PMATH 445/745 — Assignment 3 solutions

Alex Cowan

1. Consider the vector space $\mathbb{C} \otimes \mathbb{C}^2$. Is $2 \otimes (1,1) = 1 \otimes (2,2)$? Is $1 \otimes (2,2) = (2,2)$?

Solution: The answer to the first question is yes: $2 \otimes (1,1) = 2(1 \otimes (1,1)) = 1 \otimes 2(1,1) = 1 \otimes (2,2)$.

The answer to the second question is no. The vector $(2,2)$ is not an element of $\mathbb{C} \otimes \mathbb{C}^2$, so it can't possibly be equal to $1 \otimes (2,2)$. ♣

2. In $\mathbb{C}^2 \otimes \mathbb{C}^2$, write $(1,2) \otimes (3,4)$ as a linear combination of the basis

$$\{(1,0) \otimes (1,0),\ (0,1) \otimes (1,0),\ (1,0) \otimes (0,1),\ (0,1) \otimes (0,1)\}.$$

Solution: We have:

$$
\begin{aligned}
(1,2) \otimes (3,4) &= ((1,0) + (0,2)) \otimes ((3,0) + (0,4)) \\
&= (1,0) \otimes (3,0) + (1,0) \otimes (0,4) + (0,2) \otimes (3,0) + (0,2) \otimes (0,4) \\
&= 3((1,0) \otimes (1,0)) + 4((1,0) \otimes (0,1)) + 6((0,1) \otimes (1,0)) + 8((0,1) \otimes (0,1)).
\end{aligned}
$$

It's generally considered poor style to end a proof with an equation. People will often write some concluding sentence, even if it's a bit banal, to end with words instead. ♣

3. Define $T \colon \mathbb{C} \to \mathbb{C}$ and $U \colon \mathbb{C}^2 \to \mathbb{C}^2$ by $T(z) = 2z$ and $U(z,w) = (z + w, z - w)$. Compute

$$(T \otimes U)(5 \otimes (1,1) + 2 \otimes (-1,2))$$

as an element of $\mathbb{C} \otimes \mathbb{C}^2$, written with respect to the basis $\{1 \otimes (1,0),\ 1 \otimes (0,1)\}$.

Solution: We have:

$$
\begin{aligned}
(T \otimes U)(5 \otimes (1,1) + 2 \otimes (-1,2)) &= (T \otimes U)(5 \otimes (1,1)) + (T \otimes U)(2 \otimes (-1,2)) \\
&= T(5) \otimes U(1,1) + T(2) \otimes U(-1,2) \\
&= 10 \otimes (2,0) + 4 \otimes (1,-3) \\
&= 10 \otimes 2(1,0) + 4 \otimes (1,0) + 4 \otimes -3(0,1) \\
&= 20(1 \otimes (1,0)) + 4(1 \otimes (1,0)) + (-12)(1 \otimes (0,1)) \\
&= 24(1 \otimes (1,0)) - 12(1 \otimes (0,1))
\end{aligned}
$$

So there you have it. ♣

4. This problem establishes "Tensor-Hom adjunction" using the universal property. Let R be a commutative ring with 1, and let M, N, and L be left R-modules. A map $g \colon M \times N \to L$ is called R-*bilinear* iff, for all $m_1, m_2 \in M$, all $n_1, n_2 \in N$, and all $a, b, c, d \in R$,

$$g(am_1 + bm_2, cn_1 + dn_2) = acg(m_1, n_1) + adg(m_1, n_2) + bcg(m_2, n_1) + bdg(m_2, n_2).$$

The tensor product $M \otimes_R N$ is a left R-module with $r(m \otimes n) = rm \otimes n = m \otimes rn$, and the map $\iota \colon M \times N \to M \otimes_R N$ defined by $\iota(m,n) := m \otimes n$ is R-bilinear.

