PMATH 445/745 — Assignment 4

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1. Little Johnny has a group with 12 elements with 4 irreducible representations. If he gives three one-dimensional representations to his mother, what is the dimension of the representation he has left?

Solution: The sum of the squares of the dimensions of the irreducible representations must sum to 12. The three one-dimensional representations contribute one each to this sum, leaving $9 = 3^2$ for the last representation, which therefore must have dimension 3. ♣

2. Let ρ be a three-dimensional representation of a finite group such that $\langle \chi_\rho, \chi_\rho \rangle = 5$. Prove that ρ is the sum of three irreducible representations.

Solution: Since ρ is three-dimensional, it is either irreducible itself, or else it is the sum of two or three irreducible representations. Since $\langle \chi_\rho, \chi_\rho \rangle \neq 1$, ρ is clearly not irreducible. If it were the sum of two irreducible representations $\rho \cong \tau_1 \oplus \tau_2$, then we would have $\langle \chi_\rho, \chi_\rho \rangle = 1^2 + 1^2 = 2 \neq 5$, because τ_1 and τ_2 have to be non-isomorphic (one of dimension one, the other of dimension two).

So ρ must be the sum of three irreducible representations. And since $\langle \chi_\rho, \chi_\rho \rangle = 5$, exactly two of them must be isomorphic, so that $\rho \cong \tau_1 \oplus (2\tau_2)$ for one-dimensional irreducible representations τ_1 and τ_2. ♣

3. Let Q be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, and let ρ be the left multiplication representation in the space of quaternions $H = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$. That is, recall that $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$, and let ρ map each $g \in Q$ to the linear transformation $[\rho(g)](a + bi + cj + dk) = ag + bgi + cgj + dgk$. For example, $[\rho(i)](a + bi + cj + dk) = -b + ai - dj + ck$.

3. a) Compute the character χ_ρ of ρ.

Solution: This is a straightforward calculation, especially if you notice that multiplication by any of the letters permutes the basis vectors in H without fixing any:

	1	-1	i	$-i$	j	$-j$	k	$-k$
χ_ρ	4	-4	0	0	0	0	0	0

♣

3. b) Prove that ρ is isomorphic to the sum of two isomorphic irreducible representations of dimension two.

Solution: First, note that the inner product of χ_ρ with itself is $\frac{1}{8}(4^2 + (-4)^2) = 4$, so the arithmetic works out there. Sadly, the arithmetic is also consistent with ρ being the sum of four irreducible representations of dimension one, pairwise non-isomorphic. These are the only two possibilities, though, because if $\rho \cong (m_1\rho_1) \oplus \ldots \oplus (m_r\rho_r)$ for irreducible representations ρ_i, then we need $m_1 \dim \rho_1 + \ldots + m_r \dim \rho_r = 4$ and $m_1^2 + \ldots + m_r^2 = 4$. The only possibilities are therefore the two just mentioned, corresponding to $r = 1$, $m_1 = 2$, $\dim \rho_1 = 2$, and $r = 4$, $m_1 = m_2 = m_3 = m_4 = 1$, and $\dim \rho_i = 1$ for all i.

So what we need to do is to rule out the one-dimensional possibility. You could do this using a character table for Q, but that would be lame. Instead, we'll be clever, which is always more fun. Say $\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4$ for one-dimensional representations ρ_i. Since each ρ_i is one-dimensional, its image is abelian, so that $\rho_i(g_1 g_2) =$

$\rho_i(g_1)\rho_i(g_2) = \rho_i(g_2)\rho_i(g_1) = \rho_i(g_2 g_1)$ for any g_1 and g_2 in the group. Since we are assuming that ρ is the sum of the ρ_i, it would follow that $\rho(g_1 g_2) = \rho(g_2 g_1)$ as well (check it!). But in reality, ρ is injective, and so in particular since Q is not abelian, neither is the image of ρ. This means that the four one-dimensional option is impossible. Thus, as Sherlock Holmes would tell us if he weren't fictional and dead, we conclude that ρ must be isomorphic the sum of some irreducible two-dimensional representation with itself. ♣

