Cutting planes from extended LP formulations

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University of Waterloo

Lecture 1 outline

▷ Integer programming basics
- Branch-and-bound
- Cutting planes
- Split cuts
- Generalizations

▷ Overview of extended formulations

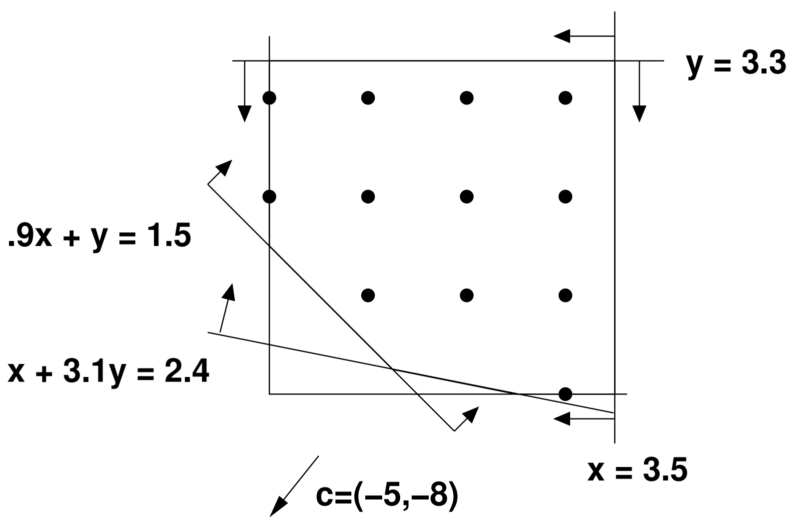
▷ Cutting planes from extended formulations
- General MIPs (Lecture 1)
- 0-1 MIPs (Lecture 2)
- Bounded MIPs (Lecture 3)

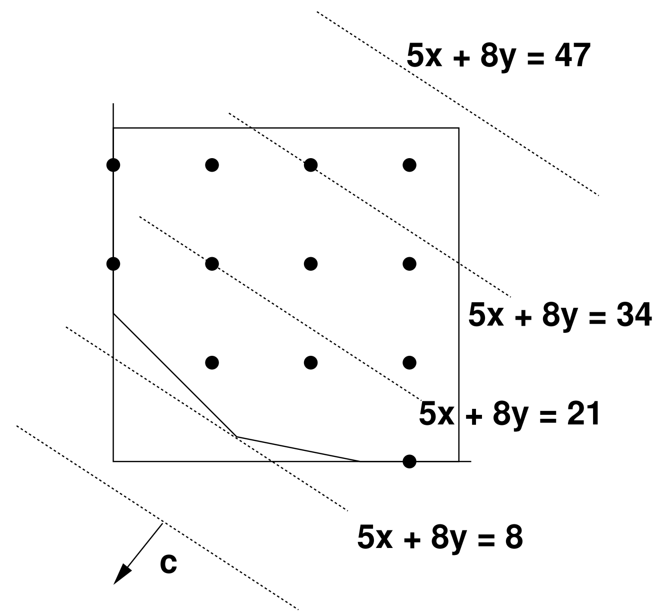
Integer programming

$\min 5x + 8y$ subject to

$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$

$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3, \;\; x, y$ integral



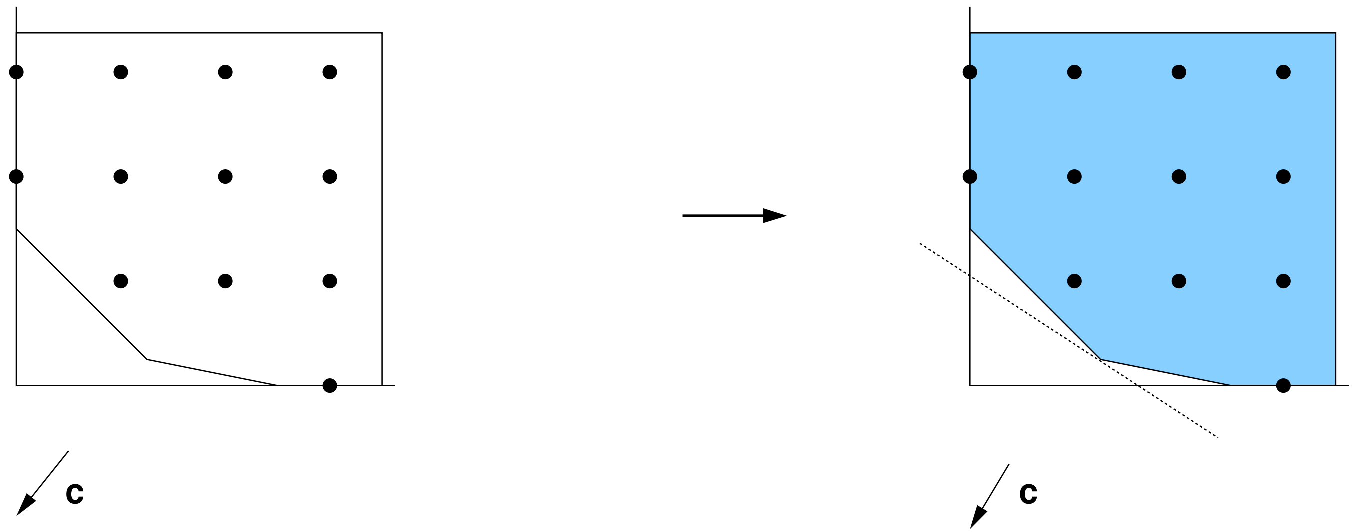


LP relaxation

$\min 5x + 8y$ subject to

$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$

$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3$



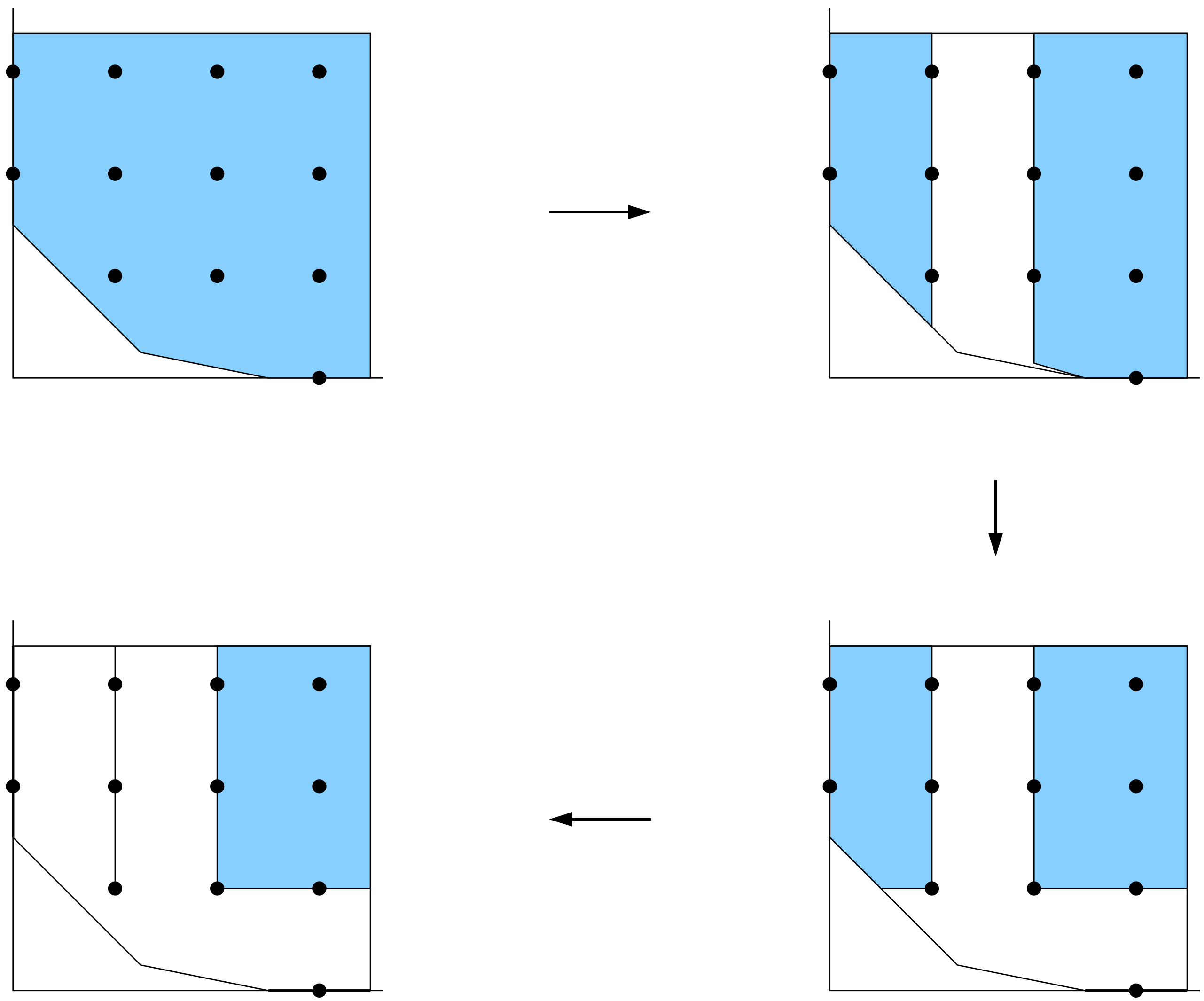
LP relaxation + branching

$$\min 5x + 8y \text{ subject to}$$
$$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$$
$$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3$$

$$\min 5x + 8y \text{ subject to}$$
$$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$$
$$0 \leq x \leq 1, \;\; 0 \leq y \leq 3.3$$

$$\min 5x + 8y \text{ subject to}$$
$$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$$
$$2 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3$$

Branch and bound



cplex-log2.txt

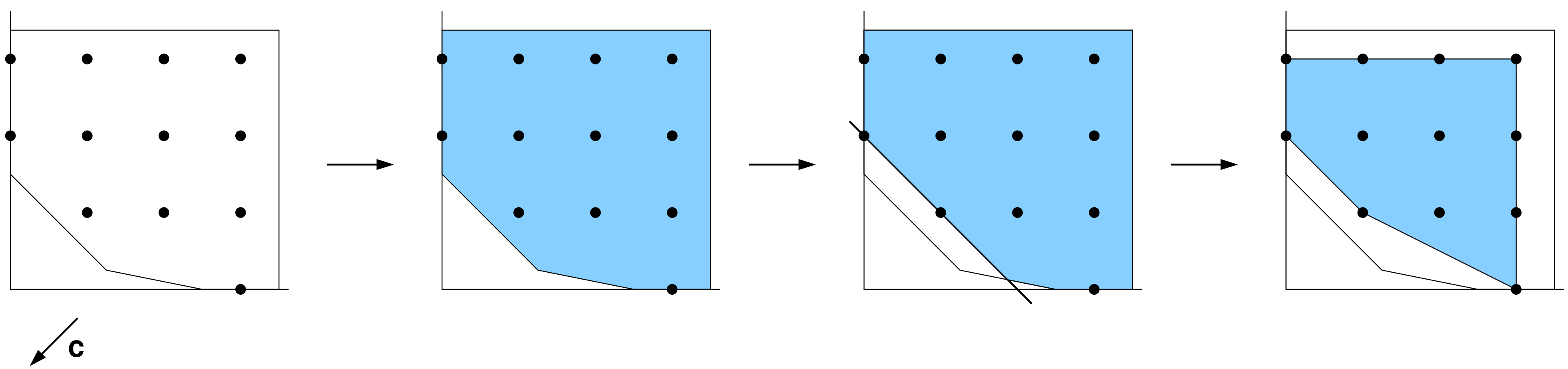
```
Problem 'pp08a' read.
....
Reduced MIP has 133 rows, 234 columns, and 468 nonzeros.
Reduced MIP has 64 binaries, 0 generals, 0 SOSs, and 0 indicators.
....
        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

*     0+    0                       27080.0000                      77     ---
      0     0     2748.3452    51   27080.0000     2748.3452      77   89.85%
*     0+    0                       14300.0000     2748.3452      77   80.78%
*     0+    0                        7950.0000     2748.3452      77   65.43%
      0     2     2748.3452    51    7950.0000     2748.3452      77   65.43%
Elapsed real time =   0.03 sec. (tree size =  0.00 MB, solutions = 3)
*   100+   94                        7860.0000     2848.3452     428   63.76%
*   100+   90                        7640.0000     2848.3452     428   62.72%
   2862  2111     6556.5595    28    7640.0000     3981.3452    9387   47.89%
   6557  5339     6788.4524    21    7640.0000     4254.2976   20447   44.32%
* 10017+ 8320                        7630.0000     4369.3452   30879   42.73%
* 10017+ 8067                        7520.0000     4369.3452   30879   41.90%
* 10017+ 8047                        7510.0000     4369.3452   30879   41.82%
* 10017+ 7947                        7480.0000     4369.3452   30879   41.59%
  10017  7949     7152.1667    16    7480.0000     4369.3452   30879   41.59%
....
 467260 381944     6279.9524    23     7480.0000     5330.2500  1336479   28.74%
Elapsed real time =  76.80 sec. (tree size = 86.82 MB, solutions = 9)
 488008 398616     6870.4881    16     7480.0000     5340.1310  1393871   28.61%
 508767 415262     7018.3810    21     7480.0000     5350.3452  1451784   28.47%
 529510 431893     5359.7738    26     7480.0000     5359.7738  1509653   28.35%
 550267 448498     5819.7024    30     7480.0000     5368.3929  1567040   28.23%
 570955 465047     7091.7738    13     7480.0000     5377.4405  1624524   28.11%
....
 760995 616110     6726.4405    24     7480.0000     5445.6548  2152219   27.20%
 778020 629628     6542.1548    30     7480.0000     5451.3214  2199840   27.12%
 794094 642371     6215.4881    25     7480.0000     5456.2024  2244463   27.06%
 811975 656559        cutoff           7480.0000     5461.4405  2294026   26.99%
 829297 670288     6740.9167    28     7480.0000     5466.6786  2342402   26.92%
 846366 683716     6716.6786    22     7480.0000     5471.6786  2389544   26.85%
Elapsed real time = 143.55 sec. (tree size = 155.11 MB, solutions = 9)
```

Cutting planes

cutting plane: an inequality satisfied by integral solutions of linear inequalities.

$$\min 5x + 8y \text{ subject to}$$
$$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$$
$$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3, \;\; x, y \;\; \text{integral}$$



Gomory-Chvátal cutting planes

$$x \leq 3.5 \Rightarrow x \leq 3$$
$$y \leq 3.3 \Rightarrow y \leq 3$$

$$(.9x + y \geq 1.5) + (.1x \geq 0) \rightarrow$$
$$x + y \geq 1.5 \Rightarrow x + y \geq 2$$

$$(x + y \geq 2) \times .6 + (x + 3.1y \geq 2.4) \times .4 \rightarrow$$
$$x + 1.84y \geq 2.16 \rightarrow$$
$$x + 2y \geq 2.16 \Rightarrow x + 2y \geq 3.$$

Every integer program can be solved by Gomory-Chvátal cuts (Gomory '60), though it may take exponential time in the worst case (Bondy), even for 0-1 problems (Pudlák '97).

cplex-log.txt

```
Problem 'pp08a' read.
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      0     0     2748.3452    51   27080.0000     2748.3452      77   89.85%
*     0+    0                     14300.0000     2748.3452      77   80.78%
      0     0     5046.0422    48   14300.0000     Cuts: 133     153   64.71%
      0     0     6749.5837    24   14300.0000     Cuts: 130     265   52.80%
*     0+    0                     10650.0000     6749.5837     265   36.62%
      0     0     7099.1233    27   10650.0000      Cuts: 53     327   33.34%
      0     0     7171.1837    28   10650.0000      Cuts: 35     356   32.66%
*     0+    0                      7540.0000     7171.1837     356    4.89%
      0     0     7176.2716    31    7540.0000      Cuts: 19     370    4.82%
      0     0     7187.8155    33    7540.0000      Cuts: 20     388    4.67%
      0     0     7188.4198    28    7540.0000       Cuts: 4     398    4.66%
      0     0     7189.5182    30    7540.0000       Cuts: 9     409    4.65%
      0     0     7189.5877    30    7540.0000   Flowcuts: 5     413    4.65%
      0     0     7189.9535    26    7540.0000   Flowcuts: 2     420    4.64%
      0     2     7189.9535    26    7540.0000     7190.0161     420    4.64%
Elapsed real time =   0.27 sec. (tree size =  0.00 MB, solutions = 4)
*    50+   40                      7530.0000     7218.8496    1733    4.13%
*    55    44      integral     0    7520.0000     7218.8496    1783    4.00%
*    60+   45                      7490.0000     7218.8496    1892    3.62%
*    60+   38                      7420.0000     7218.8496    1892    2.71%
*   110+   53                      7400.0000     7238.6753    2712    2.18%
*   210    64      integral     0    7350.0000     7255.3139    4760    1.29%

Implied bound cuts applied:  1
Flow cuts applied:  149
Flow path cuts applied:  23
Multi commodity flow cuts applied:  5
Gomory fractional cuts applied:  34
....
Total (root+branch&cut) =    0.95 sec.
```

