PMATH 445/745 — Assignment 1

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Due 2026/01/14

1. a) Let $G = \mathbb{R}$ under addition, and let $\rho\colon G \to \mathrm{GL}_2(\mathbb{C})$ be defined by

$$\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Prove that ρ is a representation of $\mathbb{R}$. That is, prove that ρ is an action of G on $\mathbb{C}^2$, and that $\rho(g)$ is a linear transformation for all g. (OK, that second thing is obvious in this case, but it's part of what you need to show in general.)

1. b) Let $G = \mathbb{Z}/7\mathbb{Z}$, and let $\rho\colon G \to \mathrm{GL}_1(\mathbb{C})$ be defined by $\rho(n) = e^{2\pi i n/7}$. (You may assume that this is well defined.) Prove that ρ is a representation of G.

2. a) For the representation of problem 1a, let $W = \mathrm{span}\{(1,0)\} \subseteq \mathbb{C}^2$. Prove that $\rho|_W$ is a subrepresentation of ρ. Equivalently, prove that W is a G-invariant subspace.

2. b) Let $G = S_3$, and let $\rho\colon G \to \mathrm{GL}_3(\mathbb{C})$ be defined by $[\rho(\sigma)](z_1, z_2, z_3) = (z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)})$. (Remember that $\rho(\sigma)$ is a linear transformation, so this formula is just telling you which linear transformation it is.) You may assume that this is a representation of G. Let $W = \{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0\}$ be a subspace of $\mathbb{C}^3$. Prove that W is G-invariant; in other words, prove that $\rho|_W$ is a subrepresentation of ρ.

3. Let $G = S_3$, and let $\rho\colon G \to \mathrm{GL}_3(\mathbb{C})$ be the representation from problem 2b. Let $\tau\colon G \to \mathrm{GL}_1(\mathbb{C})$ be the representation $\tau(\sigma) = 1$ for all σ — that is, τ is the trivial representation of G. Define $T\colon \mathbb{C}^3 \to \mathbb{C}$ by $T(z_1, z_2, z_3) = z_1 + z_2 + z_3$. Show that T is a morphism from ρ to τ.

4. Let $G = S_3$, and let ρ and τ be as in question 3. Let W be the subspace from problem 2b. Prove that $\rho \cong \rho|_W \oplus \tau$. [Hint: $W' = \{(z, z, z) : z \in \mathbb{C}\}$ is also G-invariant.]

5. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a finite group G. Prove that for any element $g \in G$, all the eigenvalues λ of the linear transformation $\rho(g)$ satisfy $|\lambda| = 1$.