4. Compute the character table of A_4.

Solution: Let's start by listing the conjugacy classes of A_4. If two permutations aren't conjugate in S_4, they won't be conjugate in A_4 either, so elements with different cycle types are in different conjugacy classes in A_4. This partitions A_4 into $\{()\}$, $\{(12)(34), (13)(24), (14)(23)\}$, $\{(123), (124), (134), (132), (142), (143), (234), (243)\}$. However, while (123) and (132) are conjugate in S_4 — $(132) = (23)(123)(23)$ — they aren't conjugate in A_4. (One way to see ahead of time that there cannot be exactly three conjugacy classes in A_4 is to note that $\#A_4 = 2^2 + 2^2 + 2^2$ is the only way to write the order of the group as a sum of three integer squares, yet there is certainly a 1-dimensional irreducible representation: the trivial representation.) One can verify that there are exactly four conjugacy classes of A_4: $\{()\}$, $\{(12)(34), (13)(24), (14)(23)\}$, $\{(123), (134), (142), (243)\}$, $\{(124), (132), (143), (234)\}$. Now let's start finding irreducible representations.

As mentioned, there is the 1-dimensional trivial representation, sending each $A_4 \ni \sigma \mapsto 1$.

The 1-dimensional representations must contain the commutator subgroup $[A_4, A_4]$ in their kernels, since their images are abelian. One calculates that $[A_4, A_4] = \{(), (12)(34), (13)(24), (14)(23)\} \cong (\mathbb{Z}/2)^2$. The quotient is isomorphic to $\mathbb{Z}/3$, the unique group of order 3. Explicitly, the map is to take any element of order 3 in A_4, e.g. (123), and map the coset $(123)[A_4, A_4]$ to $1 \in \mathbb{Z}/3$.

The irreducible representations of $\mathbb{Z}/3$ are the trivial representation, which we have already counted, as well as the two representations $\mathbb{Z}/3 \ni 1 \mapsto e^{\frac{2\pi i}{3}}$ and $\mathbb{Z}/3 \ni 1 \mapsto e^{\frac{4\pi i}{3}}$.

Summarizing our character table Sudoku so far:

	$\{()\}$	$\{(12)(34), (13)(24), (14)(23)\}$	$\{(123), (134), (142), (243)\}$	$\{(124), (132), (143), (234)\}$
χ_{triv}	1	1	1	1
χ_ω	1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
$\chi_{\overline{\omega}}$	1	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
?	?	?	?	?

The remaining irreducible representation (there is one left, since characters of irreducible representations form a basis for functions $G \to \mathbb{C}$ that are constant on conjugacy classes, and the dimension of this space is equal to the number of conjugacy classes) must be 3-dimensional: $12 = 1^2 + 1^2 + 1^2 + 3^2$. The value of the remaining character on the identity permutation $()$ is 3, the dimension of the representation. Then, one can determine the character values using the fact that irreducible characters are orthogonal, or that columns in the character table are orthogonal. The missing row in the character table above is 3, −1, 0, 0. ♣

5. a) Prove equations (46) and (47) of [Elk00] https://arxiv.org/abs/math/0005139.

Solution: Let's calculate the action of $\mathrm{Sym}^2\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$ on each of the three basis elements $e_1 \otimes e_1$, $\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$, $e_2 \otimes e_2$:

$$\begin{aligned}
\begin{bmatrix} p & q \\ r & s \end{bmatrix} (e_1 \otimes e_1) &= \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \\
&= \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= (pe_1 + re_2) \otimes (pe_1 + re_2) \\
&= (pe_1 \otimes pe_1) + (re_2 \otimes pe_1) + (pe_1 \otimes re_2) + (re_2 \otimes re_2) \\
&= p^2(e_1 \otimes e_1) + 2pr \cdot \tfrac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) + r^2(e_2 \otimes e_2),
\end{aligned}$$

so the first column of $\mathrm{Sym}^2\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$ in our basis is $\begin{bmatrix} p^2 \\ 2pr \\ r^2 \end{bmatrix}$. Similarly, the third column is $\begin{bmatrix} q^2 \\ 2qs \\ s^2 \end{bmatrix}$.