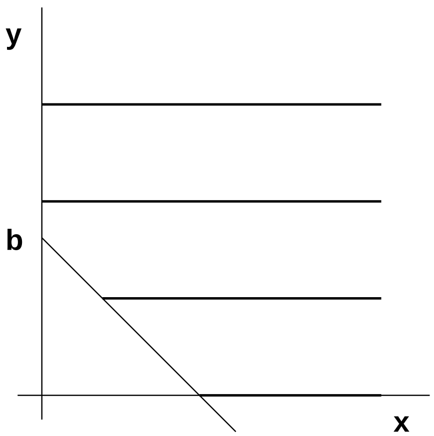
Cutting planes for general MIPs

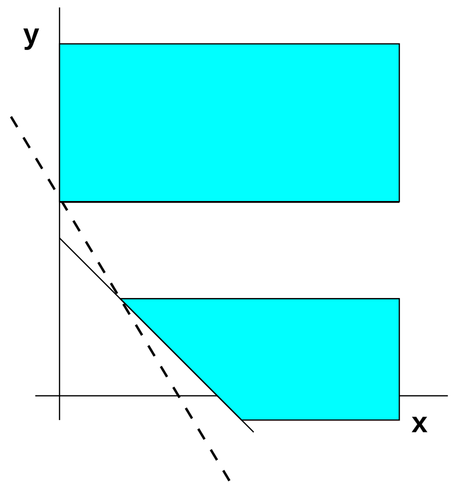
Most useful cutting planes in

CPLEX 8.0: Bixby, Rothberg 07
CPLEX 12.5: Achterberg, Wunderling '13

Disabled cut	Year	Mean Performance Degradation	
		CPLEX 8.0	CPLEX 12.5
Gomory mixed-integer	1960	2.52×	1.28×
Mixed-integer rounding	2001	1.83×	1.48×
Knapsack cover	1983	1.40×	1.14×
Flow Cover	1985	1.22×	1.22×
Implied Bound	1991	1.19×	1.07×
Flow Path	1985	1.04×	1.04×
Clique	1983	1.02×	1.04×
GUB Cover	1998	1.02×	1.02×
Zero-half GC	1996		1.08×

Generating cuts





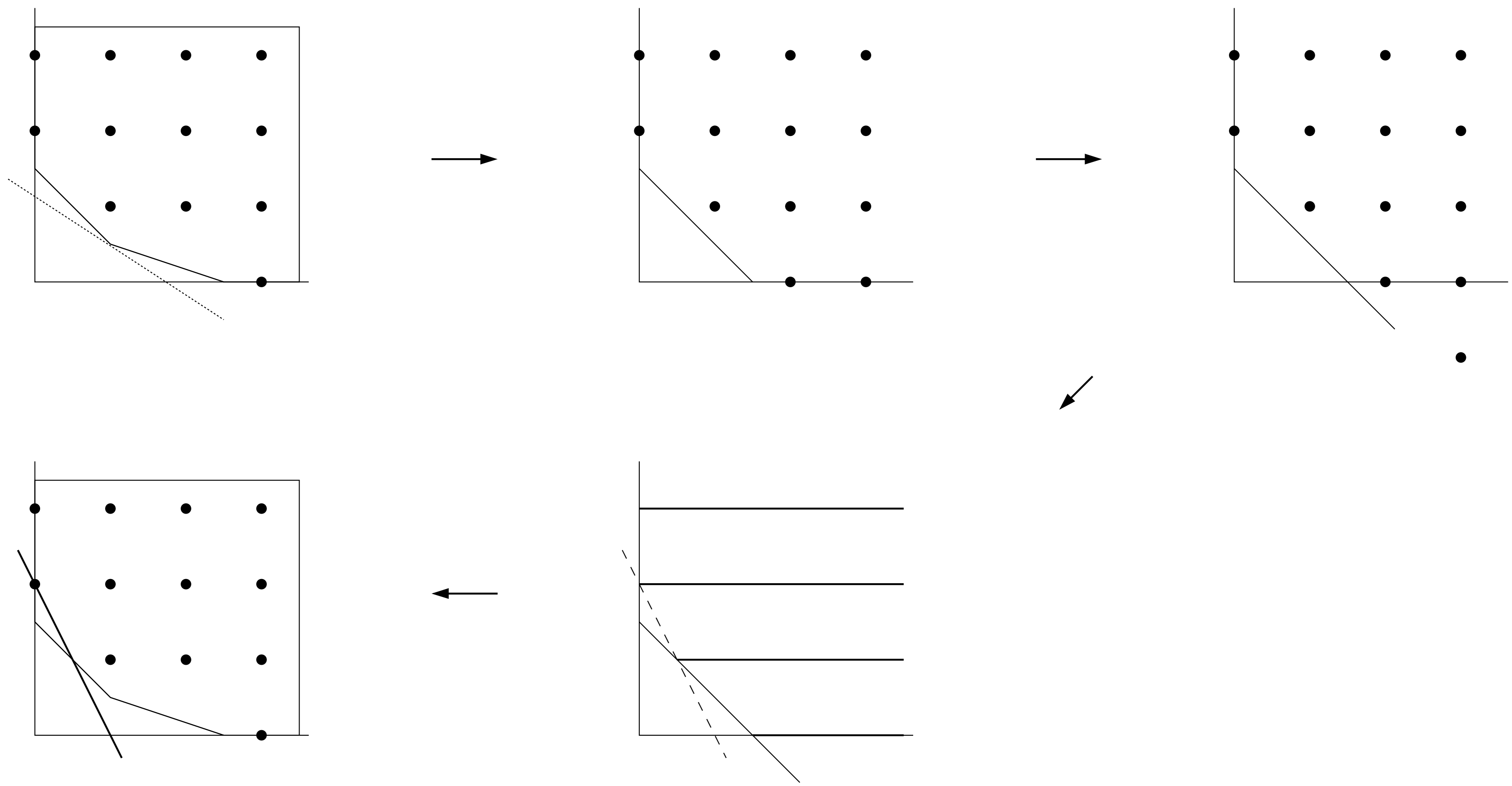
$$x + y \ \geq \ b,$$

$$x \geq 0, \quad y \text{ integral.}$$

Basic Mixed-integer Inequality:

$$\frac{x}{b - \lfloor b \rfloor} + y \geq \lceil b \rceil .$$

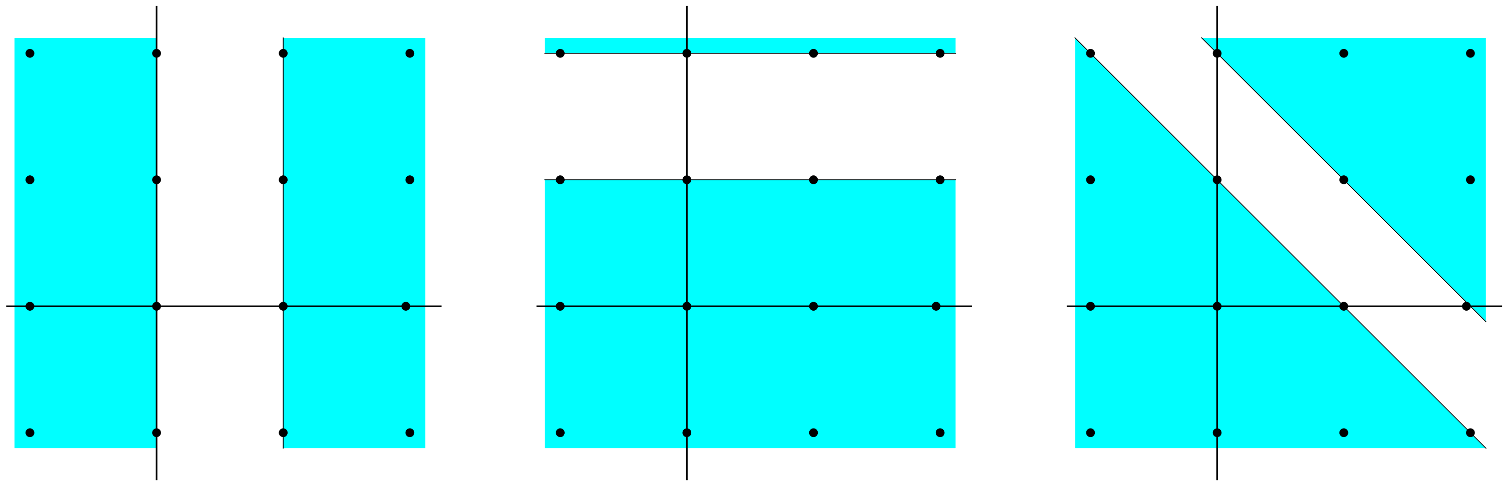
Generating cuts ..



Split sets

Defn: Given $a \in \mathbb{Z}^n$, $b \in \mathbb{Z}$, a split set $S(a, b)$ in $\mathbb{R}^n =$

$$\{x \in \mathbb{R}^n : b < a \cdot x < b + 1\}.$$



Examples:

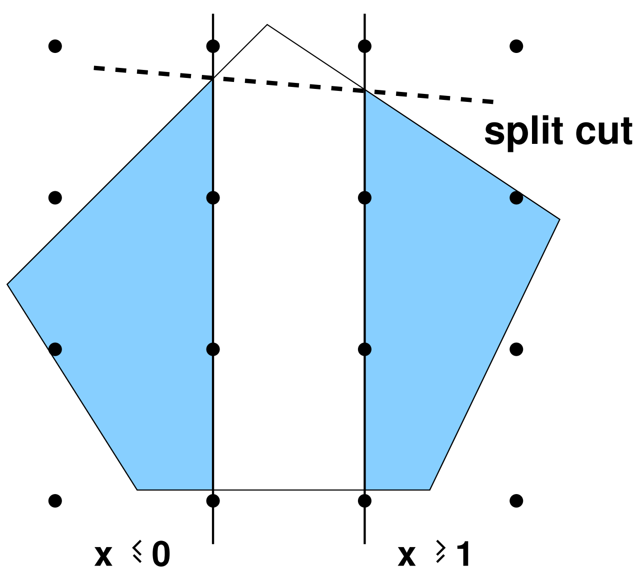
$\{x \in \mathbb{R}^2 : 0 < x_1 < 1\}$, $\{x \in \mathbb{R}^2 : 1 < x_2 < 2\}$, $\{x \in \mathbb{R}^2 : 1 < x_1 + x_2 < 2\}$.

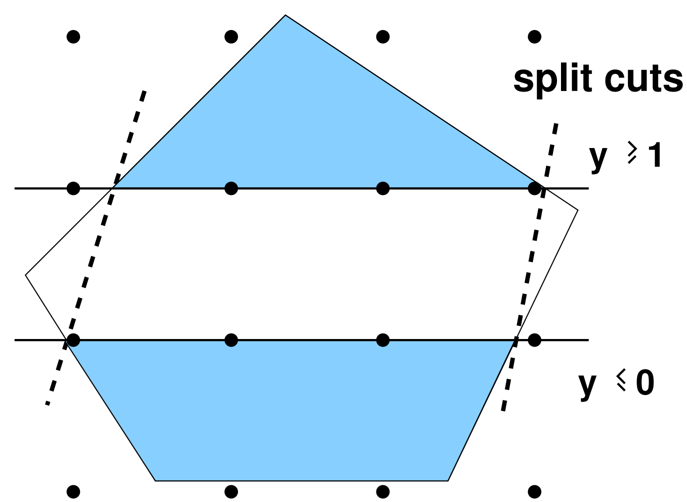
Split cuts

Let $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. A **split cut** for $P \subseteq \mathbb{R}^n$ is an inequality valid for

$$P \setminus S(a,b) = P \setminus \{x \in \mathbb{R}^n : b < a \cdot x < b+1\}$$

$$= P \cap \{a \cdot x \leq b\} \bigcup P \cap \{a \cdot x \geq b+1.\}$$





Split closure

Let $\mathcal{S}^*$ stand for all split sets, and let $L \subseteq \mathcal{S}^*$.

Split closure of P **with respect to** L := points in P satisfying all split cuts for P derived from splits in L.

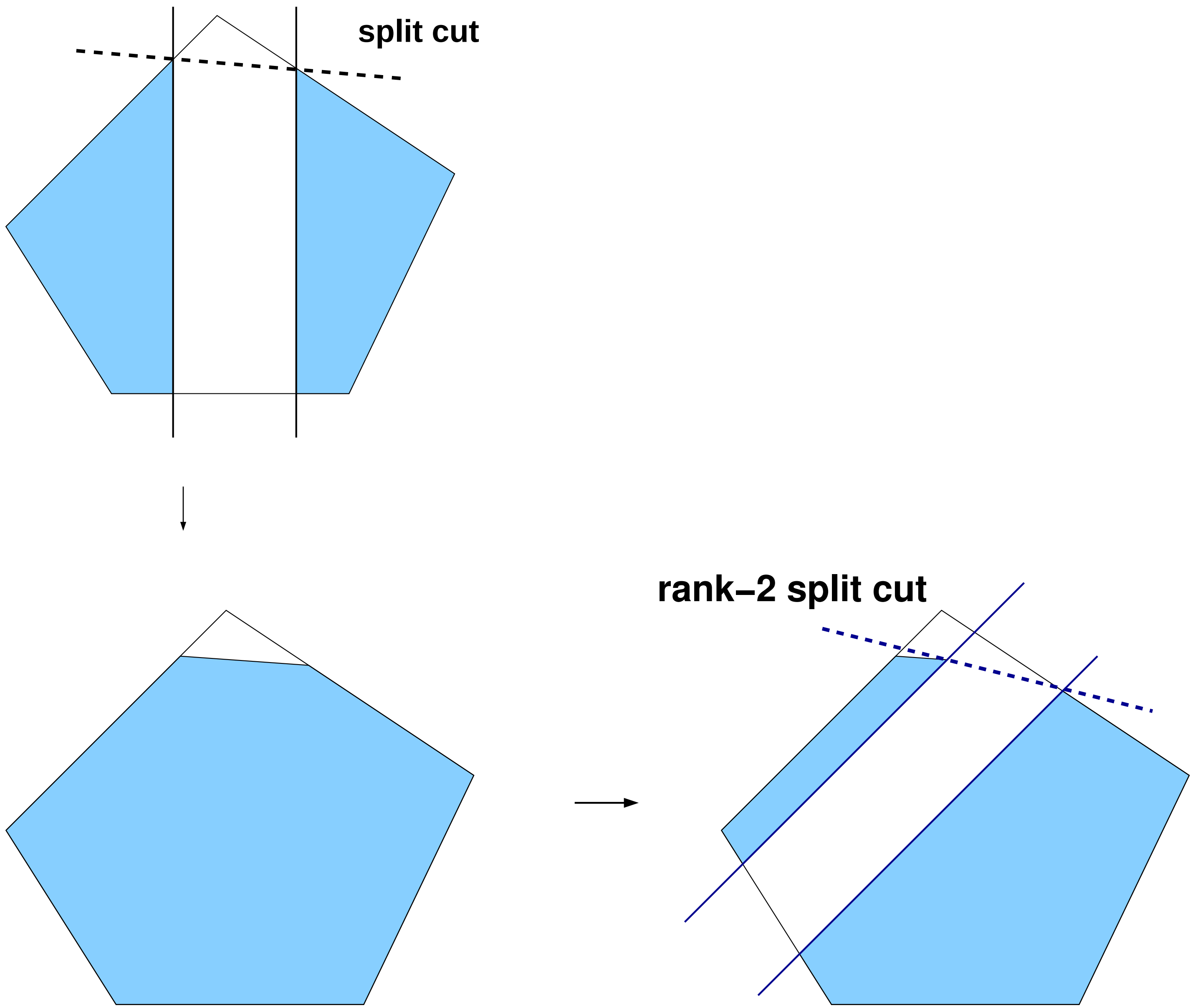
$$\mathrm{SC}(P, L) = \bigcap_{S \in L} \mathrm{conv}(P \setminus S)$$

Split closure of $P = \mathrm{SC}(P) := \mathrm{SC}(P, \mathcal{S}^*)$

$SC^k(P) = SC(SC^{k-1}(P))$.
An inequality has **rank** k, if it is valid for $SC^k(P)$ but not for $SC^{k-1}(P)$.

