PMATH 445/745 — Assignment 6

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1. Let $G = \mathbb{Z}/6$, and let H be the subgroup generated by 3. Let $\rho\colon H \to \mathrm{GL}(\mathbb{C})$ be $\rho(0) = 1$ and $\rho(3) = -1$. (Remember that the group operation in $\mathbb{Z}/6$ is addition!) Let $\tau = \mathrm{Ind}_H^G \rho\colon G \to \mathrm{GL}(\mathbb{C}^3)$. Is τ irreducible? If not, write τ as a sum of irreducible representations of G.

2. Let $G = S_3$, and let H be the subgroup generated by the three-cycle (123). Let $\rho\colon H \to \mathrm{GL}(\mathbb{C})$ be $\rho(123) = \gamma$, $\rho(321) = \gamma^2$, and $\rho(1) = 1$, where $\gamma = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$ is a primitive cube root of unity. Let $\tau = \mathrm{Ind}_H^G \rho\colon G \to \mathrm{GL}(\mathbb{C}^2)$. Is τ irreducible? If not, write τ as a sum of irreducible representations of G.

3. Let G be a finite group and H a normal subgroup of G of index d. Let $\rho\colon H \to \mathrm{GL}(\mathbb{C}^n)$ be an irreducible representation of H, and let $\tau = \mathrm{Ind}_H^G \rho\colon G \to \mathrm{GL}(\mathbb{C}^{nd})$ be the induced representation of H. Let ρ' be the representation of H given by $\rho'(h) = \tau(h)$. Prove that ρ' is isomorphic to the sum of d irreducible representations of H.

4. Let G be a finite group, and let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible representation. Let V (yes, it's the same V!) be the corresponding $\mathbb{C}[G]$-module. Prove that V is randomly singly generated as a $\mathbb{C}[G]$-module; that is, prove that any nonzero $\mathbf{v} \in V$ generates V as a $\mathbb{C}[G]$-module.

5. Let $\rho\colon S_3 \to \mathrm{GL}(\mathbb{C}^2)$ be the irreducible two-dimensional representation of S_3. Let $\tau\colon S_4 \to \mathrm{GL}(\mathbb{C}^8)$ be the induced representation of ρ, where S_3 is considered as the subgroup of S_4 of permutations that fix 4.

Compute the character of τ.