PMATH 445/745 — Assignment 6 solutions

Alex Cowan

1. Let $G = \mathbb{Z}/6$, and let H be the subgroup generated by 3. Let $\rho\colon H \to \mathrm{GL}(\mathbb{C})$ be $\rho(0) = 1$ and $\rho(3) = -1$. (Remember that the group operation in $\mathbb{Z}/6$ is addition!) Let $\tau = \mathrm{Ind}_H^G \rho\colon G \to \mathrm{GL}(\mathbb{C}^3)$. Is τ irreducible? If not, write τ as a sum of irreducible representations of G.

Solution: No, τ is not irreducible, because it has dimension two and $G = \mathbb{Z}/6$ is abelian. So it must be the sum of three irreducible one-dimensional representations.

The representation τ does not naturally have the image $\mathbb{C}^3$, but rather the vector space $(0)\mathbb{C} \oplus (1)\mathbb{C} \oplus (2)\mathbb{C}$, which is, of course, three-dimensional and therefore isomorphic to $\mathbb{C}^3$. Nevertheless, the latter vector space is more useful for understanding the G-action, so we'll use it, with the basis $\{(0)1, (1)1, (2)1\}$. The elements $\{-2, -1, 0, 1, 2, 3\}$ of $\mathbb{Z}_6$ then correspond to the following matrices:

$$\tau(-2) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$
$$\tau(-1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$
$$\tau(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\tau(1) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\tau(2) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\tau(3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

To see this for $\tau(1)$, for example, note that $\tau(1)$ sends $(0)1$ to $(1)1$, sends $(1)1$ to $(2)1$, and $(2)1$ to $(3)1 = (0)(-1) = -(0)$, since 3 acts on $\mathbb{C}$ by multiplying by -1. The other calculations are similar. This means that the character of τ is $\chi_\tau(-2,-1,0,1,2,3) = (0,0,3,0,0,-3)$, or $\chi_\tau(0,1,2,3,4,5) = (3,0,0,-3,0,0)$, if you're allergic to negative residues modulo 6.

The irreducible representations of $\mathbb{Z}/6$ are precisely $\rho_j(x) = e^{2xj\pi i/6}$, for $j = 0, \ldots, 5$. The character of ρ_j has inner product $(3 - 3(e^{j\pi i}))/6$, since the only nonzero values of χ_τ are the 0 and 3 values. This quantity is 0 if j is even, and 1 if j is odd, so we see that $\tau \cong \rho_1 \oplus \rho_3 \oplus \rho_5$. ♣

2. Let $G = S_3$, and let H be the subgroup generated by the three-cycle (123). Let $\rho\colon H \to \mathrm{GL}(\mathbb{C})$ be $\rho(123) = \gamma$, $\rho(321) = \gamma^2$, and $\rho(1) = 1$, where $\gamma = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$ is a primitive cube root of unity. Let $\tau = \mathrm{Ind}_H^G \rho\colon G \to \mathrm{GL}(\mathbb{C}^2)$. Is τ irreducible? If not, write τ as a sum of irreducible representations of G.

Solution: Yes, τ is irreducible. To see this, we compute the character of τ, by figuring out which two by two matrices each element of S_3 corresponds to. We identify the two standard basis vectors in $\mathbb{C}^2$ with $(1)1$ and

(12)1, and figure out, for each permutation, how it acts on each of the two vectors:

$$\tau(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\tau(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\tau(13) = \begin{pmatrix} 0 & \gamma \\ \gamma^2 & 0 \end{pmatrix}$$
$$\tau(23) = \begin{pmatrix} 0 & \gamma^2 \\ \gamma & 0 \end{pmatrix}$$
$$\tau(123) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^2 \end{pmatrix}$$
$$\tau(321) = \begin{pmatrix} \gamma^2 & 0 \\ 0 & \gamma \end{pmatrix}$$

For example, $\tau(13)$ takes (1)1 to (13)1, which is the same as $((12)(321))1 = (12)(\gamma^2)$, and $\tau(13)$ also takes (12)1 to $((13)(12))1 = (123)1 = (1)(\gamma)$. The other calculations are similar.

This means that the character for τ is $\chi((1),(12),(123)) = (2,0,-1)$, whose inner product with itself is one, so τ is irreducible. Indeed, τ is exactly the representation of S_3 that we recall with such fondness from the first homework assignment. ♣

3. Let G be a finite group and H a normal subgroup of G of index d. Let $\rho\colon H \to \mathrm{GL}(\mathbb{C}^n)$ be an irreducible representation of H, and let $\tau = \mathrm{Ind}_H^G \rho\colon G \to \mathrm{GL}(\mathbb{C}^{nd})$ be the induced representation of H. Let ρ' be the representation of H given by $\rho'(h) = \tau(h)$. Prove that ρ' is isomorphic to the sum of d irreducible representations of H.