$\triangleright$ $\exists$ MIP unsolvable by split cuts: Cook, Kannan, Schrijver '90

Rank-2 split cuts



0-1 Split cuts..

A **0-1 split cut** for $P \subseteq \mathbb{R}^n$ is an inequality valid for

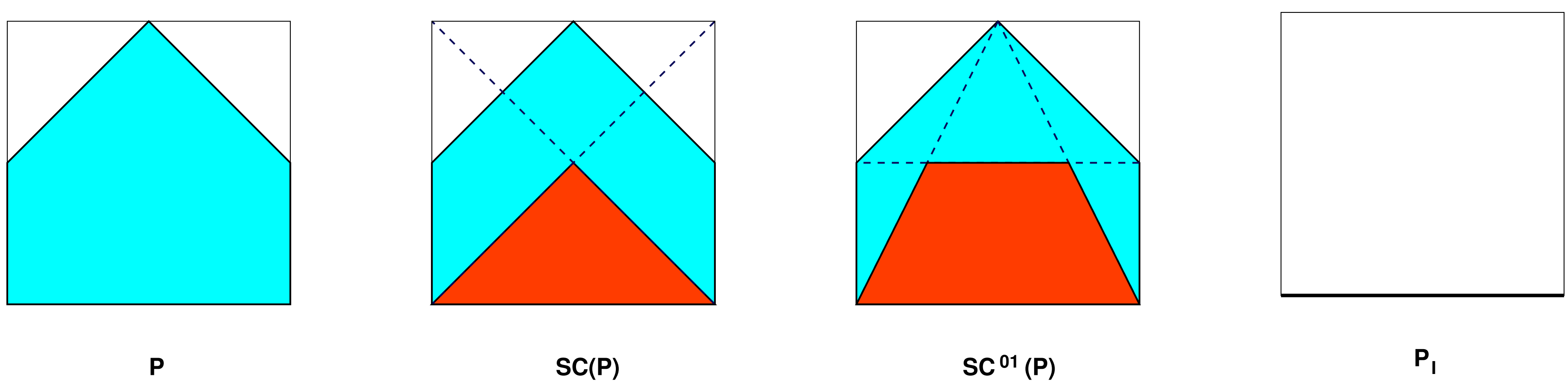
$$P \cap \{x_i \leq 0\} \bigcup P \cap \{x_i \geq 1\}$$

$$= P \setminus \{x \in \mathbb{R}^n : 0 < x_i < 1\} = P \setminus S_i \;\; (S_i \text{ is a 0-1 split set}).$$

$SC_{01}(P) =$ set of points in P satisfying 0-1 split cuts for P.

$$= \bigcap_i \text{conv}(P \setminus S_i)$$

Balas '85: $SC_{01}^n(P) = P_I = \text{conv}(P \cap \mathbb{Z}^n)$.



Split cuts..

Equivalent cuts

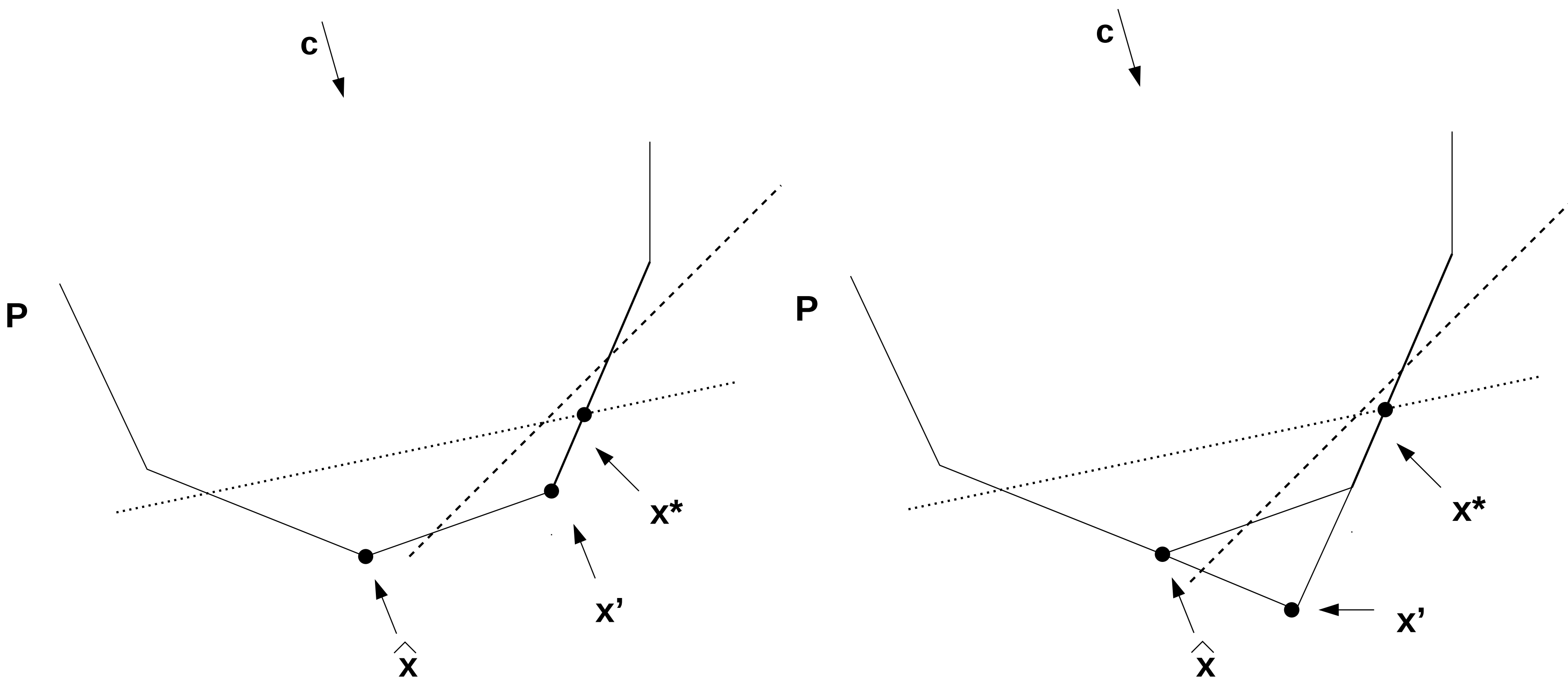
▷ Gomory mixed-integer (GMI) cuts ('60) are split cuts.
▷ Split cuts $\equiv$ intersection cuts of Balas: Andersen, Cornuéjols, Li '05.
▷ Mixed-integer rounding cuts $\equiv$ split cuts: Nemhauser, Wolsey '90.

Split closure

▷ The split closure of P is a polyhedron: Cook, Kannan, Schrijver '90.
▷ Finding violated split cuts is NP-hard: Caprara, Letchford '07.
▷ Split closure yields strong relaxations: Balas, Saxena '07, Dash, Günlük, Lodi '07.

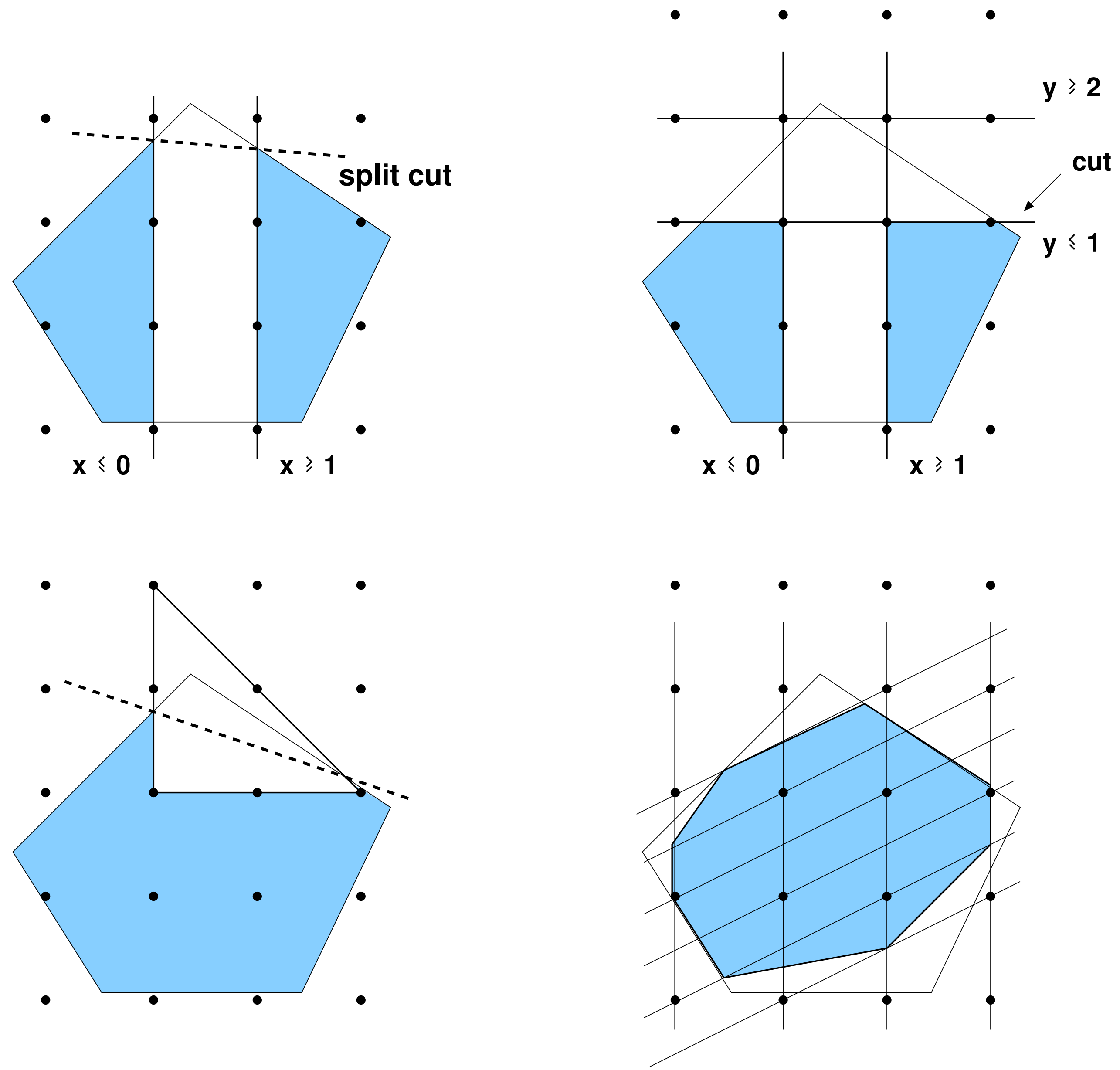
Fast approximation of split closure

Dash, Goycoolea '09 - rank-1 GMI cuts, or GMI cuts from multiple non-optimal tableau yield 64% gap closed versus 76% in Balas and Saxena '08



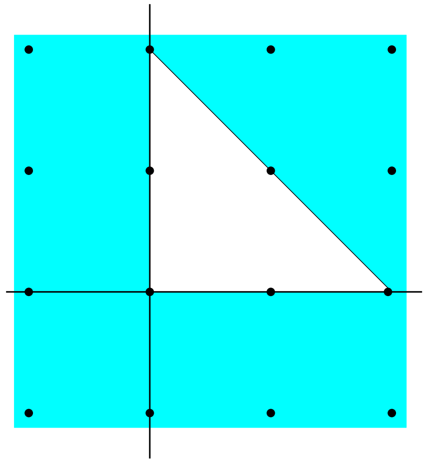
Fischetti, Salvagnin '10, Bonami '10, Nannicini, Cornuéjols '10

Generalizations of split cuts

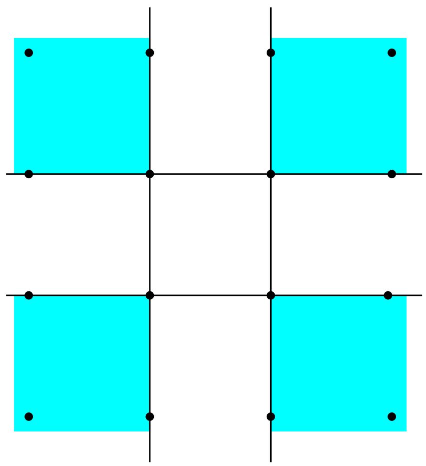


Generalizations of split cuts

Cuts from maximal, lattice-free, convex sets:
Balas '71, Johnson '81, Andersen, Louveaux, Weismantel, Wolsey '07, Cornuéjols, Margot '08, Borozan, Cornuéjols '08, Espinoza '08, Conforti, Cornuéjols, Zambelli '09, Dey, Wolsey '10, Basu, Cornjuéjols, Molinaro '10, Averkov, Wagner, Weismantel '10, Basu, Bonami, Cornuéjols, Margot '11

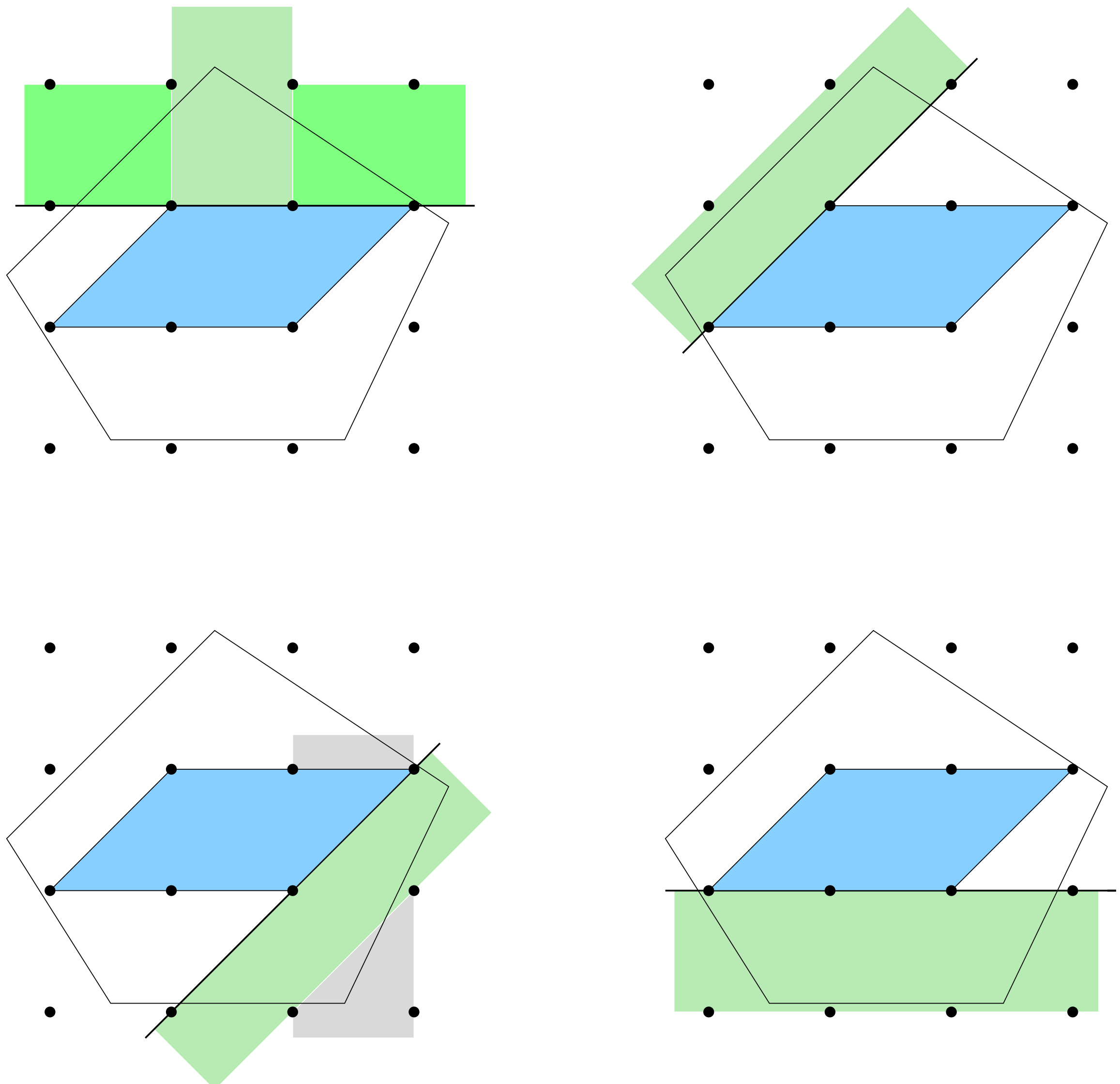


t-branch split cuts for $t \geq 2$:
Li, Richard '08: remove t split sets and convexify.
Dash, Dey, Günlük '10, Balas '09, Balas and Qualizza '09, Chen, Küçükyavuz, Sen '11, Dash, Dobbs, Günlük, Nowicki, Świrszcz '11



lattice cuts: Dash, Günlük, Moran '16

Comparing t-branch/rank-t split cuts



Extended formulations

Nonlinear programming: linearize nonlinear terms, e.g., $x_i x_j$
Fortet '60, McCormick '76

Stochastic programming: represent second-stage/recourse variables

Combinatorial optimization:

▷ Compact representation of integer hulls

▷ Lift-and-project operators
Sherali, Adams '90, Lovász, Schrijver '91, Balas, Ceria, Cornuéjols '93, Lasserre '01
Bienstock, Zuckerberg '04

Combinatorial Optimization:

▷ Compact representation of unions of polyhedra

Balas '85: Let $P^i = \{x : A^i x \leq b^i\}$. Then

$$\text{conv}(\cup_i P^i) = \text{proj}_x \Big\{ (x, \lambda) : A^i x \leq \lambda^i b_i, \sum_i \lambda_i = 1, \lambda_i \geq 0 \Big\}$$

▷ Compact representation of integer hulls

Wong '80, Balas, Pulleyblank '83, Barahona '93, Barahona, Mahjoub '94, Martin '93

▷ Non-existence of compact representations

Yannakakis '91, Fiorini, Massar, Pokutta, Tiwary, de Wolf '12, Rothvoss '14.

