PMATH 445/745 — Assignment 1 solutions

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1. a) Let $G = \mathbb{R}$ under addition, and let $\rho\colon G \to \mathrm{GL}_2(\mathbb{C})$ be defined by

$$\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Prove that ρ is a representation of $\mathbb{R}$. That is, prove that ρ is an action of G on $\mathbb{C}^2$, and that $\rho(g)$ is a linear transformation for all g. (OK, that second thing is obvious in this case, but it's part of what you need to show in general.)

Solution: By the parenthetical remark, all we need to show is that ρ is an action of G on $\mathbb{C}^2$. This amounts to showing that $\rho(x + y) = \rho(x)\rho(y)$, which is easily verified by a quick matrix calculation. □

1. b) Let $G = \mathbb{Z}/7\mathbb{Z}$, and let $\rho\colon G \to \mathrm{GL}_1(\mathbb{C})$ be defined by $\rho(n) = e^{2\pi i n/7}$. (You may assume that this is well defined.) Prove that ρ is a representation of G.

Solution: It's clear that $\rho(n)$ is a nonzero complex number for all n, which is to say that $\rho(n)$ is a well defined element of $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. Thus, it remains only to check that ρ is an action of G on $\mathbb{C}$. This amounts to checking that $\rho(n + m) = \rho(n)\rho(m)$, which is clear. □

2. a) For the representation of problem 1a, let $W = \mathrm{span}\{(1, 0)\} \subseteq \mathbb{C}^2$. Prove that $\rho|_W$ is a subrepresentation of ρ. Equivalently, prove that W is a G-invariant subspace.

Solution: We'll show that W is G-invariant. From class, it suffices to show that $[\rho(x)](1, 0) \in W$ for all x. But a simple matrix calculation shows that $[\rho(x)](1, 0) = (1, 0)$ for all x, so we're done. □

2. b) Let $G = S_3$, and let $\rho\colon G \to \mathrm{GL}_3(\mathbb{C})$ be defined by $[\rho(\sigma)](z_1, z_2, z_3) = (z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)})$. (Remember that $\rho(\sigma)$ is a linear transformation, so this formula is just telling you which linear transformation it is.) You may assume that this is a representation of G. Let $W = \{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0\}$ be a subspace of $\mathbb{C}^3$. Prove that W is G-invariant; in other words, prove that $\rho|_W$ is a subrepresentation of ρ.

Solution: It suffices to show that $[\rho(\sigma)](\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. But $\rho(\sigma)$ just permutes the coordinate of $\mathbf{w}$, so if they add up to 0 before they're permuted, then they add up to 0 after they're permuted, too. So $\rho(\sigma)(\mathbf{w}) \in W$, as desired. □

3. Let $G = S_3$, and let $\rho\colon G \to \mathrm{GL}_3(\mathbb{C})$ be the representation from problem 2b. Let $\tau\colon G \to \mathrm{GL}_1(\mathbb{C})$ be the representation $\tau(\sigma) = 1$ for all σ — that is, τ is the trivial representation of G. Define $T\colon \mathbb{C}^3 \to \mathbb{C}$ by $T(z_1, z_2, z_3) = z_1 + z_2 + z_3$. Show that T is a morphism from ρ to τ.

Solution: Since it's clear that T is a linear transformation, we just need to check the morphism property that $T \circ \rho(\sigma) = \tau(\sigma) \circ T$. Thus, pick any $\sigma \in S_3$, and any $(z_1, z_2, z_3) \in \mathbb{C}^3$, and compute:

$$T(\rho(\sigma)(z_1, z_2, z_3)) = T(z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}) = z_1 + z_2 + z_3$$

since the sum of the coordinates is the same before and after permuting the coordinates. We further compute:

$$[\tau(\sigma)](T(z_1, z_2, z_3)) = [\tau(\sigma)](z_1 + z_2 + z_3) = z_1 + z_2 + z_3$$

since σ acts trivially on $\mathbb{C}$ via τ. These two formulae are equal, so we conclude that T is a morphism, as desired. $\square$

4. Let $G = S_3$, and let ρ and τ be as in question 3. Let W be the subspace from problem 2b. Prove that $\rho \cong \rho|_W \oplus \tau$. [Hint: $W' = \{(z, z, z) : z \in \mathbb{C}\}$ is also G-invariant.]

Solution: Well, if we're going to show that two representations are isomorphic, we'd better have an isomorphism ready. This isomorphism (call it T) is supposed to be from $W \oplus \mathbb{C}$ to $\mathbb{C}^3$, so we have to define $T(\mathbf{w}, z) \in \mathbb{C}^3$ for $\mathbf{w} \in W$ and $z \in \mathbb{C}$.

Well, $\mathbf{w}$ is already an element of $\mathbb{C}^3$, so that's a good start. And the hint gives us an idea of how to deal with z: we know that morphisms take G-invariant subspaces to G-invariant subspaces, so the set of vectors of the form $T(0, z)$ should be a G-invariant subspace of $\mathbb{C}^3$. Thus, with all this inspiration at hand, we define:

$$T(\mathbf{w}, z) = \mathbf{w} + (z, z, z)$$

This is certainly a linear transformation. It has to be onto, because $\operatorname{Im} T$ contains W and $(1, 1, 1) \notin W$, so it contains three linearly independent vectors (two of them a basis of W, and $(1, 1, 1)$ the third), and hence is three-dimensional and thus equal to $\mathbb{C}^3$. But T is a linear transformation between two three-dimensional vector spaces, so since it's onto, it must be an isomorphism of vector spaces.

Thus, all we need to do is check the morphism property, and we're in the money. Let $\sigma \in G$ be any element, $(\mathbf{w}, z) \in W \oplus \mathbb{C}$ any vector, and compute:

$$\begin{aligned} T([(\rho|_W \oplus \tau)(\sigma)](\mathbf{w}, z)) &= T([\rho(\sigma)](\mathbf{w}), [\tau(\sigma)](z)) \\ &= [\rho(\sigma)](\mathbf{w}) + (z, z, z) \end{aligned}$$

since τ is the trivial representation, so $[\tau(\sigma)](z) = z$.

On the other hand, we also compute:

$$\begin{aligned} [(\rho)(\sigma)](T(\mathbf{w}, z)) &= [\rho(\sigma)](\mathbf{w} + (z, z, z)) \\ &= [\rho(\sigma)](\mathbf{w}) + [\rho(\sigma)](z, z, z) \\ &= [\rho(\sigma)](\mathbf{w}) + (z, z, z) \end{aligned}$$

and so we're done. $\square$

5. Let $\rho \colon G \to \mathrm{GL}(V)$ be a representation of a finite group G. Prove that for any element $g \in G$, all the eigenvalues λ of the linear transformation $\rho(g)$ satisfy $|\lambda| = 1$.

Solution: Since G is finite, any element g has finite order n. Since ρ is a homomorphism, it follows that $\rho(g)^n = I$. Say that λ is an eigenvalue of $\rho(g)$. Then there is some nonzero vector $\mathbf{v}$ such that $[\rho(g)](\mathbf{v}) = \lambda(\mathbf{v})$. Applying $\rho(g)$ $n - 1$ more times to both sides, we get $\mathbf{v} = [\rho(g)^n](\mathbf{v}) = \lambda^n \mathbf{v}$. Thus, $\lambda^n = 1$, so in particular $|\lambda| = 1$. $\square$