}$ , and prove it.

3.2. Figure out what the analogue of the result from problem 2 is for  $\mathbb{R}/\mathbb{Z}$ , and prove it.

4. The Fourier transform is an isomorphism

$$\mathbb{C}[G] \xrightarrow{\sim} \bigoplus_{\rho \in \text{Irr}(G)} \text{End}(V_\rho) \cong \bigoplus_{\rho \in \text{Irr}(G)} \text{Mat}_{\dim V_\rho \times \dim V_\rho}(\mathbb{C}).$$

Take  $G = S_3$ . In this problem, fix the representative of the isomorphism class of the 2-dimensional irreducible representation of  $S_3$  to be

$$\begin{aligned} \rho_{2D}(\text{id}) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \rho_{2D}((123)) &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} & \rho_{2D}((132)) &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\ \rho_{2D}((12)) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \rho_{2D}((23)) &= \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} & \rho_{2D}((31)) &= \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Let

$$A := \left( 37, -1009, \begin{bmatrix} 3 & i \\ 0 & \pi \end{bmatrix} \right) \in \bigoplus_{\rho \in \text{Irr}(S_3)} \text{Mat}_{\dim V_\rho \times \dim V_\rho}(\mathbb{C}).$$

(Ordered (trivial, sign, 2D).) Find the element  $f \in \mathbb{C}[S_3]$  whose Fourier transform is  $A$ .

5. Let  $\sigma_1, \dots, \sigma_n$  be 3-cycles in  $A_4$  chosen independently and uniformly at random. Using Fourier analysis, prove that

$$\text{Prob}(\sigma_1 \cdots \sigma_n = h) = \frac{1}{12} \left( 1 + \frac{1}{(-2)^n} \begin{cases} -1 & \text{if } h \text{ is a 3-cycle} \\ 2 & \text{if } h \text{ is not a 3-cycle} \end{cases} \right).$$