▷ Construction techniques:

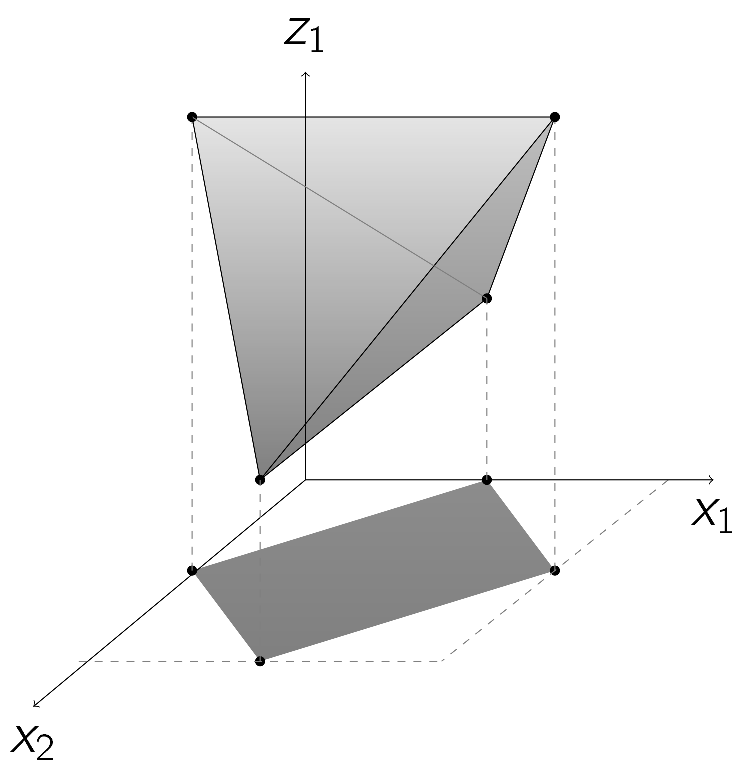
disjunctive programming (Balas '85), dynamic programming (Martin, Radin, Campbell '90), Reflection relations (Kaibel, Pashkovich '11), Factorization of slack matrices (Yannakakis '91)

Extended formulation

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a (pointed) polyhedron.
$Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : Cx + Gy \leq d\}$ is an *extended formulation* of P if

$$P = \text{proj}_x (Q) .$$

$\text{conv}(P \cap \mathbb{Z}^n) = \text{proj}_x \text{conv}(Q \cap (\mathbb{Z}^n \times \mathbb{R}^k))$



Cuts from extended LP formulations

For $a_i \in \mathbb{Z}^n$ and $b_i \in \mathbb{Z}$, let

$$S_i = \{x \in \mathbb{R}^n : b_i < a_i x < b_i + 1\}$$

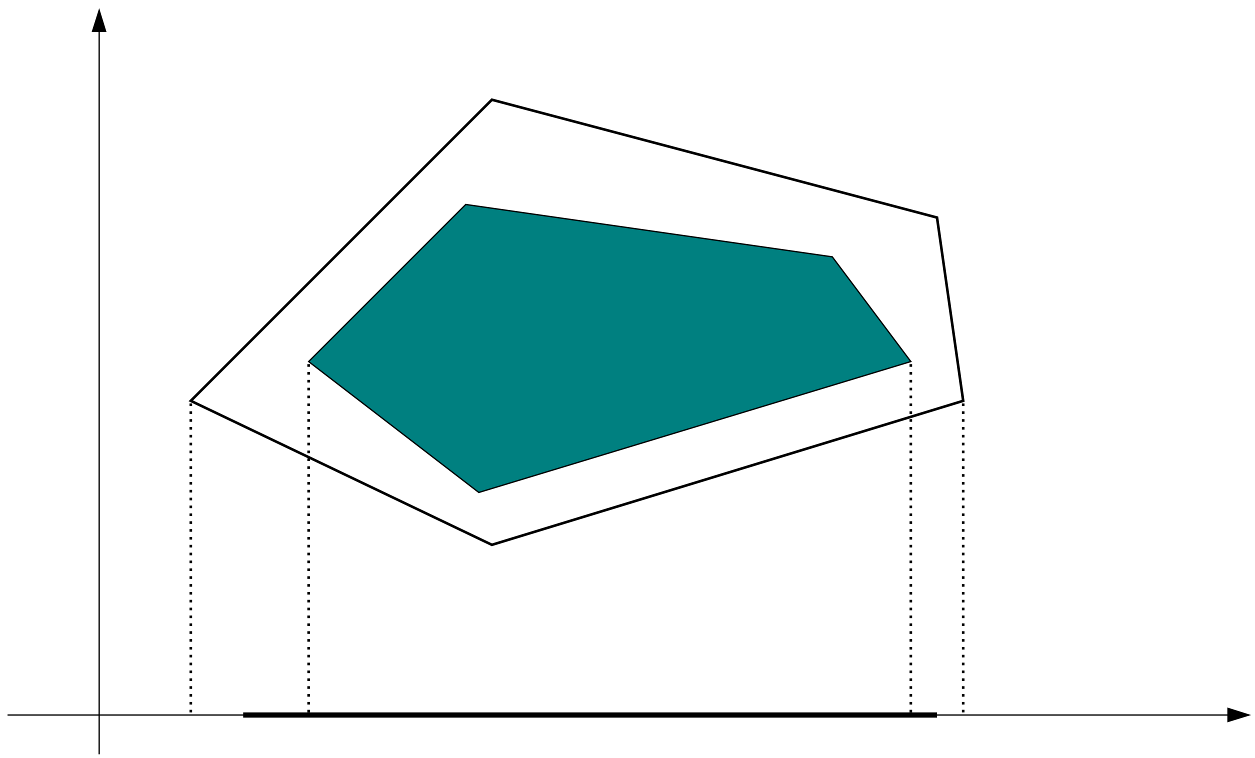
$$S_i \times \mathbb{R}^k = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : b_i < a_i x < b_i + 1\}$$

Goal: Compare the strength of split cuts applied to P and to Q, ie.,

$$\bigcap_{S_i \in L} \mathrm{conv}(P \setminus S_i) \text{ with } \mathrm{proj}_x \bigcap_{S_i \in L} \mathrm{conv}(Q \setminus (S_i \times \mathbb{R}^k)).$$

In other words, we compare $\mathrm{SC}(P, L)$ with $\mathrm{proj}_x \mathrm{SC}(Q, L)$.
When L is the list of all splits, we compare $\mathrm{SC}(P)$ with $\mathrm{proj}_x \mathrm{SC}(Q)$.

Cuts from extended LP formulations



Thm (Bodur, Dash, Günlük, Luedtke '14)

(i) When $|L| = 1$, $\text{SC}(P, L) = \text{proj}_x \text{SC}(Q, L)$.

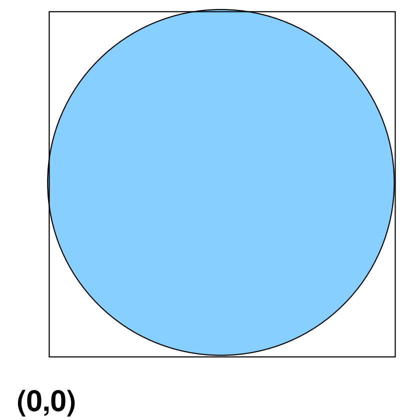
(ii) When $|L| > 1$, $\text{SC}(P, L) \supseteq \text{proj}_x \text{SC}(Q, L)$.

(iii) When $|L| > 1$, $\text{SC}(P, L) \neq \text{proj}_x \text{SC}(Q, L)$ for some instances.

Related results

▷ Bonami (unpublished): Few split cuts added to an extended formulation of the example yield integer hull.

For sphere contained in the unit cube and touching all faces:



▷ Modaresi, Kilinç, Vielma '14: Applying multiple split disjunctions to polyhedral portion of extended formulation in Atamturk, Narayanan's conic MIR idea yields stronger cuts than in the original space.

▷ Bonami, ISMP 2015: Few split cuts added to an extended formulation yield integer hull.

Affine transformation lemma

Lemma Let $Q = \{(x, y) : Ax + By \leq c\} \subseteq \mathbb{R}^m$ and let $P \subseteq \mathbb{R}^n$ be a polyhedron. Let T, U be matrices and ℓ be a vector such that $(x, y) \in Q \Rightarrow (x, Tx + Uy + \ell) \in P$. Then $\text{proj}_x SC(Q, L) \subseteq \text{proj}_x SC(P, L)$.

Proof: Let $\bar{x} \in \text{proj}_x SC(Q, L)$. Then, $\exists\ \bar{y}$ such that $(\bar{x}, \bar{y}) \in SC(Q, L) \Rightarrow$

$$(\bar{x}, \bar{y}) \in \bigcap_{(a_i, b_i) \in L} \text{conv}(Q \setminus \{(x, y) : b_i < a_i x < b_i + 1\})$$

$$\Rightarrow \text{ for each } i, \exists (x^j, y^j) \in Q \setminus \{(x, y) : b_i < a_i x < b_i + 1\}$$

$$(\bar{x}, \bar{y}) = \sum_j \lambda_j (x^j, y^j),\ \sum_j \lambda_j = 1, \lambda_j \geq 0 \forall j$$

$$\Rightarrow (\bar{x}, T\bar{x} + U\bar{y} + \ell) = \sum_j \lambda_j (x^j, Tx^j + Uy^j + \ell).$$

As $(x^j, Tx^j + Uy^j + \ell) \in P \setminus \{(x, z) : b_i < a_i x < b_i + 1\}$

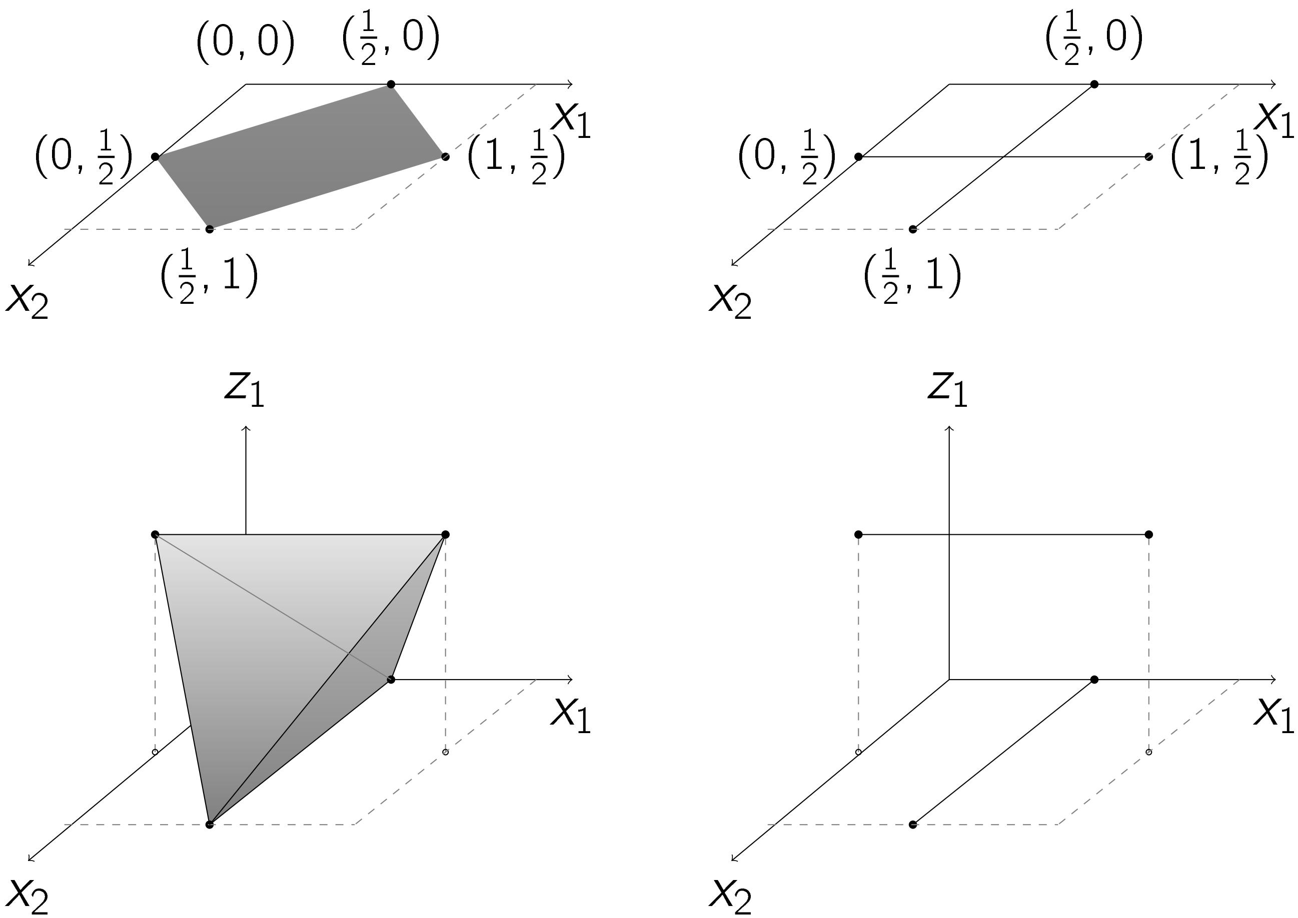
$$\Rightarrow (\bar{x}, T\bar{x} + U\bar{y} + \ell) \in SC(P, L) \Rightarrow \bar{x} \in \text{proj}_x SC(P, L).$$

Cuts from extended LP formulations

Thm (Bodur, Dash, Günlük, Luedtke '14)
(i) When $|L| = 1$, $\mathrm{SC}(P, L) = \mathrm{proj}_x \mathrm{SC}(Q, L)$.
(ii) When $|L| > 1$, $\mathrm{SC}(P, L) \supseteq \mathrm{proj}_x \mathrm{SC}(Q, L)$.
(iii) When $|L| > 1$, $\mathrm{SC}(P, L) \neq \mathrm{proj}_x \mathrm{SC}(Q, L)$ for some instances.

Proof of (ii): Apply the affine transformation lemma with $T = 0$, $U = 0$, $\ell = 0$.

Example proving (iii)



- ▷ $SC(P) \neq \emptyset$: Cornuéjols, Li '01.
- ▷ $SC_{01}(Q) = \emptyset$: (with Luedtke '14).