Theorem 1 (Universal property of the tensor product)**.** *Let $R, M, N, L,$ and ι be as above. There is a bijection*

$$\left\{ \begin{array}{c} R\text{-bilinear maps} \\ f : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} R\text{-module homomorphisms} \\ F : M \otimes_R N \to L \end{array} \right\}$$

where the correspondance is given via the commutative diagram

$$M \times N \overset{\iota}{\longrightarrow} M \otimes_R N$$

$$\underset{f}{\searrow} \quad \downarrow F$$

$$L$$

Care must be taken when working with tensors, since each $m \otimes n$ represents a coset in some quotient group, and so we may have $m \otimes n = m' \otimes n'$ where $m \neq m'$ or $n \neq n'$. More generally, an element of $M \otimes N$ may be expressible in many different ways as a sum of simple tensors. In particular, care must be taken when defining maps from $M \otimes_R N$ to another group or module, since a map from $M \otimes_R N$ which is described on the generators $m \otimes n$ in terms of m and n is not well defined unless it is shown to be independent of the particular choice of $m \otimes n$ as a coset representative. Theorem 1 is extremely useful in defining homomorphisms on $M \otimes_R N$ since it replaces the often tedious check that maps defined on simple tensors $m \otimes n$ are well defined with a check that a related map defined on ordered pairs (m, n) is bilinear.

Over the course of this problem, "we" (you) will use theorem 1 to establish that

$$\operatorname{Hom}(U \otimes_R V, W) \cong \operatorname{Hom}(U, \operatorname{Hom}(V, W)) \tag{1}$$

for any three R-modules U, V, W. (In (1), Hom means R-module homomorphism, and $\cong$ means R-module isomorphism.)

4. a) Suppose $F : U \otimes_R V \to W$ is an R-module homomorphism. What map f does theorem 1 guarantee exists?

Solution: Theorem 1 guarantees that there exists a unique $f : U \times V \to W$ such that f is R-bilinear. Explicitly, $f = F \circ \iota$, and this is guaranteed to be well defined. ♣

4. b) Let f be an R-bilinear map $f : U \times V \to W$. Define $\varphi_f : U \to \operatorname{Hom}(V, W)$ by $\varphi_f(u)(v) = f(u, v)$. In other words, φ_f is the map $u \mapsto f(u, \cdot)$. Show that $\varphi_f \in \operatorname{Hom}(U, \operatorname{Hom}(V, W))$, i.e. that φ_f is an R-module homomorphism.

Solution: We must check that $\varphi_f(au_1 + bu_2) = a\varphi_f(u_1) + b\varphi_f(u_2)$ for all $u_1, u_2 \in U$ and all $a, b \in R$. For every $v \in V$,

$$\begin{aligned} \varphi_f(au_1 + bu_2)(v) &= f(au_1 + bu_2, v) && \text{(by definition of } \varphi_f) \\ &= af(u_1, v) + bf(u_2, v) && \text{(because } f \text{ is bilinear)} \\ &= a\varphi_f(u_1)(v) + b\varphi_f(u_2)(v) && \text{(by definition of } \varphi_f). \end{aligned}$$

The calculation above shows that $\varphi_f(au_1 + bu_2) = a\varphi_f(u_1) + b\varphi_f(u_2)$, since these maps are equal pointwise. ♣

4. c) Show that $\varphi : \operatorname{Hom}(U \otimes_R V, W) \to \operatorname{Hom}(U, \operatorname{Hom}(V, W))$ sending $F \mapsto \varphi_f$ as above is an R-module homomorphism.

Solution: We must check that $\varphi(aF_1 + bF_2) = a\varphi(F_1) + b\varphi(F_2)$ for all $F_1, F_2 \in \operatorname{Hom}(U \otimes_R V, W)$ and all $a, b \in R$. By theorem 1, there exist unique R-bilinear maps $f_1, f_2 : U \times V \to W$ such that $f_1 = F_1 \circ \iota$ and $f_2 = F_2 \circ \iota$. Let $f : U \times V \to W$ be defined

$$f(u, v) := af_1(u, v) + bf_2(u, v).$$

Since f_1 and f_2 are R-bilinear, f is as well. Then, by theorem 1, there exists a unique $F \in \mathrm{Hom}(U \otimes_R V, W)$ such that $f = F \circ \iota$. Because F_1 and F_2 are R-module homomorphisms, for all $\xi \in U \otimes_R V$,

$$(aF_1 + bF_2)(\xi) = a\big(F_1(\xi)\big) + b\big(F_2(\xi)\big).$$

Since $(aF_1 + bF_2) \circ \iota = f$, by uniqueness we must have $F = aF_1 + bF_2$.

