PMATH 445/745 — Assignment 5

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1. Show that $\chi_{V^*} = \overline{\chi_V}$. (Here $V^* := \mathrm{Hom}(V, \mathbb{C})$ is the dual of V.)

2. Let $\chi_1, \ldots, \chi_n$ be a complete list of the characters of the irreducible representations of G. For $g \in G$, let $C_g := \{kgk^{-1} : k \in G\}$ denote the conjugacy class of g. Show that, for any $g, h \in G$,

$$\sum_{i=1}^{n} \chi_i(g)\chi_i(h) = \begin{cases} \#G/\#C_g & \text{if } C_g = C_h \\ 0 & \text{if } C_g \neq C_h. \end{cases}$$

[Hint: write the indicator function of C_g as a linear combination of characters of irreducible representations.]

3. This problem classifies the characters of a direct product. Let $G = H \times K$ and let $\rho : H \to \mathrm{GL}(V)$ be an irreducible representation of H with character χ. Then $G \overset{\pi_H}{\to} H \overset{\rho}{\to} \mathrm{GL}(V)$ gives an irreducible representation of G, where π_H is the natural projection; the character, $\tilde{\chi}$, of this representation is $\tilde{\chi}((h,k)) = \chi(h)$. Likewise any irreducible character ψ of K gives an irreducible character $\tilde{\psi}$ of G with $\tilde{\psi}((h,k)) = \psi(k)$.

3. a) Prove that the product $\tilde{\chi}\tilde{\psi}$ is an irreducible character of G.

3. b) Prove that every irreducible character of G is obtained from such products of irreducible characters of the direct factors.

4. Show that the character of any irreducible representation of dimension greater than 1 assumes the value 0 on some conjugacy class of the group.

5. Consider the group $\mathbb{Z}/3$ acting on $\mathbb{R}^2$ where 1 acts as rotation by 120 degrees. We may view $\mathbb{R}^2$ as $\mathbb{C}$, then the action of 1 becomes multiplication by $\exp(2\pi i/3)$, giving us a complex representation.

Now consider the (real) standard representation V_{std} of S_{2n+1}. (The natural representation of S_{2n+1} on $\mathbb{R}^{2n+1}$, permuting the basis vectors in the natural way, is the direct sum of the trivial representation and V_{std}; see assignment 1 for the case $n = 1$.) You may take for granted that V_{std} is an irreducible representation of dimension $2n$. Show that it is not possible to identify $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ in a compatible way so that the real standard representation V_{std} gives you a complex representation.