PMATH 445/745 — Assignment 5 solutions

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1. Show that $\chi_{V^*} = \overline{\chi_V}$. (Here $V^* := \mathrm{Hom}(V, \mathbb{C})$ is the dual of V.)

Solution: If V and W are representations of G, then the space $\mathrm{Hom}(V, W)$ of linear maps (not necessarily G-equivariant) inherits a G-action by requiring that the following diagram commute for all $\varphi \in \mathrm{Hom}(V, W)$ and all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{g\cdot\varphi} & W \end{array}$$

In the definition $V^* = \mathrm{Hom}(V, \mathbb{C})$, the codomain $\mathbb{C}$ is the trivial representation of G. The action of G on V^* is thus given by the commutativity of the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & \mathbb{C} \\ {\scriptstyle g}\downarrow & \nearrow_{g\cdot\varphi} & \\ V & & \end{array}$$

The element g acts as an automorphism of V, with inverse $g^{-1} \in G$. Commutativity of the diagram then gives $(g \cdot \varphi)(v) = \varphi(g^{-1}v)$.

Let $e_1, \ldots, e_n$ be a basis for V, with dual basis $\varphi_1, \ldots, \varphi_n$ (i.e. $\varphi_j(e_k) = \mathbb{1}\{j = k\}$). One has the canonical isomorphism $V^{**} \cong V$, given by evaluation: $v \in \mathrm{Hom}(V^*, \mathbb{C})$, $v(\varphi) := \varphi(v)$. This canonical isomorphism identifies the basis of V^{**} with the basis of V.

For any linear endomorphism $A : V \to V$, the trace of A is

$$\mathrm{Tr}(A) = \sum_{j=1}^{n} \varphi_j(Ae_j),$$

as can be seen by inspecting the matrix corresponding to A, viewing φ_j as the row vector which is the transpose of the column vector e_j. Similarly, for the linear endomorphism $g : V^* \to V^*$,

$$\mathrm{Tr}(g|_{V^*}) = \sum_{j=1}^{n} e_j(g \cdot \varphi_j) = \sum_{j=1}^{n} (g \cdot \varphi_j)(e_j) = \sum_{j=1}^{n} \varphi_j(g^{-1}e_j) = \chi(g^{-1}).$$

The trace of a linear map is the sum of its eigenvalues. Letting $\lambda_1, \ldots, \lambda_d$ denote the eigenvalues of the linear map $g|_V$, we have

$$\mathrm{Tr}(g^{-1}) = \sum_{j=1}^{d} \lambda_j^{-1}$$

(as can be seen by applying $g^{-1}g$ to an eigenvector of g.)

As was shown in assignment 1, any eigenvalue λ of the linear map $g|_V$ satisfies $\lambda^{\#G} = 1$, i.e. the eigenvalues of matrices in the image of a linear representation of a finite group are roots of unity. If $|\lambda| = 1$, then $\lambda^{-1} = \overline{\lambda}$. Hence

$$\chi(g^{-1}) = \sum_{j=1}^{d} \overline{\lambda_j} = \overline{\sum_{j=1}^{d} \lambda_j} = \overline{\chi(g)}.$$

2. Let $\chi_1, \ldots, \chi_n$ be a complete list of the characters of the irreducible representations of G. For $g \in G$, let $C_g \coloneqq \{kgk^{-1} : k \in G\}$ denote the conjugacy class of g. Show that, for any $g, h \in G$,

$$\sum_{i=1}^{n} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} \#G/\#C_g & \text{if } C_g = C_h \\ 0 & \text{if } C_g \neq C_h. \end{cases}$$

[Hint: write the indicator function of C_g as a linear combination of characters of irreducible representations.]