Calculating the middle column,

$$\begin{aligned} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \tfrac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) &= \frac{1}{2} \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_2 + \frac{1}{2} \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_2 \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \\ &= \frac{1}{2}(pe_1 + re_2) \otimes (qe_1 + se_2) + \frac{1}{2}(qe_1 + se_2) \otimes (pe_1 + re_2) \\ &= pq(e_1 \otimes e_1) + (ps + qr) \cdot \tfrac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) + rs(e_2 \otimes e_2). \end{aligned}$$

This verifies [Elk00, (46)].

Setting $(p, q, r, s) = (0, X^{-1/4}, X^{1/4}, -X^{1/4}b/2)$ yields

$$\mathrm{Sym}^2 \begin{bmatrix} 0 & X^{-1/4} \\ X^{1/4} & -X^{1/4}b/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & X^{-1/2} \\ 0 & 1 & -b \\ X^{1/2} & -X^{1/2}b/2 & X^{1/2}b^2/4 \end{bmatrix}$$

Note that there is a typo in [Elk00, (47)]: the bottom right entry of the 2×2 matrix is given as $X^{1/4}b/2$, without the minus sign.

5. b) Immediately after equation (47), Elkies writes

> *"This is why we went after $4x^3 - 3y^2$ rather than pursuing $x^3 - y^2$ directly: an analogous approach to $x^3 - y^2$ would yield a matrix that is still a symmetric square but with respect to a different basis, requiring a definition of* Sym^2 *with fractional coefficients and complicating the lattice reduction."*

Let X, g, and ρ be positive real numbers. Define

$$M := \begin{pmatrix} & & X^{-\frac{\rho}{2}} \\ & X^{\frac{\rho}{2}-\frac{1}{2}} & -X^{\frac{\rho}{2}-\frac{1}{2}}g \\ X^{\frac{3}{2}\rho-1} & -X^{\frac{3}{2}\rho-1}\frac{g}{6} & X^{\frac{3}{2}\rho-1}\frac{g^2}{12} \end{pmatrix} = \begin{pmatrix} X^{-\frac{\rho}{2}} & & \\ & X^{\frac{\rho}{2}-\frac{1}{2}} & \\ & & X^{\frac{3}{2}\rho-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & -g \\ 1 & -\frac{g}{6} & \frac{g^2}{12} \end{pmatrix}.$$

(Blank entries denote 0.) This problem asks you to recognize M as a symmetric square. This will involve choosing an unusual basis.

Let V be the 2-dimensional $\mathbb{R}$ vector space with basis $\{e_1, e_2\}$. Let

$$\mathrm{Sym}^2(V) \cong V \otimes V / \langle e_i \otimes e_j - e_j \otimes e_i \rangle$$

be the 3-dimensional $\mathbb{R}$ vector space with basis $\{a(e_1 \otimes e_1), b(e_1 \otimes e_2 + e_2 \otimes e_1), c(e_2 \otimes e_2)\}$. Here a, b, c are unknown real numbers, for you to determine. There is a homomorphism $\mathrm{GL}(V) \to \mathrm{GL}(\mathrm{Sym}^2(V))$ given by $T \mapsto \mathrm{Sym}^2 T$ with $\mathrm{Sym}^2 T(e_i \otimes e_j) := T(e_i) \otimes T(e_j)$, extended linearly.

Find $a, b, c \in \mathbb{R}$ and $N \in \mathrm{GL}(V)$ such that $M = \mathrm{Sym}^2 N$.

Solution: Let's calculate the equivalent of [Elk00, (46)] for this new basis:

$$\begin{aligned} \begin{bmatrix} p & q \\ r & s \end{bmatrix} a(e_1 \otimes e_1) &= a \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \\ &= a(pe_1 + re_2) \otimes (pe_1 + re_2) \\ &= p^2 a(e_1 \otimes e_1) + \frac{apr}{b} \cdot b(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{ar^2}{c} \cdot c(e_2 \otimes e_2), \end{aligned}$$

so the first column of $\mathrm{Sym}^2 \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is $\begin{bmatrix} p^2 \\ apr/b \\ ar^2/c \end{bmatrix}$. Similarly

