PMATH 445/745 — Assignment 2 solutions

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1. An important step in the proof of Maschke's theorem is the following lemma.

Lemma 1. *Let (V, ρ) be a complex representation of a finite group G. Let W be a G-invariant subspace of V. Then there exists a subspace W^c of V such that W^c is G-invariant and $V = W \oplus W^c$ as vector spaces.*

In class, we proved lemma 1 by constructing a G-invariant inner product and taking W^c to be the orthogonal complement of W. This question asks you to give a different proof of lemma 1.

Let U be an arbitrary subspace of V such that $V = W \oplus U$ as vector spaces. Let $\pi : V \to W$ be the corresponding projection onto W. I.e., every vector $v \in V$ can be written uniquely as $v = w + u$ with $w \in W$ and $u \in U$, and $\pi(v) := w$. Now define

$$\pi^c := \frac{1}{\#G} \sum_{g \in G} \rho(g)\pi\rho(g)^{-1}.$$

Show that π^c is also a projection $V \to W$, and that the corresponding complement W^c is G-invariant. Conclude that lemma 1 follows.

Solution: For $x \in W$, we have $\rho(g)^{-1}x \in W$ also, since W is G-invariant by assumption. Moreover, π, when restricted to W, is the identity map. Hence, $\pi\rho(g)^{-1}x = \rho(g)^{-1}x$ for every $x \in W$, and

$$\begin{aligned}
\pi^c x &= \frac{1}{\#G} \sum_{g \in G} \rho(g)\pi\rho(g)^{-1}x \\
&= \frac{1}{\#G} \sum_{g \in G} \rho(g)\rho(g)^{-1}x \\
&= x.
\end{aligned}$$

Now we show that π^c commutes with $\rho(h)$ for every $h \in G$:

$$\begin{aligned}
\rho(h)\pi^c\rho(h)^{-1} &= \frac{1}{\#G} \sum_{g \in G} \rho(h)\rho(g)\pi\rho(g)^{-1}\rho(h)^{-1} \\
&= \frac{1}{\#G} \sum_{g \in G} \rho(hg)\pi\rho(gh)^{-1} \\
&= \frac{1}{\#G} \sum_{g \in G} \rho(hg)\pi\rho((hg)^{-1}) \\
&= \frac{1}{\#G} \sum_{g' \in G} \rho(g')\pi\rho(g')^{-1} \\
&= \pi^c.
\end{aligned}$$

(In the penultimate step, the change of variables $g' = hg$; left multiplication is a group automorphism.)

Suppose $x \in \ker \pi^c$. Then $0 = \rho(g)\pi^c x = \pi^c\rho(g)x$, i.e. $\rho(g)x \in \ker \pi^c$ also. This proves that $\ker \pi^c =: W^c$ is G-invariant. Hence $V \cong \operatorname{Im} \pi^c \oplus \ker \pi^c = W \oplus W^c$ with W and W^c each G-invariant. ♣

2. Prove the following converse to Schur's lemma. Let (V, ρ) be a complex representation of a finite group G with positive dimension. Suppose every $\varphi \in \mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$ is of the form $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the identity matrix. Then V is irreducible.

Solution: Suppose that V is a reducible representation of G with positive dimension. Being reducible, there exist two proper subrepresentations W_1 and W_2 with positive dimension, such that $V \cong W_1 \oplus W_2$. Consider the projection map $\pi_1 : V \to W_1$. This map is G-equivariant. Its kernel is $W_2 \neq 0$, so it is not a scalar multiple of the identity. We have proved the contrapositive of the statement from the problem: if V is reducible, then there exists an endomorphism of V which is not a scalar multiple of the identity. Hence, if every endomorphism is a scalar multiple of the identity, then V cannot be reducible. $\clubsuit$

3. Let G be a finite abelian group. Prove that every irreducible complex representation of G is 1-dimensional.

Solution: Let (V, ρ) be an irreducible representation of G. For all $g \in G$, the linear map $\rho(g)$ is an endomorphism of the vector space V. Let's verify that $\rho(g) \in \mathrm{End}_G(V)$, i.e. that it is G-equivariant. For any $h \in G$, consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ {\scriptstyle\rho(h)}\downarrow & & \downarrow{\scriptstyle\rho(h)} \\ V & \xrightarrow{\rho(g)} & V. \end{array}$$

The top and right arrows compose to give the map $\rho(h)\rho(g) = \rho(hg)$ from V to V, while the left and bottom arrows compose to give $\rho(gh)$. Since G is abelian by assumption, $\rho(gh) = \rho(hg)$, i.e. the diagram commutes.

By Schur's lemma, every $\varphi \in \mathrm{End}_G(V)$, and hence every $\rho(g)$, is a scalar multiple of the identity. Every vector subspace W of V is invariant under scalar multiples of the identity, so every vector subspace of V is in fact a subrepresentation. By definition, V is irreducible if and only if its only G-invariant proper subspace is the 0-dimensional subspace. Hence V must be 1-dimensional. $\clubsuit$

4. Exhibit all finite-dimensional (not necessarily irreducible) complex representations of $\mathbb{Z}/n\mathbb{Z}$, for arbitrary $n \in \mathbb{Z}_{>0}$. Make sure to decide which are inequivalent.

