PMATH 445/745 — Assignment 2

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Due 2026/01/21

1. An important step in the proof of Maschke's theorem is the following lemma.

Lemma 1. *Let (V, ρ) be a complex representation of a finite group G. Let W be a G-invariant subspace of V. Then there exists a subspace W^c of V such that W^c is G-invariant and $V = W \oplus W^c$ as vector spaces.*

In class, we proved lemma 1 by constructing a G-invariant inner product and taking W^c to be the orthogonal complement of W. This question asks you to give a different proof of lemma 1.

Let U be an arbitrary subspace of V such that $V = W \oplus U$ as vector spaces. Let $\pi : V \to W$ be the corresponding projection onto W. I.e., every vector $v \in V$ can be written uniquely as $v = w + u$ with $w \in W$ and $u \in U$, and $\pi(v) \coloneqq w$. Now define

$$\pi^c \coloneqq \frac{1}{\#G} \sum_{g \in G} \rho(g) \pi \rho(g)^{-1}.$$

Show that π^c is also a projection $V \to W$, and that the corresponding complement W^c is G-invariant. Conclude that lemma 1 follows.

2. Prove the following converse to Schur's lemma. Let (V, ρ) be a complex representation of a finite group G with positive dimension. Suppose every $\varphi \in \mathrm{End}_G(V) \coloneqq \mathrm{Hom}_G(V, V)$ is of the form $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the identity matrix. Then V is irreducible.

3. Let G be a finite abelian group. Prove that every irreducible complex representation of G is 1-dimensional.

4. Exhibit all finite-dimensional (not necessarily irreducible) complex representations of $\mathbb{Z}/n\mathbb{Z}$, for arbitrary $n \in \mathbb{Z}_{>0}$. Make sure to decide which are inequivalent.

5. Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$. Prove that there is a matrix P such that for every $M \in G$, the matrix $P^{-1}MP$ is diagonal. That is, prove that the matrices in G are simultaneously diagonalizable. [Note: I have given the TA permission to mock any solution to this question that does not involve representation theory.]

Hints:

1. This is the proof Serre gives in §1.3 of his book *Linear representations of finite groups*. Alternatively, this is done in Dummit and Foote §18.1.

2. Suppose that V is reducible. Apply Maschke's theorem. Consider projection onto one of the summands.

3. Suppose (V, ρ) is an irreducible representation of G. Start by using Schur's lemma to deduce that, for every $g \in G$, the linear map $\rho(g) : V \to V$ is a scalar multiple of the identity.

4. Classify all irreducible representations, and then use Maschke's theorem.

5. Consider the inclusion homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$; this is a representation of G.