Solution: Let's follow the hint. Write $\mathbb{1}_g(h) \coloneqq \mathbb{1}\{h \in C_g\}$. Since the characters of the irreducible representations of G form an orthonormal basis of the space of class functions $\mathcal{C}(G) \coloneqq \{f : G \to \mathbb{C} : f(kgk^{-1}) = f(g) \text{ for all } g, k \in G\}$ (and $\mathbb{1}_g$ is evidently a class function),

$$\mathbb{1}_g = \sum_{i=1}^{n} \langle \mathbb{1}_g, \chi_i \rangle \chi_i$$

with

$$\langle \mathbb{1}_g, \chi_i \rangle = \frac{1}{\#G} \sum_{k \in G} \mathbb{1}_g(k)\overline{\chi_i(k)} = \frac{1}{\#G} \sum_{k \in C_g} \overline{\chi_i(k)} = \frac{\#C_g}{\#G}\overline{\chi_i(g)},$$

the last equality following from the facts that $\chi_i \in \mathcal{C}(G)$ and that $g \in C_g$. Substituting and evaluating at h,

$$\mathbb{1}_g(h) = \sum_{i=1}^{n} \frac{\#C_g}{\#G}\overline{\chi_i(g)}\chi_i(h).$$

The left hand side is 0 or 1 according to $h \notin C_g$ or $h \in C_g$. The result follows. ♣

3. This problem classifies the characters of a direct product. Let $G = H \times K$ and let $\rho : H \to \mathrm{GL}(V)$ be an irreducible representation of H with character χ. Then $G \overset{\pi_H}{\to} H \overset{\rho}{\to} \mathrm{GL}(V)$ gives an irreducible representation of G, where π_H is the natural projection; the character, $\tilde{\chi}$, of this representation is $\tilde{\chi}((h,k)) = \chi(h)$. Likewise any irreducible character ψ of K gives an irreducible character $\tilde{\psi}$ of G with $\tilde{\psi}((h,k)) = \psi(k)$.

3. a) Prove that the product $\tilde{\chi}\tilde{\psi}$ is an irreducible character of G.

Solution: The character $\tilde{\chi}\tilde{\psi}$ is irreducible iff $\langle \tilde{\chi}\tilde{\psi}, \tilde{\chi}\tilde{\psi} \rangle_G = 1$, the subscript G indicating that the inner product is in $\mathcal{C}(G)$. Calculating,

$$\begin{aligned} \langle \tilde{\chi}\tilde{\psi}, \tilde{\chi}\tilde{\psi} \rangle_G &= \frac{1}{\#H \times K} \sum_{(h,k) \in H \times K} \tilde{\chi}\tilde{\psi}((h,k)) \\ &= \frac{1}{\#H}\frac{1}{\#K} \sum_{h \in H}\sum_{k \in K} \tilde{\chi}((h,k))\tilde{\psi}((h,k)) \\ &= \frac{1}{\#H}\sum_{h \in H} \chi(h) \cdot \frac{1}{\#K}\sum_{k \in K} \psi(k) \\ &= \langle \chi, \chi \rangle_H \langle \psi, \psi \rangle_K \\ &= 1, \end{aligned}$$

since χ and ψ are irreducible characters of H and K respectively by assumption. ♣

3. b) Prove that every irreducible character of G is obtained from such products of irreducible characters of the direct factors.

Solution: Let's check that $\#G$ is equal to the sums of the squares of the dimensions of the irreducible representations with characters $\tilde{\chi}\tilde{\psi}$.

Recall that, for any representation (V, ρ) of G with character χ_ρ,

$$\dim V = \chi_\rho(e),$$

where $e \in G$ is the identity element (since $\rho(e) = I$, the identity matrix). Moreover, the identity element of $G = H \times K$ is $e_G = (e_H, e_K)$.

With these preliminaries, we calculate the sum of the squares of the dimensions of the representations with characters $\tilde{\chi}\tilde{\psi}$:

$$\sum_\chi \sum_\psi (\tilde{\chi}\tilde{\psi}(e_G))^2 = \sum_\chi \sum_\psi \chi(e_H)^2 \psi(e_K)^2 = \sum_\chi \chi(e_H)^2 \cdot \sum_\psi \psi(e_K)^2 = \#H \cdot \#K = \#G.$$

(The sums are over all irreducible characters χ, ψ of H and K respectively.) ♣

4. Show that the character of any irreducible representation of dimension greater than 1 assumes the value 0 on some conjugacy class of the group.