The definition of f is precisely

$$f(u, v) = a\varphi_{f_1}(u)(v) + b\varphi_{f_2}(u)(v) = a\varphi(F_1)(u)(v) + b\varphi(F_2)(u)(v).$$

On the other hand, $F = aF_1 + bF_2$ gives

$$f(u, v) = \varphi_f(u)(v) = \varphi(F)(u)(v) = \varphi(aF_1 + bF_2)(u)(v).$$

We conclude $\varphi(aF_1 + bF_2) = a\varphi(F_1) + b\varphi(F_2)$, as they are equal pointwise. ♣

4. d) Let $\psi \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$. Show that the map $g_\psi : U \times V \to W$ defined by $g_\psi(u, v) = \psi(u)(v)$ is R-bilinear.

Solution: The following calculation verifies linearity in the V-argument:

$$
\begin{aligned}
g_\psi(u, cv_1 + dv_2) &= \psi(u)(cv_1 + dv_2) && \text{(by definition of } g_\psi) \\
&= c\psi(u)(v_1) + d\psi(u)(v_2) && \text{(because } \psi(u) \in \mathrm{Hom}(V, W) \text{ is an } R\text{-module homomorphism)} \\
&= cg_\psi(u, v_1) + dg_\psi(u, v_2) && \text{(by definition of } g_\psi).
\end{aligned}
$$

The following calculation verifies linearity in the U-argument:

$$
\begin{aligned}
g_\psi(au_1 + bu_2, v) &= \psi(au_1 + bu_2)(v) && \text{(by definition of } g_\psi) \\
&= [a\psi(u_1) + b\psi(u_2)](v) && \text{(because } \psi : U \to \mathrm{Hom}(V, W) \text{ is an } R\text{-module homomorphism)} \\
&= a\psi(u_1)(v) + b\psi(u_2)(v) && \text{(by definition of the } R\text{-module structure on } \mathrm{Hom}(U, \mathrm{Hom}(V, W))) \\
&= ag_\psi(u_1, v) + bg_\psi(u_2, v) && \text{(by definition of } g_\psi).
\end{aligned}
$$

The map g_ψ is therefore R-bilinear. ♣

4. e) Applied to g_ψ above, what map does theorem 1 guarantee the existence of?

Solution: Theorem 1 guarantees that there exists a unique R-module homomorphism $G_\psi : U \otimes_R V \to W$ such that $g_\psi = G_\psi \circ \iota$. ♣

4. f) Prove that $\mathrm{Hom}(U \otimes_R V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$.

Solution: In the previous parts of this problem, we've established the existence of well defined maps

$$\varphi : \mathrm{Hom}(U \otimes_R V, W) \to \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$
$$F \mapsto \varphi_f$$

and

$$G : \mathrm{Hom}(U, \mathrm{Hom}(V, W)) \to \mathrm{Hom}(U \otimes_R V, W)$$
$$\psi \mapsto G_\psi.$$

Moreover, we proved that φ was an R-module homomorphism. Now we'll show that φ and G are mutual inverses. This will prove that φ is in fact an isomorphism.

Let's first calculate $\varphi \circ G$. Let $\psi \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$. Then $g_\psi(u, v) := \psi(u)(v) : U \times V \to W$ is bilinear, so lifts uniquely to $G_\psi \in \mathrm{Hom}(U \otimes_R V, W)$. Let us now apply φ to G_ψ. The image g of G_ψ under the bijection of theorem 1 is the unique bilinear map satisfying $g = G_\psi \circ \iota : U \times V \to W$. Since G_ψ was the unique lift of g_ψ, by construction we have $G_\psi \circ \iota = g_\psi$. Hence, as bilinear maps on $U \times V$, we have the equalities

$$\varphi(G_\psi)(u)(v) = g(u, v) = g_\psi(u, v) = \psi(u)(v).$$

This implies that $\varphi(G_\psi) = \psi$, as they agree pointwise.

Now let's calculate the other direction, $G \circ \varphi$. Let $F \in \mathrm{Hom}(U \otimes_R V, W)$, and let $F|_\iota$ denote the restriction of F to the image of $U \times V \overset{\iota}{\hookrightarrow} U \otimes_R V$. Then $F|_\iota : U \times V \to W$ is a bilinear map, and $\varphi(F) \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$ is the map satisfying $\varphi(F)(u)(v) = F|_\iota(u, v)$. Now we apply G to $\varphi(F)$. The bilinear map associated to $\varphi(F)$ is $g_{\varphi(F)}(u, v) = F|_\iota(u, v)$. By theorem 1, there is a unique lift $G_{\varphi(F)}$ of $g_{\varphi(F)}$ to an R-module homomorphism, and since F is a lift of $g_{\varphi(F)} = F|_\iota$, we find that $G(\varphi(F)) = F$. ♣