Stochastic programming

$$\min cx + p_1 d^1 y^1 + p_2 d^2 y^2 + \ldots + p_k d^k y^k$$

Subject to

$$Ax \geq b, \quad x \in \mathbb{Z}^n$$

$$T^1 x + W^1 y^1 \geq h^1$$

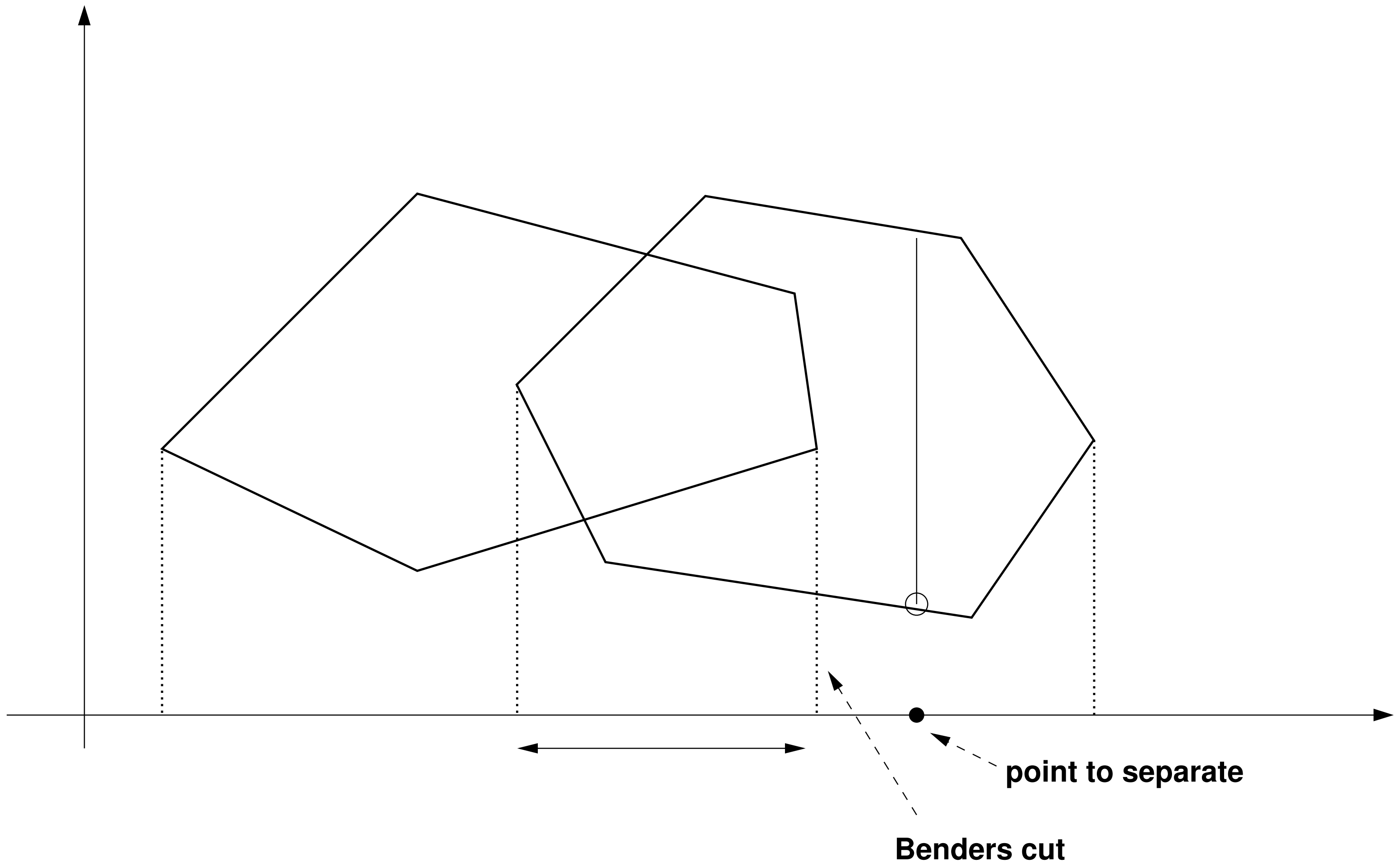
$$\vdots$$

$$T^k x + \qquad W^k y^k \geq h^k$$

$$y^1, \ldots, y^k \geq 0$$

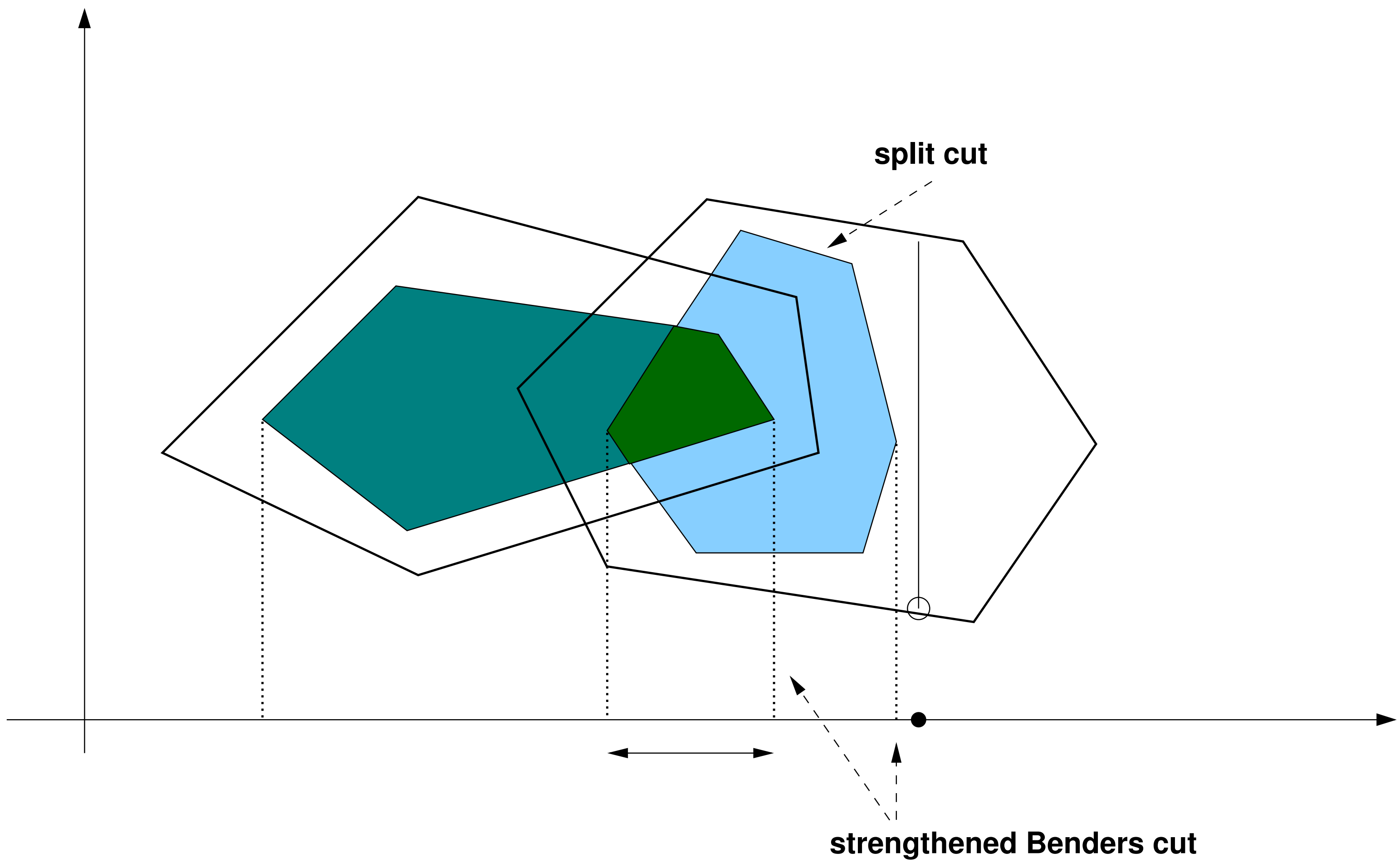
- $p_1, \ldots, p_k$: scenario probabilities
- x: integer first-stage variable; $y^1, \ldots, y^k$: second-stage variables

Benders Decomposition



▷ project out y variables dynamically using Benders decomposition.

Cut-and-project

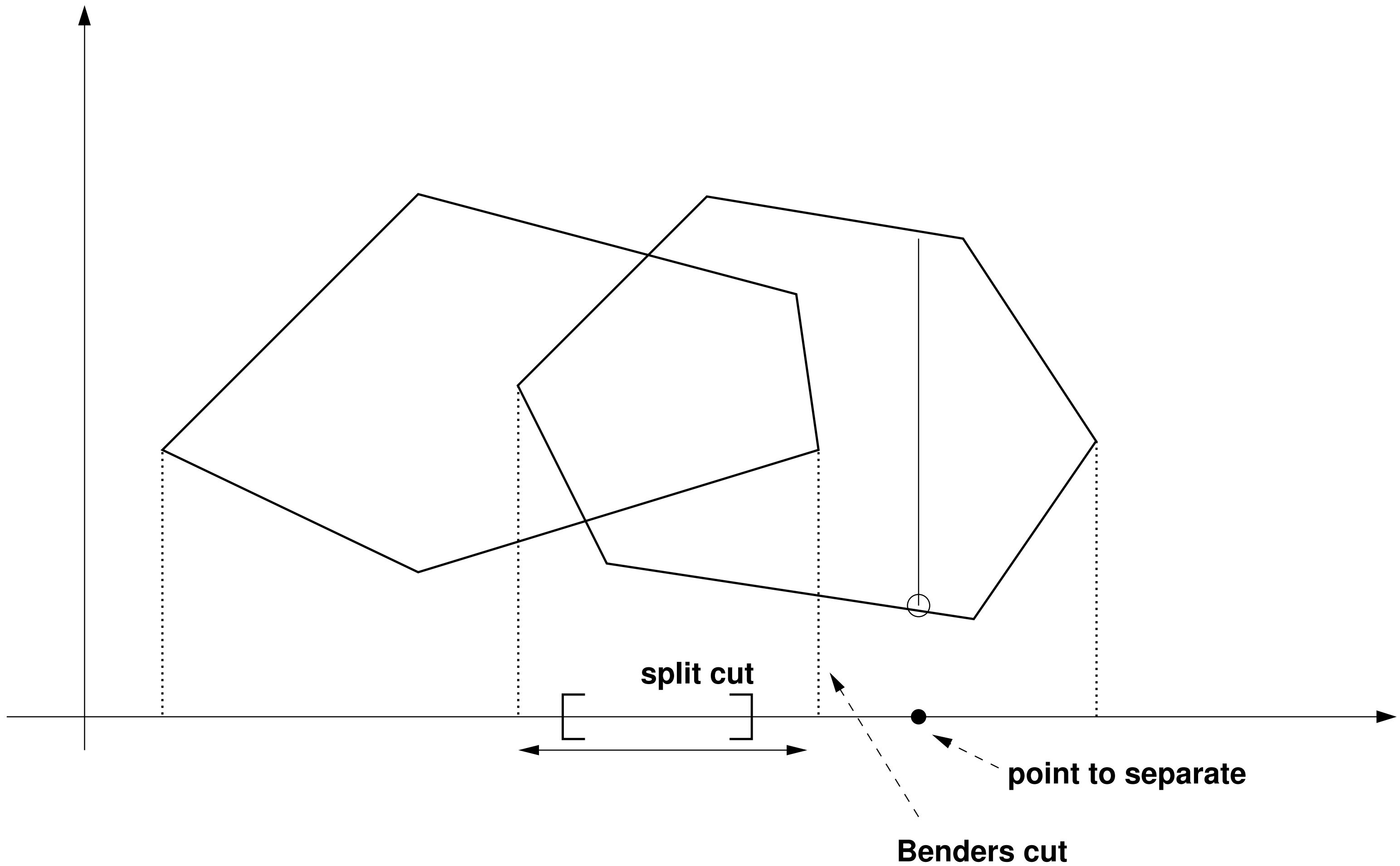


▷ Add cuts to scenarios, then project out y variables.

▷ Mixed 0-1 stochastic programs: Ntaimo '10.

▷ Rank-1 GMI cuts approximate the split closure well: Dash, Goycoolea '09.

Project-and-cut



▷ Project out y variables via Benders cuts, then derive split cuts based on Benders cuts.

Computation

Table 1: Average % root gap for CAP instances with 250 scenarios.

CAP #	BEN	MP	SP
101-104	22.40	20.74	0.07
111-114	8.72	8.01	0.41
121-124	18.92	18.26	0.99
131-134	25.23	24.18	0.30
Mean	18.82	17.80	0.44

- ▷ Stochastic CAP facility location instances based on Beasley's OR-Library.
- ▷ No instance was solved by pure Benders within the time limit.
- ▷ Nearly all instances were solved with Benders + SP GMI.

Minimality

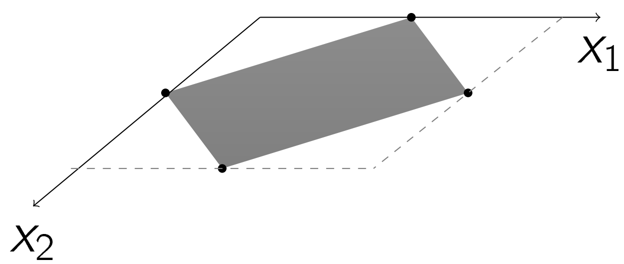
Let Q, Q' be two extended formulations of P. Then

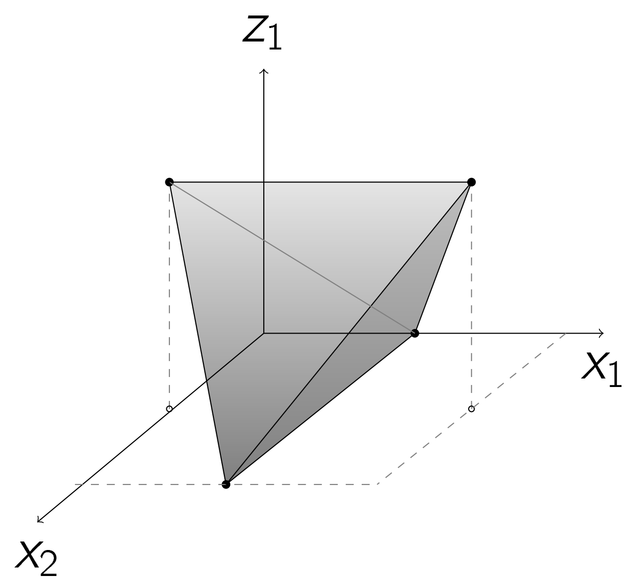
$$Q' \subset Q \Rightarrow SC(Q') \subseteq SC(Q).$$

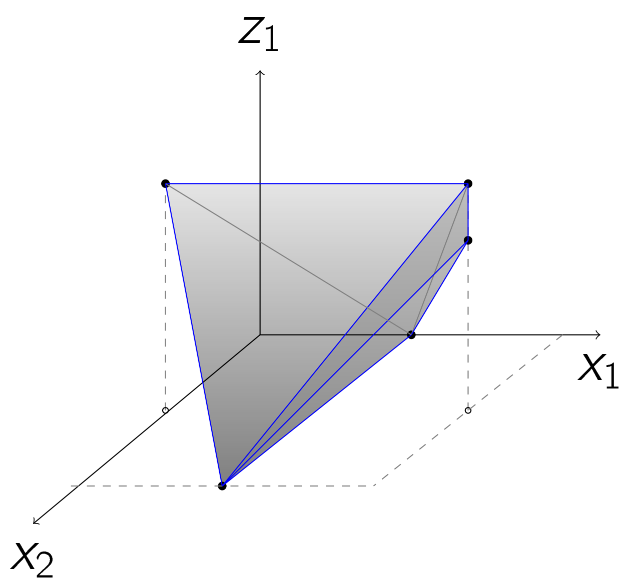
Q is a minimal extended formulation of P if there is no $Q' \subset Q$ where Q' is an extended formulation of P:

The "pre-image" of each vertex (ray) of P is a vertex (ray) of Q.