Solution: Let $g_1H, \ldots, g_dH$ be a complete set of cosets of H in G. Then $\mathbb{C}^{nd} = g_1\mathbb{C}^n \oplus \ldots \oplus g_d\mathbb{C}^n$ as vector spaces, and since $Hg_i\mathbb{C}^n = g_iH\mathbb{C}^n = g_i\mathbb{C}^n$ for all $h \in H$ (H is normal!), we see that each of the subspaces $g_i\mathbb{C}^n$ is H-invariant. All we need to show is that they're all irreducible, and we'll be done.

Thus, assume that W is an H-invariant subspace of $g_i\mathbb{C}^n$. Then for every $\mathbf{w} \in W$, $[\rho'(h)](\mathbf{w}) \in W$, which is equivalent to $[\tau(h)](\mathbf{w}) \in W$ for all $\mathbf{w} \in W$, which in turn is equivalent to $[\rho(g_i^{-1}hg_i)](\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. But H is a normal subgroup, so $g_i^{-1}Hg_i = H$. This means that $[\rho(g_i^{-1}hg_i)](\mathbf{w}) \in W$ for all $\mathbf{w} \in W$ is equivalent to $[\rho(h)](\mathbf{w}) \in W$ for all $\mathbf{w} \in W$, which happens only when W is zero or $\mathbb{C}^n$. We therefore triumphantly conclude that $g_i\mathbb{C}^n$ is irreducible, as desired. ♣

4. Let G be a finite group, and let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible representation. Let V (yes, it's the same V!) be the corresponding $\mathbb{C}[G]$-module. Prove that V is randomly singly generated as a $\mathbb{C}[G]$-module; that is, prove that any nonzero $\mathbf{v} \in V$ generates V as a $\mathbb{C}[G]$-module.

Solution: Since ρ is irreducible, V is a simple $\mathbb{C}[G]$-module. The set $\mathbb{C}[G]\mathbf{v} = \{\sum_{g\in G} a_g g\mathbf{v} \mid a_g \in \mathbb{C}\}$ is a $\mathbb{C}[G]$-submodule of V, so since V is simple and $\mathbf{v}$ is nonzero, we get $\mathbb{C}[G]\mathbf{v} = V$. That is, we see that V is generated by $\mathbf{v}$, which was an arbitrary nonzero element of V. ♣

5. Let $\rho\colon S_3 \to \mathrm{GL}(\mathbb{C}^2)$ be the irreducible two-dimensional representation of S_3. Let $\tau\colon S_4 \to \mathrm{GL}(\mathbb{C}^8)$ be the induced representation of ρ, where S_3 is considered as the subgroup of S_4 of permutations that fix 4.

Compute the character of τ.

Solution: We use the formula: $\chi_\tau(g) = \frac{1}{6}\sum_{t\in G t^{-1}gt\in S_3} \chi_\rho(t^{-1}gt)$. We need to compute one value for each conjugacy class of S_4, of which there are five.

Obviously $\chi_\tau(1) = \dim(\mathbb{C}^8) = 8$ (and by the way, the 8 was computed by multiplying the dimension of ρ – namely 2 – by the index of S_3 in S_4 – namely 4). Still, it's potentially educational to realize that the formula does indeed give the same answer, since the sum is over all $t \in G$ – every element of G gives $t^{-1}1t \in S_3$ – and $\chi_\rho(1) = 2$, so $\chi_\tau(1) = \frac{1}{6}(2)(24) = 8$.

Next up is a 2-cycle. All the 2-cycles in S_3 are conjugate in S_3 and in S_4. But $\chi_\rho(g) = 0$ for any 2-cycle g, so we deduce that $\chi_\tau(g) = 0$, since it's the sum of a bunch of zeroes.

Let $g \in S_4$ be any 3-cycle. Any conjugate of it is also a 3-cycle, of course, which means that $\chi_\rho(t^{-1}gt) = -1$ for any $t \in G$. How many t satisfy $t^{-1}gt \in S_3$? There are two possible values for $t^{-1}gt$, namely (123) and (321). Thus, if we figure out how many solutions there are to the equation $t^{-1}gt = (123)$, and multiply that by two, we'll get our answer. The equation $t^{-1}gt = (123)$ has solution set $t_0\text{stab}(g)$, where t_0 is any fixed element satisfying $t_0^{-1}gt_0 = (123)$ and $\text{stab}(g)$ is the stabilizer of g under the conjugation action of G on itself. (In other words, $\text{stab}(g) = \{t \in G \mid t^{-1}gt = g\}$. This stabilizer is exactly the set of t that commute with g. By the orbit-stabilizer relation (or the class equation), since there are 8 conjugates of g (the six 3-cycles!), the stabilizer has size $24/8 = 3$. Thus, we get $\chi_\tau(g) = \frac{1}{6}(-1)(3)(2) = -1$.

No conjugate of a 4-cycle or double-flip lies in S_3, so $\chi_\tau(g) = 0$ for any 4-cycle or double-flip g.

And that's them all! ♣