$$\begin{aligned} \begin{bmatrix} p & q \\ r & s \end{bmatrix} c(e_2 \otimes e_2) &= c \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_2 \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_2 \\ &= c(qe_1 + se_2) \otimes (qe_1 + se_2) \\ &= \frac{cq^2}{a} \cdot a(e_1 \otimes e_1) + \frac{cqs}{b} \cdot b(e_1 \otimes e_2 + e_2 \otimes e_1) + s^2 c(e_2 \otimes e_2), \end{aligned}$$

so the third column of $\mathrm{Sym}^2 \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is $\begin{bmatrix} cq^2/a \\ cqs/b \\ s^2 \end{bmatrix}$.

Calculating the middle column,

$$\begin{aligned}
\begin{bmatrix} p & q \\ r & s \end{bmatrix} b(e_1 \otimes e_2 + e_2 \otimes e_1) &= b \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_2 + b \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_2 \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} e_1 \\
&= b(pe_1 + re_2) \otimes (qe_1 + se_2) + b(qe_1 + se_2) \otimes (pe_1 + re_2) \\
&= \frac{2bpq}{a} \cdot a(e_1 \otimes e_1) \;+\; (ps + qr) \cdot b(e_1 \otimes e_2 + e_2 \otimes e_1) \;+\; \frac{2brs}{c} \cdot c(e_2 \otimes e_2).
\end{aligned}$$

Hence, with this basis,

$$\mathrm{Sym}^2 \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p^2 & 2bpq/a & cq^2/a \\ apr/b & ps + qr & cqs/b \\ ar^2/c & 2brs/c & s^2 \end{bmatrix}.$$

Note that the above is invariant under $(a, b, c) \mapsto (\lambda a, \lambda b, \lambda c)$ for all $\lambda \neq 0$.

Taking the symmetric square is a homomorphism, and the diagonal of $\mathrm{Sym}^2 \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ doesn't depend on a, b, c. By inspection,

$$\begin{bmatrix} X^{-\frac{\rho}{2}} & & \\ & X^{\frac{\rho}{2}-\frac{1}{2}} & \\ & & X^{\frac{3}{2}\rho-1} \end{bmatrix} = \mathrm{Sym}^2 \begin{bmatrix} X^{-\frac{\rho}{4}} & \\ & X^{\frac{3}{4}\rho-\frac{1}{2}} \end{bmatrix}.$$

Now let's find a, b, c, p, q, r, s such that

$$\begin{bmatrix} & & 1 \\ & 1 & -g \\ 1 & -\frac{g}{6} & \frac{g^2}{12} \end{bmatrix} = \mathrm{Sym}^2 \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Looking at the diagonal, $p = 0$, $s = \frac{g}{2\sqrt{3}}$, and $qr = 1$. We may as well set $b = 1$, by the above scale-invariance observation. Next, dividing the middle entry of the last column by the top entry gives $-g = sa/q$, hence $q = -a/2\sqrt{3}$. Since $qr = 1$, we have $r = -2\sqrt{3}/a$. The bottom left entry then gives the relation $ac = 12$. The only remaining entry to leverage is the bottom row middle column, and this gives the tautological $-g/6 = -g/6$.

Overall, for any triple a, b, c of nonzero real numbers such that $ac = 12b^2$, we have

$$\begin{bmatrix} X^{-\frac{\rho}{2}} & & \\ & X^{\frac{\rho}{2}-\frac{1}{2}} & \\ & & X^{\frac{3}{2}\rho-1} \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & -g \\ 1 & -\frac{g}{6} & \frac{g^2}{12} \end{bmatrix} = \mathrm{Sym}^2 \left(\begin{bmatrix} X^{-\frac{\rho}{4}} & \\ & X^{\frac{3}{4}\rho-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 0 & -\frac{a}{2\sqrt{3}} \\ -\frac{2\sqrt{3}}{a} & \frac{g}{2\sqrt{3}} \end{bmatrix} \right).$$

♣

References

[Elk00] Noam D. Elkies. Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction. In *Algorithmic number theory (Leiden, 2000)*, volume 1838 of *Lecture Notes in Comput. Sci.*, pages 33–63. Springer, Berlin, 2000.