Minimality



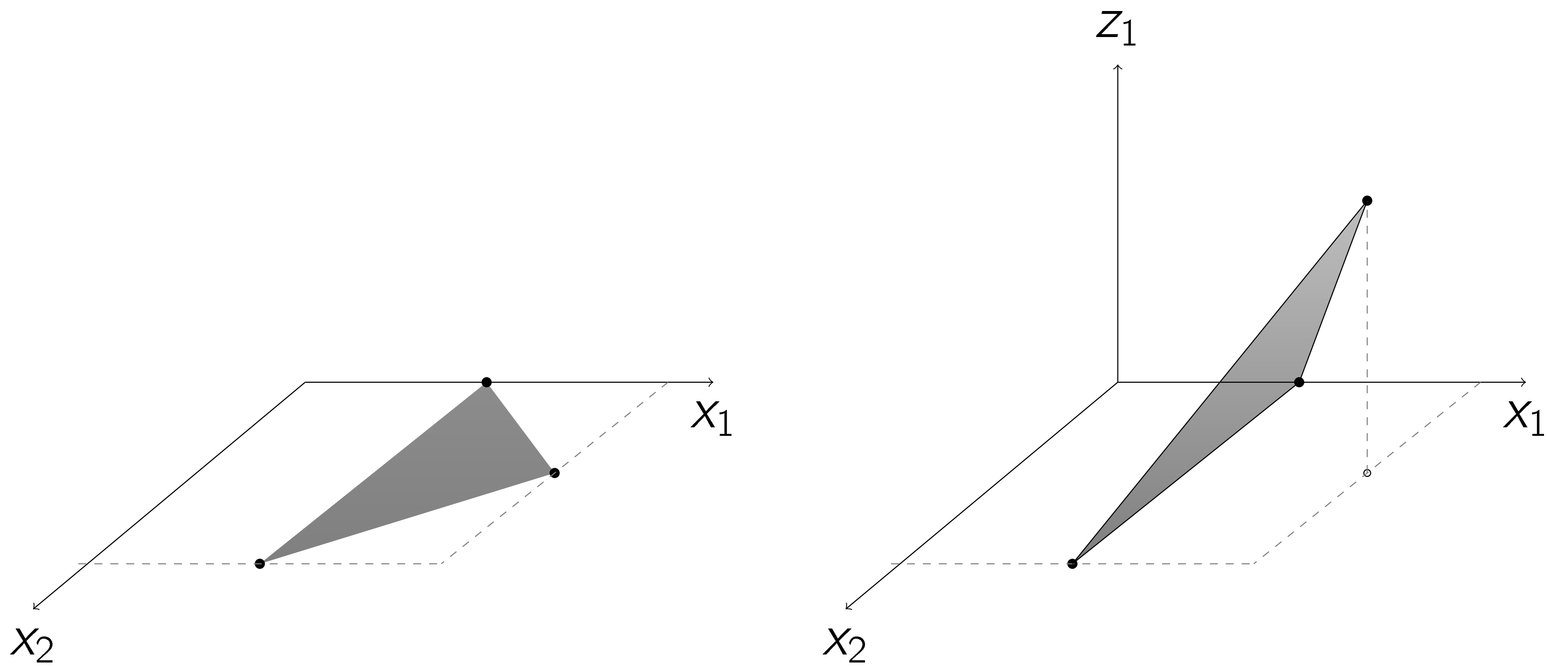




Adding extra variables

Thm (Bodur, Dash, Günlük '14) Let $Q \subseteq \mathbb{R}^{n+k}$ be an extended formulation of $P \subseteq \mathbb{R}^n$. Let $t = \dim Q - \dim P$. If $t < k$, then there exists an extended formulation $Q' \subseteq \mathbb{R}^{n+t}$ of P such that $\text{proj}_{\mathbb{R}^n}(SC(Q)) = \text{proj}_{\mathbb{R}^n}(SC(Q'))$.

▷ Adding extra variables does not help, unless dimension increases too.



Proof when $t = 0$

Let P be a polytope with m vertices which are columns of V, and let Let $\Delta_m = \{\lambda \in \mathbb{R}^m : \sum_i \lambda_i = 1, \lambda_i \geq 0 \quad \forall i\}$. Let Q be a minimal extended formulation of P, then

$$\begin{aligned} P &= \{x \in \mathbb{R}^n : \exists \lambda \in \Delta_m \text{ s.t. } x = V\lambda\}. \\ Q &= \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^k \ : \ \exists \lambda \in \Delta_m \text{ s.t. } \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} V \\ V' \end{pmatrix} \lambda\}. \end{aligned}$$

$\dim(P) = t \Rightarrow \exists n \times t$ submatrix U of V with affinely independent columns such that $P \subseteq \operatorname{aff}(U)$. Let U' be the corresponding submatrix of V'. Then

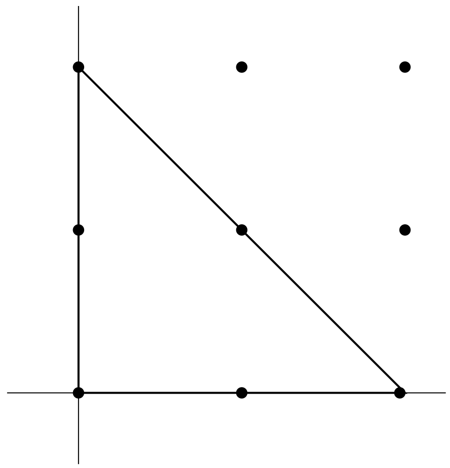
$$(\mathbf{1}, U)^T \text{ and } (\mathbf{1}, U, U')^T$$

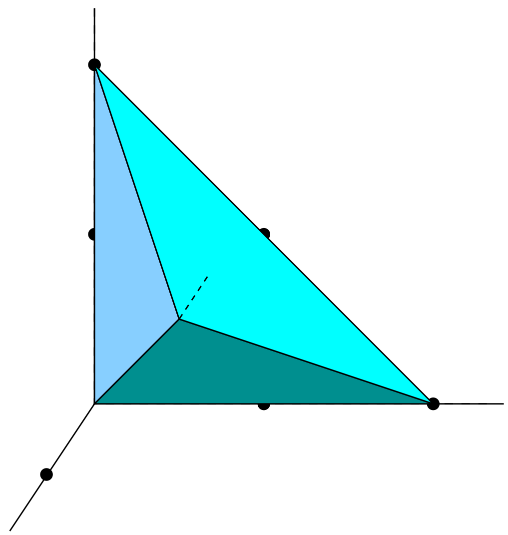
have rank t. Therefore $U' = \mu_0 \mathbf{1}^T + \mu U$. Let f be the affine mapping defined by $f(x) = (x, \mu_0 + \mu x)$. Then $f(P) \subseteq Q$, and therefore $\operatorname{proj}_x SC(P) \subseteq \operatorname{proj}_x SC(Q)$.

Cook-Kannan-Schrijver example

$$P = \text{conv}\left(\left\{(0,0,0),(2,0,0),(0,2,0),(1/2,1/2,\epsilon)\right\}\right)$$

$$I = \{1,2\} \Rightarrow P_I = \{(x,w) \in \mathbb{R}^2 \times \mathbb{R} : (x,w) \in P, w \le 0\}.$$





Thm (CKS '90) $SC^k(P) \not\subseteq P_I$ for any finite $k \ge 1$.

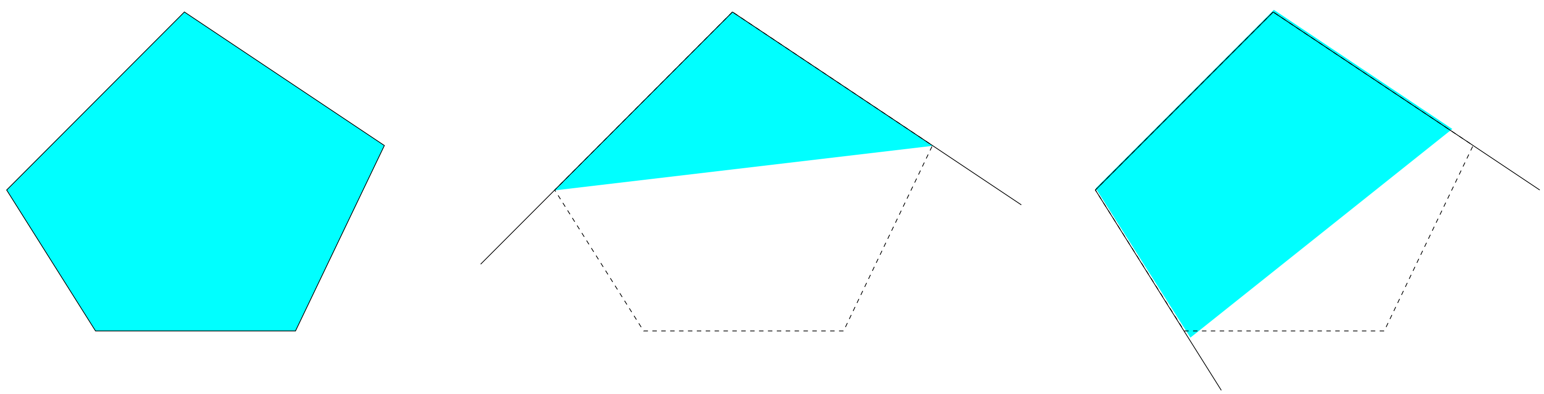
Del Pia '13: $SC^k(Q) \neq Q_I$ for any extended formulation of P.

$\text{proj}_{(x,w)}(SC(Q)) = SC(P)$ for any extended formulation of P.

Simplices

If $P \subseteq \mathbb{R}^n$ is a full-dimensional polyhedron with $n+1$ vertices and rays, then for any extended formulation Q, $\mathrm{SC}(P) = \mathrm{proj}_x \mathrm{SC}(Q) \Rightarrow$ no useful extended formulation can be built from a corner relaxation.

Dash, Günlük, Yildiz '16: Extended formulations can be built from relaxations involving multiple vertices



Strongest extended formulation

Let P be a polyhedron with m extreme points which are columns of V, and ℓ extreme rays which are columns of R.

$$P = \{x \in \mathbb{R}^n : \exists \lambda, \mu \text{ s.t. } x = V\lambda + R\mu,\ \mathbf{1}^T\lambda = 1,\ \lambda, \mu \geq 0\}.$$

A natural extended formulation:

$$E(P) = \{(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \ :\ x = V\lambda + R\mu,\ \mathbf{1}^T\lambda = 1,\ \lambda, \mu \geq 0\}.$$

This corresponds to appending a unit vector in $\mathbb{R}^{m+\ell}$ to each vertex or ray.

Thm (Bodur, Dash, Günlük '14) For any extended formulation Q of P, $\text{proj}_x SC(E(P)) \subseteq \text{proj}_x SC(Q)$.

Affine transformation lemma

Lemma Let $Q = \{(x, y) : Ax + By \leq c\} \subseteq \mathbb{R}^m$ and let $P \subseteq \mathbb{R}^n$ be a polyhedron. Let T, U be matrices and ℓ be a vector such that $(x, y) \in Q \Rightarrow (x, Tx + Uy + \ell) \in P$. Then $\mathrm{proj}_x \mathrm{SC}(Q, L) \subseteq \mathrm{proj}_x \mathrm{SC}(P, L)$.

Proof when P is a polytope

Let $\Delta_m = \{\lambda \in \mathbb{R}^m : \sum_i \lambda_i = 1, \lambda_i \geq 0 \ \ \forall i\}$, and let P and $E(P)$ be given as

$$\begin{aligned} P &= \{x \in \mathbb{R}^n : \exists \lambda \in \Delta_m \text{ s.t. } x = V\lambda\}. \\ E(P) &= \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} V \\ I \end{pmatrix} \lambda, \ \lambda \in \Delta_m\}. \end{aligned}$$

Let Q be a minimal extended formulation of P, then

$$Q = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^k \ : \ \exists \lambda \in \Delta_m \text{ s.t. } \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} V \\ V' \end{pmatrix} \lambda\}.$$

Let f be the affine mapping defined by $f(x, \lambda) = (x, V'\lambda)$. Then

$$(x, \lambda) \in E(P) \Rightarrow x = V\lambda \text{ and } \lambda \in \Delta_m \Rightarrow f(x, \lambda) = (V\lambda, V'\lambda) \in Q.$$

Therefore $f(E(P)) \subseteq Q$, and therefore $\text{proj}_x SC(E(P)) \subseteq \text{proj}_x SC(Q)$.

Mixing Set

Mixing set of Günlük and Pochet '01:

$$
\begin{aligned}
S: \quad & s + x_1 \geq b_1 \\
& s + x_2 \geq b_2 \\
& \dots \\
& s + x_n \geq b_n \\
& s \geq 0, \quad x_i \text{ integral.}
\end{aligned}
$$

Let $0 < b_1 < b_2 < \cdots < b_n \leq 1$. Mixing inequalities of type I for S are:

$$s + b_1 x_1 + \sum_{k=2}^{n} (b_k - b_{k-1}) x_k \ \geq \ b_n \quad (\text{mix1}(x_1, \dots, x_n))$$

Let S^{LP} = linear relaxation of S.
Dash, Günlük '09: Split-rank of $mix1(x_1, \dots, x_n)$ w.r.t. $S^{LP} \geq 2$ and $\leq n$.
Dey '11: Split-rank $\geq \log_2(n+1)$.

Mixing Set..

S^{LP} has $n+1$ variables, 1 extreme point and $n+1$ extreme rays $\Rightarrow$
No extended formulation yields better split cuts.

Impose $x \geq 0 \rightarrow n+1$ extreme points and $n+1$ extreme rays.
Projecting out variables from $E(S^{LP})$, we get an extended formulation of S^{LP}:

$$
\begin{aligned}
X = \Big\{(s,x,y) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n :\ & \sum_{i=1}^n y_i \leq 1, \\
& s + \sum_{i=1}^n (b_n - b_{i-1}) y_i \geq b_n, \\
& x_i \geq \sum_{j=1}^i (b_i - b_{j-1}) y_j, \qquad i = 1,\dots,n, \\
& y_i \geq 0, \qquad i = 1,\dots,n \Big\}.
\end{aligned}
$$

Thm $\text{proj}_x\left(SC_{01}(X)\right) = S$.

Lecture 2 outline

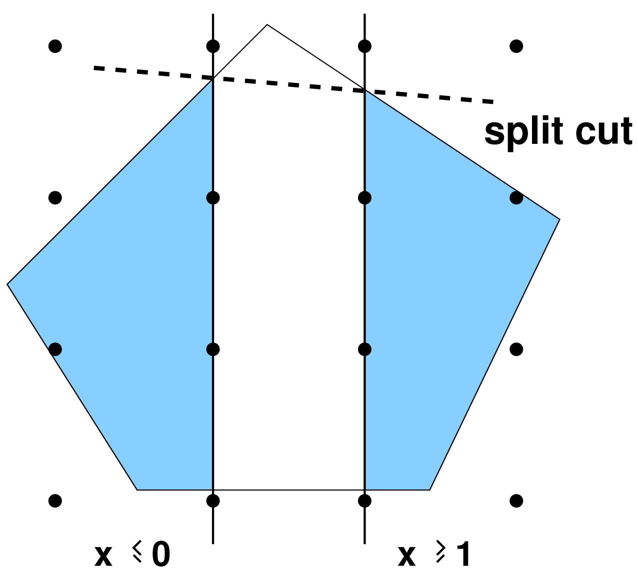
- ▷ Recap of Lecture 1
- ▷ 0-1 split closure
- ▷ Stongest possible split closure for 0-1 problems
- ▷ Lift-and-project operators

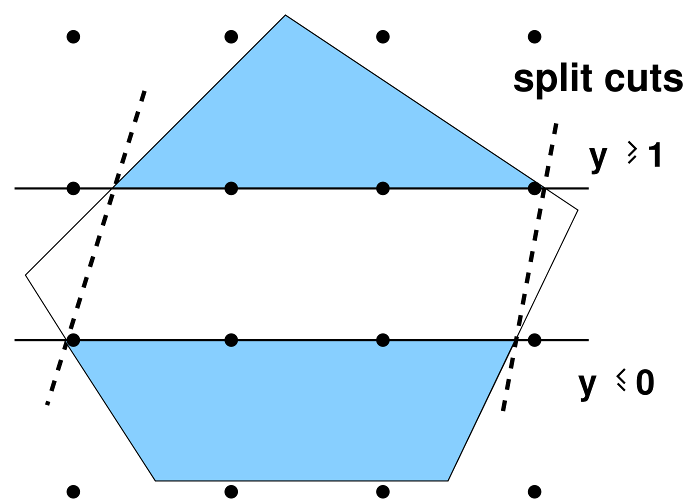
Split cuts

Let $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. A **split cut** for $P \subseteq \mathbb{R}^n$ is an inequality valid for

$$P \setminus S(a,b) = P \setminus \{x \in \mathbb{R}^n : b < a \cdot x < b+1\}$$

$$= P \cap \{a \cdot x \leq b\} \bigcup P \cap \{a \cdot x \geq b+1.\}$$





Split closure

Let $\mathcal{S}^*$ stand for all split sets, and let $L \subseteq \mathcal{S}^*$.

Split closure of P **with respect to** L := points in P satisfying all split cuts for P derived from splits in L.

$$\mathrm{SC}(P, L) = \bigcap_{S \in L} \mathrm{conv}(P \setminus S)$$

Split closure of $P = \mathrm{SC}(P) := \mathrm{SC}(P, \mathcal{S}^*)$

▷ $\mathrm{SC}^k(P) = \mathrm{SC}(\mathrm{SC}^{k-1}(P))$.

▷ An inequality has **rank** k, if it is valid for $\mathrm{SC}^k(P)$ but not for $\mathrm{SC}^{k-1}(P)$.

▷ P has rank k if $\mathrm{SC}^k(P) = \mathrm{conv}(()P \cap \mathbb{Z}^n)$ but $\mathrm{SC}^{k-1}(P)$ is not.

0-1 Split cuts..