Solution: This problem is very hard. Below I've transcribed the solution from Fulton–Harris's *Representation theory, A first course*, solution to exercise 2.39 on page 517.

In case the character χ is $\mathbb{Z}$-valued, the equation $\sum_g |\chi(g)|^2 = |G|$ shows that $|G|$ is the sum of $|G|$ non-negative integers, one of which, $|\chi(e)|^2$, is greater than 1, so at least one must be 0. In general, the values of χ are algebraic integers, since they are sums of roots of unity. Let $\chi_1, \dots, \chi_m$ be the characters obtained from χ by the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$) on χ; these characters are also characters of irreducible representations of G. Now if $\chi(g) \neq 0$, then $\prod_i \chi_i(g)$ is a nonzero integer, so $|\prod_i \chi_i(g)|^2 \geqslant 1$. Since the arithmetic mean is at least the geometric mean, $\sum_i |\chi_i(g)|^2 \geqslant m$. Therefore,

$$m|G| = \sum_{i=1}^m \sum_{g \in G} |\chi_i(g)|^2 \geqslant m|G|,$$

and we must have equality for every $g \in G$. In particular, if d is the degree of the representation, $md^2 = \sum_i |\chi_i(e)|^2 = m$, so $d = 1$. ♣

5. Consider the group $\mathbb{Z}/3$ acting on $\mathbb{R}^2$ where 1 acts as rotation by 120 degrees. We may view $\mathbb{R}^2$ as $\mathbb{C}$, then the action of 1 becomes multiplication by $\exp(2\pi i/3)$, giving us a complex representation.

Now consider the (real) standard representation V_{std} of S_{2n+1}. (The natural representation of S_{2n+1} on $\mathbb{R}^{2n+1}$, permuting the basis vectors in the natural way, is the direct sum of the trivial representation and V_{std}; see assignment 1 for the case $n = 1$.) You may take for granted that V_{std} is an irreducible representation of dimension $2n$. Show that it is not possible to identify $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ in a compatible way so that the real standard representation V_{std} gives you a complex representation.

Solution: We have covered this in class, but I would still prefer you write down precisely what it means to "identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$" and what it means for it to be "compatible with the representation of the group". This is formalized as follows.

A linear complex structure on $V \cong \mathbb{R}^{2n}$ is an endomorphism J such that $J^2 = \mathrm{Id}$ (J acts in the role of the imaginary number i, hence a good notion of complex structure). A linear complex structure is further compatible with a G-representation if it is G-equivariant, meaning $J \in \mathrm{End}_G(V)$.

With the above notion, your job is to show that there does not exists any S_n-equivariant endomorphism J on V_{std} such that $J^2 = -I$.

Let us consider V_{std} as the subspace of $\mathbb{R}^{2n-1}$ whose coordinates sum to 0 with basis $e_1, \dots, e_{2n}$ such that $e_i = (0, \dots, -1, 0, \dots, 0, 1)$ where the -1 is in the i-th spot. Suppose such a J exists and sends e_1 to $\sum_{i=1}^{2n} a_i e_i$. Consider the action by the permutation $\sigma = (1, 2n+1)$, we have $\sigma(e_1) = -e_1$ and $\sigma(e_i) = -e_1 + e_i$ for $i > 1$. We necessarily have

$$-J(e_1) = J(\sigma e_1) = \sigma J e_1 = -(\sum a_i) e_1 + \sum_{i>1} a_i e_i$$

and we conclude by comparing coefficients $a_2 = \cdots = a_n = 0$, so $Je_1 = a_1 e_1$ and e_1 is an eigenvector of J. But $J^2 = -I$ implies its eigenvalues must be complex. You can not have a real matrix with a real eigenvector whose eigenvalue is complex! This is the contradiction, so J can not exist in the first place.

You could use Schur lemma to say that J must be a multiple of Id, but remember, Schur lemma requires your base field to be algebraically closed. So you should complexify the standard representation if you use this method, but then you have two "complexification", one by tensoring $\mathbb{C}$, and one by choosing a complex structure. This can get confusing really quick so I do not recommend this approach.