PMATH 445/745 — Assignment 4

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1. Little Johnny has a group with 12 elements with 4 irreducible representations. If he gives three one-dimensional representations to his mother, what is the dimension of the representation he has left?

2. Let ρ be a three-dimensional representation of a finite group such that $\langle \chi_\rho, \chi_\rho \rangle = 5$. Prove that ρ is the sum of three irreducible representations.

3. Let Q be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, and let ρ be the left multiplication representation in the space of quaternions $H = \{a + bi + cj + dk \mid a,\, b,\, c,\, d \in \mathbb{R}\}$. That is, recall that $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$, and let ρ map each $g \in Q$ to the linear transformation $[\rho(g)](a + bi + cj + dk) = ag + bgi + cgj + dgk$. For example, $[\rho(i)](a + bi + cj + dk) = -b + ai - dj + ck$.

3. a) Compute the character χ_ρ of ρ.

3. b) Prove that ρ is isomorphic to the sum of two isomorphic irreducible representations of dimension two.

4. Compute the character table of A_4.

5. a) Prove equations (46) and (47) of [Elk00] https://arxiv.org/abs/math/0005139.

5. b) Immediately after equation (47), Elkies writes

> *"This is why we went after $4x^3 - 3y^2$ rather than pursuing $x^3 - y^2$ directly: an analogous approach to $x^3 - y^2$ would yield a matrix that is still a symmetric square but with respect to a different basis, requiring a definition of* Sym^2 *with fractional coefficients and complicating the lattice reduction."*

Let X, g, and ρ be positive real numbers. Define

$$M := \begin{pmatrix} & & X^{-\frac{\rho}{2}} \\ & X^{\frac{\rho}{2}-\frac{1}{2}} & -X^{\frac{\rho}{2}-\frac{1}{2}}g \\ X^{\frac{3}{2}\rho-1} & -X^{\frac{3}{2}\rho-1}\frac{g}{6} & X^{\frac{3}{2}\rho-1}\frac{g^2}{12} \end{pmatrix} = \begin{pmatrix} X^{-\frac{\rho}{2}} & & \\ & X^{\frac{\rho}{2}-\frac{1}{2}} & \\ & & X^{\frac{3}{2}\rho-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & -g \\ 1 & -\frac{g}{6} & \frac{g^2}{12} \end{pmatrix}.$$

(Blank entries denote 0.) This problem asks you to recognize M as a symmetric square. This will involve choosing an unusual basis.

Let V be the 2-dimensional $\mathbb{R}$ vector space with basis $\{e_1, e_2\}$. Let

$$\mathrm{Sym}^2(V) \cong V \otimes V / \langle e_i \otimes e_j - e_j \otimes e_i \rangle$$

be the 3-dimensional $\mathbb{R}$ vector space with basis $\{a(e_1 \otimes e_1),\, b(e_1 \otimes e_2 + e_2 \otimes e_1),\, c(e_2 \otimes e_2)\}$. Here a, b, c are unknown real numbers, for you to determine. There is a homomorphism $\mathrm{GL}(V) \to \mathrm{GL}(\mathrm{Sym}^2(V))$ given by $T \mapsto \mathrm{Sym}^2 T$ with $\mathrm{Sym}^2 T(e_i \otimes e_j) := T(e_i) \otimes T(e_j)$, extended linearly.

Find $a, b, c \in \mathbb{R}$ and $N \in \mathrm{GL}(V)$ such that $M = \mathrm{Sym}^2 N$.

References

[Elk00] Noam D. Elkies. Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction. In *Algorithmic number theory (Leiden, 2000)*, volume 1838 of *Lecture Notes in Comput. Sci.*, pages 33–63. Springer, Berlin, 2000.