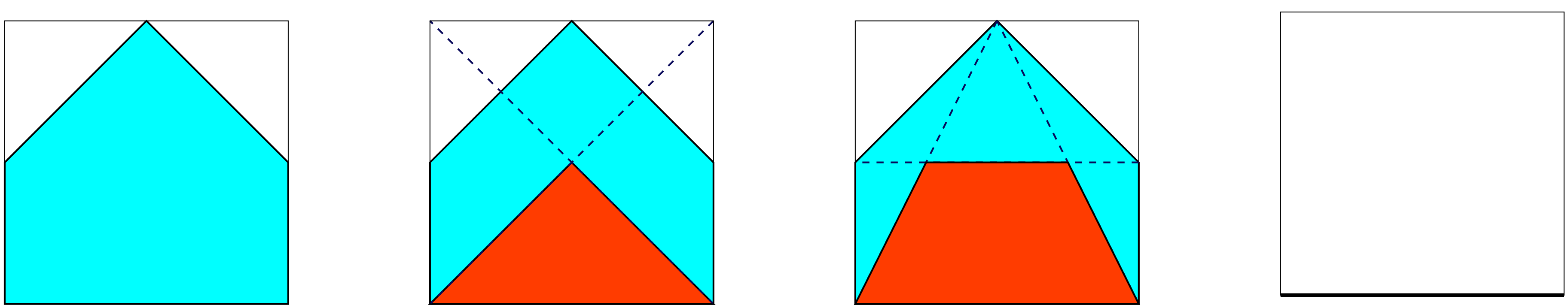
A **0-1 split cut** for $P \subseteq \mathbb{R}^n$ is an inequality valid for

$$P \cap \{x_i \le 0\} \bigcup P \cap \{x_i \ge 1\}$$

$$= P \setminus \{x \in \mathbb{R}^n : 0 < x_i < 1\} = P \setminus S_i \quad (S_i \text{ is a 0-1 split set}).$$

$\mathrm{SC}_{01}(P)$ = set of points in P satisfying 0-1 split cuts for P.

$$= \bigcap_i \mathrm{conv}(P \setminus S_i)$$



P **SC(P)** **SC ^{01} (P)** **P_I**

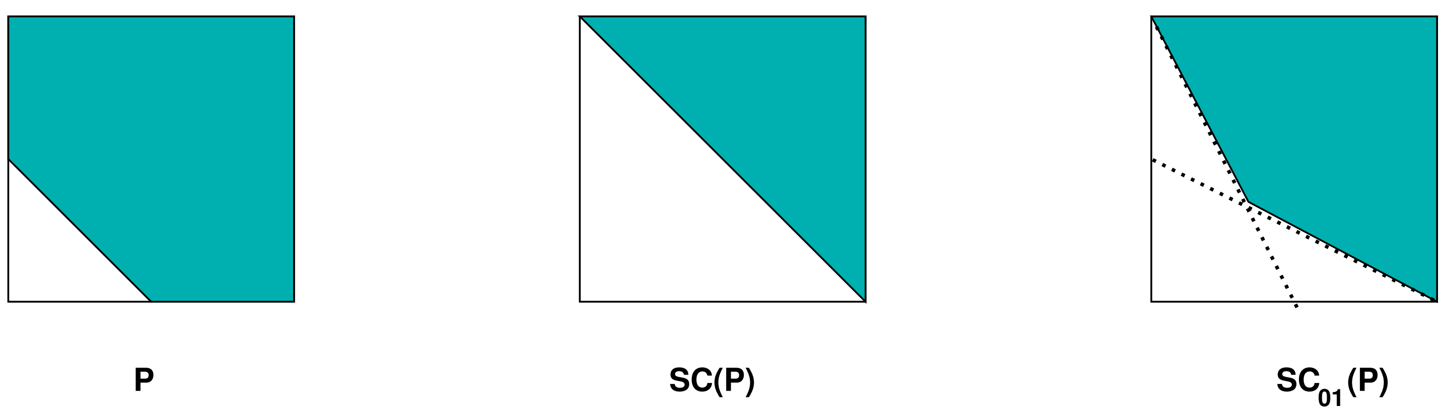
Split cuts for 0-1 problems

$\exists$ MIP unsolvable by split cuts: Cook, Kannan, Schrijver '90

Thm (Balas '85) If $P \subseteq [0,1]^n]$, then $SC_{01}^n(P) = P_I = \text{conv}(P \cap \mathbb{Z}^n)$.

There can be significant difference between 0-1 split cuts and split cuts:

$P^n = \{x \in [0,1]^n : x_1 + \cdots + x_n \geq 0.5\} \Rightarrow SC(P^n) = P_I,\ SC_{01}^{n-1}(P^n) \neq P_I$.



Lecture 3: Reformulate bounded MIP by 0-1 MIP, by replacing each bounded integer variable by 0-1 variables.

0-1 split closure via extended formulations

Balas '85: Let $P^i = \{x : A^i x \leq b^i\}$. Then

$$\text{conv}(\cup_i P^i) = \text{proj}_x \Big\{(x, \lambda) : A^i x \leq \lambda^i b_i, \sum_i \lambda_i = 1, \lambda_i \geq 0\Big\}$$

Let $P = \{x \in [0,1]^n : Ax \leq b\}$ and let $N = \{1, \ldots, n\}$.

$$\mathrm{SC}_{01}(P) = \left\{ \begin{array}{lllll} x \in \mathbb{R}^n & : & \exists\, \bar{x}^i, \bar{\bar{x}}^i \in \mathbb{R}^n, & \bar{\lambda}^i, \bar{\bar{\lambda}}^i \in \mathbb{R}_+ & \text{s.t.} \\ & & x = \bar{x}^i + \bar{\bar{x}}^i, & \bar{\lambda}^i + \bar{\bar{\lambda}}^i = 1, & i \in N, \\ & & A\bar{x}^i \leq \bar{\lambda}^i b, & A\bar{\bar{x}}^i \leq \bar{\bar{\lambda}}^i b, & i \in N, \\ & & \bar{x}^i_i = 1, & \bar{\bar{x}}^i_i = 0 & i \in N \end{array} \right\}$$

General MIPs

Let P be a polyhedron with m extreme points which are columns of V, and ℓ extreme rays which are columns of R.

$$P = \{x \in \mathbb{R}^n : \exists \lambda, \mu \text{ s.t. } x = V\lambda + R\mu,\ \mathbf{1}^T\lambda = 1,\ \lambda, \mu \geq 0\}.$$

Strongest extended formulation for general MIPs in $n + m + l$ variables:

$$E(P) = \{(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \ :\ x = V\lambda + R\mu,\ \mathbf{1}^T\lambda = 1,\ \lambda, \mu \geq 0\}.$$

Main 0/1 result

Let $P = \text{conv}(\{x^1, \ldots, x^m\}) \subseteq [0, 1]^n$.

Let $Q^t(P)$ be an extended formulation of P such that

$$Q^t(P) = \text{conv}\left(\left\{ \begin{pmatrix} x^1 \\ y^1 \end{pmatrix}, \ldots, \begin{pmatrix} x^m \\ y^m \end{pmatrix} \right\}\right) \text{ where}$$

$$y_i^j = \begin{cases} 1, & \text{if } x_i^j \text{ fractional} \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \ldots, t.$$

Example: $x^1 = \begin{pmatrix} 1/2 \\ 1 \\ 1/3 \\ 0 \end{pmatrix} \Rightarrow y^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Thm (Bodur, Dash, Günlük '14): $\text{proj}_x \text{SC}_{01}(Q^{n-1}(P)) = P_I$.

Proof of main 0/1 result

Thm Let $P \subseteq [0,1]^n$ and $Q^n = Q^n(P)$. Then $\text{proj}_x \text{SC}_{01}(Q^n) = P_I$.

Proof Let V be the set of vertices of Q^n. Consider

$$Q^n \setminus S_i \;=\; (Q^n \cap \{x_i = 0\}) \;\bigcup\; (Q^n \cap \{x_i = 1\}).$$

As $x_i \geq 0$ is a face of Q^n, any point $(\bar{x}, \bar{y})$ in $(Q^n \cap \{x_i = 0\})$ satisfies $\bar{y}_i = 0$ and the same is true for $(Q^n \cap \{x_i = 1\})$. Therefore

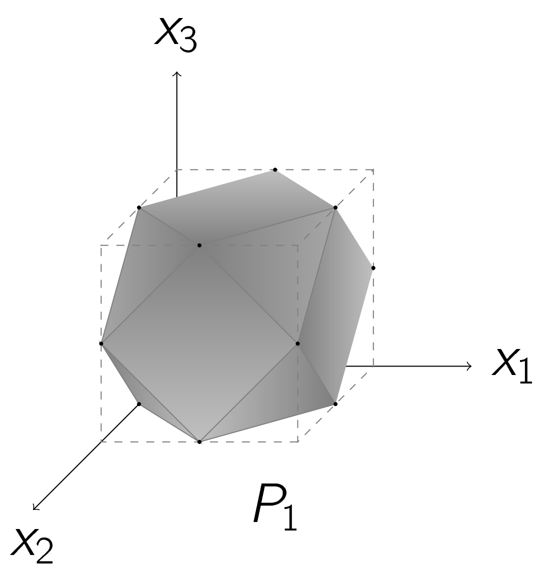
$$\text{conv}(Q^n \setminus S_i) \subseteq \{(x,y) : y_i = 0\} \Rightarrow \text{SC}_{01}(Q^n) \subseteq \{(x,y) : y_i = 0 \ \forall i\}.$$

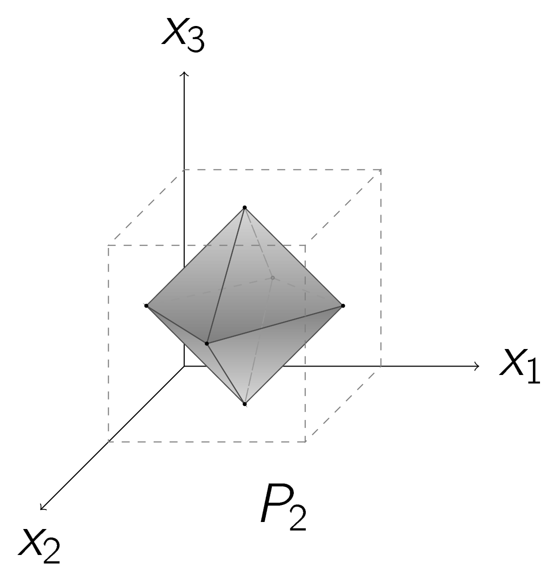
As $y_i \geq 0$ is valid for Q^n, we have $Q^n \cap \{y_i = 0\}$ is a face F_i of Q^n. Therefore

$$F_i = \text{conv}(V \cap F_i) = \text{conv}\{\begin{pmatrix} x^j \\ y^j \end{pmatrix} : y_i^j = 0\} = \text{conv}\{\begin{pmatrix} x^j \\ y^j \end{pmatrix} : x_i^j \in \{0,1\},$$

$$\text{SC}_{01}(Q^n) \subseteq Q^n \cap_i F_i \Rightarrow \text{proj}_x \text{SC}_{01}(Q^n) = P_I.$$

Cropped cube





Let $N = \{1, \ldots, n\}$.

$$P^n = \left\{ x \in \mathbb{R}^n : \sum_{i \in I} x_i + \sum_{i \in N \setminus I} (1 - x_i) \geq 1/2, \quad \forall I \subseteq N,\ 0 \leq x_i \leq 1, \quad \forall i \in N \quad \right\}$$

All $2^n + 2n$ inequalities defining P are facet defining.
In every vertex of P, one component has value $1/2$ and the rest are $0/1$.

Rank of cropped cube

Rank of P^n with respect to:

Gomory-Chvátal cuts: n (Chvátal, Cook, Hartmann '89).
Lovász-Schrijver N_+ cuts: n (Goemans, Tunçel '01, Cook, Dash '01).
Sherali-Adams hierarchy: n
Lasserre hierarchy: $O(n)$
Bienstock-Zuckerberg hierarchy: $O(n)$ (Bienstock, Zuckerberg '04)

Extended LP formulation for the cropped cube

Thm

$$Q^n = \Big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \quad \begin{aligned} & y_i \leq 2x_i, \ \forall i \in N \\ & y_i + 2x_i \leq 2, \ \forall i \in N \\ & y_i \geq 0, \ \forall i \in N \\ & \sum_{i \in N} y_i = 1 \Big\} \end{aligned}$$

$\mathrm{SC}_{01}(Q^n)$ = integer hull of Q^n: $x_i = 0 \Rightarrow y_i = 0$, and $x_i = 1 \Rightarrow y_i = 0$.
$\therefore \ y_i = 0$ is valid for $Q^n \setminus \{(x,y) : 0 < x_i < 1\} \Rightarrow \mathrm{SC}_{01}(Q^n) \subseteq \{y_i = 0 \forall i\}$.

Similar extended formulation and result given by Bonami.

Basu: Is this best possible?
We show that there is no extended formulation Q' with $2n-2$ variables such that $\mathrm{SC}_{01}(Q') = Q'_I$.

Clique inequalities

Let $G = (V, E)$ be a graph. Let K_n be the complete graph on n nodes.

Stable set polytope:

$$\text{STAB}(G) = \text{conv}(\{x \in \{0,1\}^n : x_i + x_j \leq 1,\ \forall\{i,j\} \in E\}).$$

Fractional stable set polytope:

$$\text{FSTAB}(G) = \text{conv}(\{x \in [0,1]^n : x_i + x_j \leq 1,\ \forall\{i,j\} \in E\}).$$

The *clique inequality* for $\text{STAB}(K_n)$ is $\sum_{i=1}^n x_i \leq 1$.

Thm The split rank of the clique inequality with respect to $\text{FSTAB}(K_n)$ for $n \geq 3$ is at least $\lceil \log_2(n-1) \rceil$.

Hartmann '88, Chvátal, Cook, Hartmann '89: same bound for GC cuts

Clique inequalities..

Vertices of FSTAB(K_n): Balinski '70, Nemhauser, Troter '74:

Thm $Q^n(\text{FSTAB}(K_n))$ is defined by $2n$ variables and $4n$ constraints in $\mathbb{R}^{2n}$:

$$\Big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n :$$

$$2x_i \geq y_i, \qquad \forall i \in N,$$

$$\sum_{j \in N} y_j \geq 3y_i, \qquad \forall i \in N,$$

$$2 \geq \sum_{j \in N} (2x_j - y_j) + 2y_i, \qquad \forall i \in N,$$

$$y_i \geq 0, \qquad \forall i \in N\Big\}.$$

Clique inequalities...

$\mathrm{SC}_{01}(Q^n)$ = integer hull of Q^n.

Proof:

$$
\begin{aligned}
x_i = 0 \Rightarrow y_i = 0, \qquad & \text{as } 2x_i \geq y_i \geq 0. \\
x_i = 1 \Rightarrow y_i = 0, \qquad & \text{as } 2 \geq \sum_{j \in N} (2x_j - y_j) + 2y_i \\
& = 2x_i + y_i + \sum_{j \in N, j \neq i} (2x_j - y_j) \\
\Rightarrow y_i = 0 \text{ is valid for} \qquad & Q^n \setminus \{(x, y) : 0 < x_i < 1\} \\
\Rightarrow \mathrm{SC}_{01}(Q^n) \subseteq \{y_i = 0 \quad & \forall i\}.
\end{aligned}
$$

Lovasz-Schrijver and Sherali-Adams level 1

Let $P = \{x \in [0,1]^n : b^\ell - a^\ell \cdot x \geq 0,\ \ell \in L\}$.

