PMATH 445/745 — Assignment 3

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1. Consider the vector space $\mathbb{C} \otimes \mathbb{C}^2$. Is $2 \otimes (1,1) = 1 \otimes (2,2)$? Is $1 \otimes (2,2) = (2,2)$?

2. In $\mathbb{C}^2 \otimes \mathbb{C}^2$, write $(1,2) \otimes (3,4)$ as a linear combination of the basis

$$\{(1,0) \otimes (1,0),\ (0,1) \otimes (1,0),\ (1,0) \otimes (0,1),\ (0,1) \otimes (0,1)\}.$$

3. Define $T\colon \mathbb{C} \to \mathbb{C}$ and $U\colon \mathbb{C}^2 \to \mathbb{C}^2$ by $T(z) = 2z$ and $U(z,w) = (z+w, z-w)$. Compute

$$(T \otimes U)(5 \otimes (1,1) + 2 \otimes (-1,2))$$

as an element of $\mathbb{C} \otimes \mathbb{C}^2$, written with respect to the basis $\{1 \otimes (1,0),\ 1 \otimes (0,1)\}$.

4. This problem establishes "Tensor-Hom adjunction" using the universal property. Let R be a commutative ring with 1, and let M, N, and L be left R-modules. A map $g : M \times N \to L$ is called R-*bilinear* iff, for all $m_1, m_2 \in M$, all $n_1, n_2 \in N$, and all $a, b, c, d \in R$,

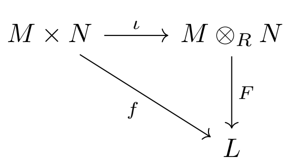
$$g(am_1 + bm_2, cn_1 + dn_2) = acg(m_1, n_1) + adg(m_1, n_2) + bcg(m_2, n_1) + bdg(m_2, n_2).$$

The tensor product $M \otimes_R N$ is a left R-module with $r(m \otimes n) = rm \otimes n = m \otimes rn$, and the map $\iota : M \times N \to M \otimes_R N$ defined by $\iota(m,n) \coloneqq m \otimes n$ is R-bilinear.

Theorem 1 (Universal property of the tensor product)**.** *Let R, M, N, L, and ι be as above. There is a bijection*

$$\left\{ \begin{array}{c} R\text{-bilinear maps} \\ f : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} R\text{-module homomorphisms} \\ F : M \otimes_R N \to L \end{array} \right\}$$

where the correspondance is given via the commutative diagram



Care must be taken when working with tensors, since each $m \otimes n$ represents a coset in some quotient group, and so we may have $m \otimes n = m' \otimes n'$ where $m \neq m'$ or $n \neq n'$. More generally, an element of $M \otimes N$ may be expressible in many different ways as a sum of simple tensors. In particular, care must be taken when defining maps from $M \otimes_R N$ to another group or module, since a map from $M \otimes_R N$ which is described on the generators $m \otimes n$ in terms of m and n is not well defined unless it is shown to be independent of the particular choice of $m \otimes n$ as a coset representative. Theorem 1 is extremely useful in defining homomorphisms on $M \otimes_R N$ since it replaces the often tedious check that maps defined on simple tensors $m \otimes n$ are well defined with a check that a related map defined on ordered pairs (m,n) is bilinear.

Over the course of this problem, "we" (you) will use theorem 1 to establish that

$$\mathrm{Hom}(U \otimes_R V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W)) \tag{1}$$

for any three R-modules U, V, W. (In (1), Hom means R-module homomorphism, and $\cong$ means R-module isomorphism.)

4. a) Suppose $F : U \otimes_R V \to W$ is an R-module homomorphism. What map f does theorem 1 guarantee exists?

4. b) Let f be an R-bilinear map $f : U \times V \to W$. Define $\varphi_f : U \to \mathrm{Hom}(V, W)$ by $\varphi_f(u)(v) = f(u, v)$. In other words, φ_f is the map $u \mapsto f(u, \cdot)$. Show that $\varphi_f \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$, i.e. that φ_f is an R-module homomorphism.

4. c) Show that $\varphi : \mathrm{Hom}(U \otimes_R V, W) \to \mathrm{Hom}(U, \mathrm{Hom}(V, W))$ sending $F \mapsto \varphi_f$ as above is an R-module homomorphism.

4. d) Let $\psi \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$. Show that the map $g_\psi : U \times V \to W$ defined by $g_\psi(u, v) = \psi(u)(v)$ is R-bilinear.

4. e) Applied to g_ψ above, what map does theorem 1 guarantee the existence of?

4. f) Prove that $\mathrm{Hom}(U \otimes_R V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$.