PMATH 445/745 — Assignment 7

Alex Cowan

Due 2026/03/11

1. Let p be a prime. Prove that every group of order p^2 is abelian. [Note: I have given the TA permission to mock *mercilessly* anyone who does not use representation theory to solve this problem.]

2. Let p be a prime, and let G be a group of order p^3. Prove that either G is abelian, or else G has exactly $p-1$ irreducible representations of dimension p and p^2 irreducible representations of dimension one.

3. Let p be a prime, and let G be the group of invertible affine transformations $x \mapsto ax+b$ modulo p. This group can be realized as

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_p^\times,\, b \in \mathbb{F}_p \right\}.$$

Let H denote the subgroup of G generated by translations $x \mapsto x+b$. Let ρ be a nontrivial representation of H of your choice. Determine whether or not $\mathrm{Ind}_H^G(\rho)$ is irreducible.

4. Let H be a subgroup of a finite group G, and let $\rho\colon H \to \mathrm{GL}(V)$ be an irreducible representation. Prove that there is some representation τ of H such that $\mathrm{Res}_H \mathrm{Ind}_H^G(\rho) \cong \tau \oplus \rho$.

5. Let G be a group with exactly five irreducible characters, corresponding to irreducible representations of dimensions 1, 3, 3, 4, and 5. Prove that G is simple.