Solution: Let's begin by classifying all irreducible representations of $\mathbb{Z}/n$. By problem 1, these irreducible representations are all 1-dimensional, i.e. homomorphisms $\rho : \mathbb{Z}/n \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. For every $x \in \mathbb{Z}/n$, we have $nx = 0$. Since ρ is a homomorphism, $\rho(nx) = \rho(x)^n$ and $\rho(0) = 1$. Hence $\rho(x)$ is an n^{th} root of unity. Explicitly, let $\mu_n := \exp\left(\frac{2\pi i}{n}\right)$, so that $\{1, \mu_n, \mu_n^2, \ldots, \mu_n^{n-1}\}$ is the set of all complex numbers z with the property that $z^n = 1$. Since $\mathbb{Z}/n$ is cyclic, the representation ρ is determined entirely by its value on the generator $1 \in Z/n$. Hence there are n irreducible representations of $\mathbb{Z}/n$:

$$\rho_a(x) = \mu_n^{ax} = \exp\left(\frac{2\pi a x}{n}\right)$$

with $a \in \{0, 1, \ldots, n-1\}$. By inspection, we see that these are indeed all homomorphisms, i.e. representations.

The representations ρ_a are pairwise inequivalent: suppose $\varphi\rho_a = \rho_b\varphi$ as maps $\mathbb{Z}/n \to \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is abelian, $\rho_b\varphi = \varphi\rho_b$, and since the image of φ is in $\mathbb{C}^\times$, $\varphi\rho_a = \varphi\rho_b$ implies $\rho_a = \rho_b$.

By Maschke's theorem, every finite-dimensional representation (V, ρ) of $\mathbb{Z}/n$ is isomorphic to a direct sum of irreducible representations, all of which we've just described. I.e. there exist non-negative integers m_a, for each $a \in \{0, 1, \ldots, n-1\}$ "=" $\mathbb{Z}/n$, such that

$$\rho \cong \bigoplus_{a \in \mathbb{Z}/n} m_a \rho_a,$$

i.e.

$$\rho \cong \begin{bmatrix} \rho_0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \rho_0 & & & & & & \\ & & & \rho_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \rho_1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \rho_{n-1} & \\ & & & & & & & & \ddots \\ & & & & & & & & & \rho_{n-1} \end{bmatrix} \begin{matrix} \left.\right\} m_0 \text{ copies} \\ \left.\right\} m_1 \text{ copies} \\ \vdots \\ \left.\right\} m_{n-1} \text{ copies.} \end{matrix}$$

Two such representations, say $\rho \cong \bigoplus_{a\in\mathbb{Z}/n} m_a\rho_a$ and $\rho' \cong \bigoplus_{a\in\mathbb{Z}/n} m'_a\rho_a$ are isomorphic iff all of the multiplicities match, i.e. $m_0 = m'_0$, $m_1 = m'_1$, $\ldots$, $m_{n-1} = m'_{n-1}$. Note that this description also captures the 0-dimensional representation, with $m_a = 0$ for all $a \in \mathbb{Z}/n$. The 0-dimensional representation is not irreducible, though calling it reducible is a little weird, like calling the integer 1 "composite". ♣

5. Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$. Prove that there is a matrix P such that for every $M \in G$, the matrix $P^{-1}MP$ is diagonal. That is, prove that the matrices in G are simultaneously diagonalizable. [Note: I have given the TA permission to mock any solution to this question that does not involve representation theory.]

Solution: Since G is abelian, every irreducible representation of G has dimension one. Let $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ be the inclusion homomorphism – this is a representation of G! So it must be isomorphic to the sum of n one-dimensional representations: $\rho \cong \tau_1 \oplus \ldots \oplus \tau_n$. Write $\mathbb{C}^n = V_1 \oplus \ldots \oplus V_n$ as the sum of G-invariant subspaces of dimension one, and for each i let $\mathbf{v}_i$ be a basis of V_i. Since V_i is G-invariant, we have $[\rho(g)](\mathbf{v}_i) \in V_i$ for each i. This means, for each i, and for each $g \in G$, there is some $\lambda_i(g) \in \mathbb{C}$ such that $[\rho(g)](\mathbf{v}_i) = \lambda_i(g)\mathbf{v}_i$. In other words, v_i is an eigenvector of $\rho(g)$ for every $g \in G$ (although the corresponding eigenvalues may be different for different g).

Let P be the n by n matrix whose ith column is v_i, written in the standard basis. Then for each $g \in G$, the columns of P are all eigenvectors of the matrix $\rho(g) = M$, so the matrix $P^{-1}MP$ is diagonal for all $M \in G$. ♣