The following inequalities are valid for P:

$$x_i(b^\ell - a^\ell \cdot x) \geq 0, \qquad i \in N,\ \ell \in L,$$
$$(1 - x_i)(b^\ell - a^\ell \cdot x) \geq 0, \qquad i \in N,\ \ell \in L,$$

Replace each quadratic term $x_i x_j$ by a new variable y_{ij} for all $i \leq j \in N$ to get $\hat{M}(P)$:

$$b^\ell x_i - \sum_{j \in N} a_j^\ell y_{ij} \geq 0, \qquad i \in N,\ \ell \in L,$$
$$b^\ell - \sum_{j \in N} a_j^\ell x_j - b^\ell x_i + \sum_{j \in N} a_j^\ell y_{ij} \geq 0, \qquad i \in N,\ \ell \in L.$$

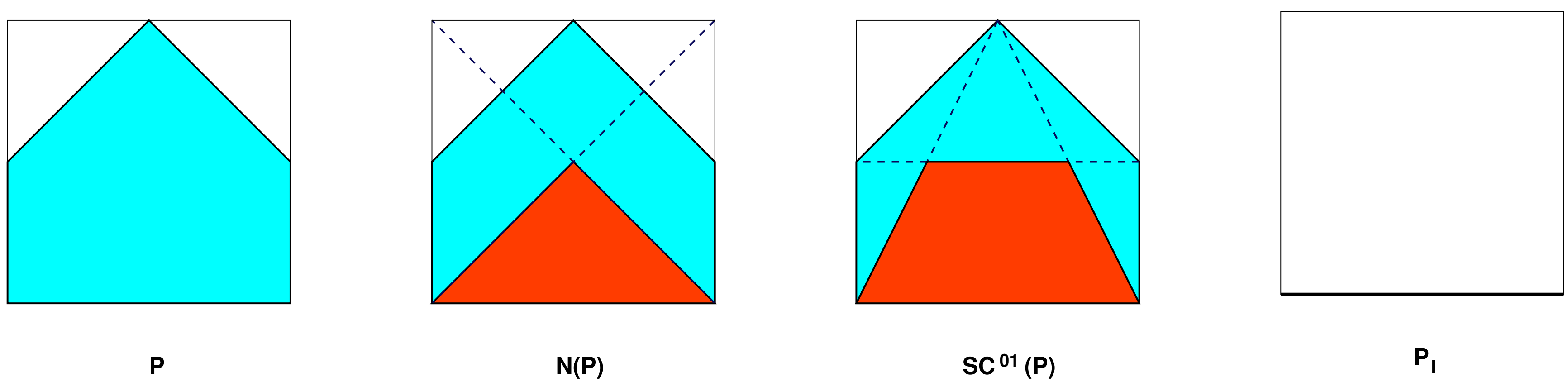
Add constraints $y_{ii} = x_i$ for $i \in N$ to $\hat{M}(P)$ to get $M(P)$.
Let $N(P) = SA^1(P) = \text{proj}_x M(P)$.

Properties

Lovász and Schrijver ('01): $N(P) \subseteq \mathrm{SC}_{01}(P)$.

Balas: $(\mathrm{SC}_{01})^n(P) = P_I \Rightarrow N^n(P) = P_I$.

If $P =$ the cropped cube, then $N^{n-1}(P) \neq P_I$ (Cook and Dash '01, Goemans and Tuncel '01), $SA^{n-1}(P) \neq P_I$ (Laurent '03).



Lovasz-Schrijver and 0-1 split cuts

$M(P) = \hat{M}(P) \cap \{y_{ii} = x_i \text{ for all } i \in I\}$.

Lemma $\hat{M}(P)$ is an extended LP formulation of P

Thm(Bodur, Dash, Günlük '15) $y_{ii} = x_i$ is a **split cut** for $\hat{M}(P)$.

Proof Multiplying the bound constraints on x_j variables with x_i and $(1 - x_i)$ we get the following nonlinear inequalities and associated linearizations:

$$
\begin{aligned}
x_i x_j \geq 0 \quad &\longrightarrow \quad y_{ij} \geq 0, \\
x_i(1 - x_j) \geq 0 \quad &\longrightarrow \quad x_i \geq y_{ij}, \\
(1 - x_i)x_j \geq 0 \quad &\longrightarrow \quad x_j \geq y_{ij}, \\
(1 - x_i)(1 - x_j) \geq 0 \quad &\longrightarrow \quad 1 - x_i - x_j + y_{ij} \geq 0.
\end{aligned}
$$

Setting $i = j$, we have

$$0 \leq y_{ii} \leq x_i \text{ and } 2x_i - 1 \leq y_{ii} \Rightarrow$$

$$y_{ii} = x_i \text{ when } x_i = 0 \text{ or } x_i = 1 \Rightarrow$$

$$y_{ii} = x_i \text{ is valid for } P \setminus \{x : 0 < x_i < 1\} \Rightarrow$$

$$\mathrm{SC}_{01}(\hat{M}(P)) \subseteq \hat{M}(P) \cap \{(x, y) : y_{ii} = x_i \text{ for } i \in I\} \Rightarrow$$

$$\mathrm{SC}_{01}(\hat{M}(P)) \subseteq M(P)$$

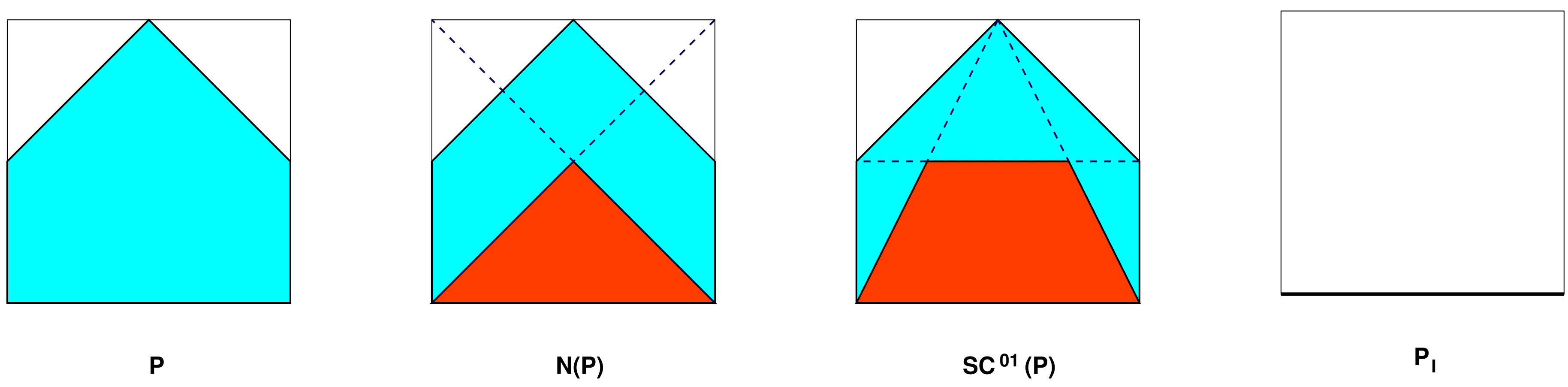
New lift-and-project operator

Definition: $\tilde{M}(P) = \mathrm{SC}_{01}(\hat{M}(P))$ and $\tilde{N}(P) = \mathrm{proj}_x \tilde{M}(P)$.

Thm (Bodur, Dash, Günlük '14) $\tilde{N}(P) \subseteq N(\mathrm{SC}_{01}(P))$.

Corollary $\tilde{N}^{\lceil n/2 \rceil}(P) = P_I$.

Proof As $N(P) \subseteq \mathrm{SC}_{01}(P)$, we have $\tilde{N}(P) \subseteq \mathrm{SC}_{01}(\mathrm{SC}_{01}(P))$. Then $\mathrm{SC}_{01}^n(P) = P_I$ gives the corollary.



Complexity and lower bounds

Thm (Bodur, Dash, Günlük '15) It is possible to optimize over $\tilde{N}(P)$ in polynomial time.

Thm $SA^2(P) \subseteq \tilde{N}(P)$.

Discussion

$SA^2(P)$:

1) Multiply a constraint defining P by the nonnegative terms $x_i x_j$, $x_i(1-x_j)$, $(1-x_i)(1-x_j)$ where $i, j \in N$ are distinct:

$$
\begin{aligned}
x_i x_j (b^\ell - (a^\ell)^T x) &\geq 0, \qquad && i, j \in N, i < j,\ \ell \in L, \\
x_i (1-x_j)(b^\ell - (a^\ell)^T x) &\geq 0, \qquad && i, j \in N, i \neq j,\ \ell \in L, \\
(1-x_i)(1-x_j)(b^\ell - (a^\ell)^T x) &\geq 0, \qquad && i, j \in N, i < j,\ \ell \in L.
\end{aligned}
$$

2) Linearize by replacing $x_i x_j x_k$ where $i \leq j \leq k$ with $y_{(i,j,k)}$, and replacing $x_i x_j$ by $y_{(i,j)}$.

3) Strengthen by replacing $y_{(i,i,k)}$ and $y_{(i,k,k)}$ with $y_{(i,k)}$ and $y_{(i,i)}$ with $y_{(i)}$.

Sherali-Adams level k operator

Step 1: Create the system of inequalities $(PSA^k(P))$:

$$\prod_{i \in J_1} x_i \prod_{j \in J_2} (1 - x_j)(b^\ell - (a^\ell)^T x) \geq 0,$$

for $\ell \in L$ and all J_1, J_2 where J_1 and J_2 are disjoint subsets of $I = \{1, \ldots, n\}$ and $|J_1 \cup J_2| = k$.

Step 2: Replace each polynomial term $\prod_{j=1}^{|S|} x_{S_j}$ in $PSA^k(P)$ with a real variable y_S where S is a tuple of up to $k+1$ terms to create $ESA^k(P)$.
Note: we are renaming the original variables $x_1, \ldots, x_n$ as $y_{(1)}, \ldots, y_{(n)}$.

Step 3: For a tuple S, let $[S]$ consist of unique elements in S in the same order: e.g., if $S_1 = (1, 1, 2)$, $S_2 = (1, 2, 3)$, then $[S_1] = (1, 2)$ and $[S_2] = (1, 2, 3)$.

$$LSA^k(P) := ESA^k(P) \cap \{y_S = y_{[S]} \quad \text{for all } S\}.$$

Sherali-Adams and 0-1 split cuts

Defn $SA^k(P)$ = projection of $LSA^k(P)$ onto space of variables $y_{(1)}, \ldots, y_{(n)}$.

Thm (Bodur, Dash, Günlük '16) The inequalities $y_S = y_{[S]}$ are 0-1 split cuts for $ESA^k(P)$.

Computational results

Let $G = (V, E)$ be a graph.

The stable set polytope of G:

$$\mathrm{STAB}(G) = \mathrm{conv}(\{x \in \{0,1\}^{|V|} : x_i + x_j \leq 1,\ \forall \{i,j\} \in E\}).$$

The fractional stable set polytope:

$$\mathrm{FSTAB}(G) = \{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1,\ \forall \{i,j\} \in E,\ \ 0 \leq x_i \leq 1,\ \forall i \in V\}.$$

$$\mathrm{STAB}(G) \subseteq SA^2(P) \subseteq \tilde{N}(P) \subseteq N^2(P) \subseteq N(P) \subseteq P = \mathrm{FSTAB}(G).$$

Each of the above inclusions can be strict.

If $G = K_n$ and $P = FRAC(G)$, then $SA^2(P) = N^2(P)$.

Strength of new operator

D	$\lvert V \rvert$	N	$N^2 - N$	$\tilde{N} - N^2$	$SA^2 - \tilde{N}$	Remaining
0.25	20	100	0	0	0	0
	30	97.5086	2.4914	0	0	0
	40	89.4596	10.3762	0.0843	0.0003	0.0796
	50	80.5525	18.3314	0.0862	0.0001	1.0299
0.5	20	80.2491	19.3807	0.0073	0	0.3628
	30	66.7473	27.7985	0	0	5.4542
	40	58.3730	29.1865	0	0	12.4404
	50	51.8947	25.9474	0	0	22.1579
0.75	20	61.2747	27.8776	0	0	10.8476
	30	50.4497	25.2249	0	0	24.3254
	40	46.3941	23.1971	0	0	30.4088
	50	43.9074	21.9537	0	0	34.1389

Table 2: Incremental Average Gap Closed Percentages

Summary

▷ Strongest extended LP formulation w.r.t split cuts for $P \subseteq [0,1]^n$ has n extra variables.

▷ In some simple cases, this formulation can be constructed explicitly.

▷ Lovász-Schrijver and Sherali-Adams operators add specific 0-1 split cuts to extended LP formulations.

▷ Adding all 0-1 split cuts to these extended formulations yields stronger operators.

▷ A similar modification is possible for the N_+ operators which incorporate semidefiniteness constraints.

▷ The new operators remain tractable.

▷ Coming up with practical implementations is an open question.

Lecture 3 outline

▷ Recap of Lecture 2

▷ Reformulate bounded MIP as 0-1 MIP ("Binarization")

▷ Relative strength of different binarizations.

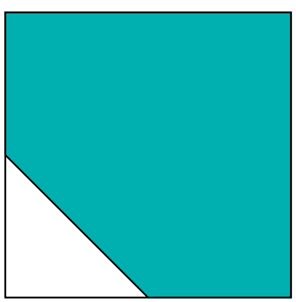
Split cuts for 0-1 problems

$\exists$ bounded MIP unsolvable by split cuts: Cook, Kannan, Schrijver '90

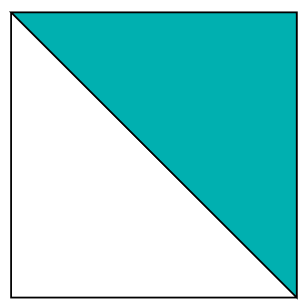
Thm (Balas '85) If $P \subseteq [0,1]^n]$, then $\mathrm{SC}_{01}^n(P) = P_I = \mathrm{conv}(P \cap \mathbb{Z}^n)$.

There can be significant difference between 0-1 split cuts and split cuts:

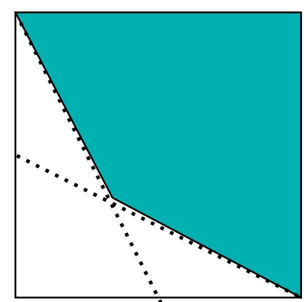
$$P^n = \{x \in [0,1]^n : x_1 + \cdots + x_n \geq 0.5\} \Rightarrow \mathrm{SC}(P^n) = P_I,\ \mathrm{SC}_{01}^{n-1}(P^n) \neq P_I.$$



P



SC(P)



SC_{01}(P)

Summary of lectures 1 & 2

▷ The strongest extended LP formulation w.r.t split cuts for $P \subseteq [0,1]^n$ has $\leq n$ extra variables, and its split closure gives the integer hull of P; the last claim is not true for general MIPs.

▷ Lovász-Schrijver and Sherali-Adams operators add specific 0-1 split cuts to extended LP formulations and can be strengthened by adding all 0-1 split cuts.

▷ Coming up with practical implementations is an open question.

Bounded IPs

Let $\ell \leq n$, and let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

$$\begin{aligned} \min \quad & c \cdot x \\ & Ax \leq b \\ & x_1, \ldots, x_\ell \in \{0, \ldots, u\} \end{aligned}$$

Binarization

Let $x \in \{0, \ldots, u\}$

Full binarization

Miller '63, Glover '75

Sherali, Adams '99

Owen and Mehrotra '02

Angulo, Van Vyve '17

$$x = \sum_{i=1}^{u} i z_i$$

$$\sum_{i=1}^{u} z_i \leq 1$$

$$z_1, \ldots, z_u \in \{0, 1\}$$

Logarithmic expansion

(let $k = \lfloor \log_2(u+1) \rfloor$)

Glover '75

$$x = \sum_{i=0}^{k} 2^i z_i$$

$$u \geq \sum_{i=0}^{k} 2^i z_i$$

$$z_1, \ldots, z_k \in \{0, 1\}$$

Binarization..

Unary binarization

Roy '07,

Bonami, Margot '15

$$x = \sum_{i=1}^{u} z_i$$

$$1 \geq z_1 \geq \ldots \geq z_{u-1} \geq z_u \geq 0$$

$$z_1, \ldots, z_u \in \{0, 1\}$$

If P = LP relaxation of original problem, let $L(P)$, $F(P)$ and $U(P)$ be the LP relaxations corresponding to the logarithmic, full and unary binarizations.

Let $x \in \{0, \dots, 4\}$

Full binarization

$$\begin{aligned} x = \quad & z_1 + 2z_2 + 3z_3 + 4z_4 \\ & z_1 + z_2 + z_3 + z_4 \leq 1 \\ & z_1, \dots, z_4 \in \{0, 1\} \end{aligned}$$

Logarithmic binarization

$$\begin{aligned} x = \quad & z_1 + 2z_2 + 4z_3 \leq 4 \\ & z_1, \dots, z_3 \in \{0, 1\} \end{aligned}$$

Unary binarization

$$\begin{aligned} x = \quad & z_1 + z_2 + z_3 + z_4 \\ & 1 \geq z_1 \geq z_2 \geq z_3 \geq z_4 \geq 0 \\ & z_1, \dots, z_4 \in \{0, 1\} \end{aligned}$$

Example

$$\max x_1 + x_2$$
$$x_1 + 10x_2 \leq 20$$
$$10x_1 + x_2 \leq 20$$
$$x_i \in \{0, 1, 2\}$$
$$\downarrow$$
$$\max x_1 + x_2$$
$$x_1 + 10x_2 \leq 20$$
$$10x_1 + x_2 \leq 20$$
$$x1 = z11 + 2z12, \quad z11 + z12 <= 1$$
$$x2 = z21 + 2z22, \quad z21 + z22 <= 1$$
$$x_i \in \{0, 1, 2\}, z_{ij} \in \{0, 1\}$$

Literature

▷ Owen and Mehrotra ('02): For full and logarithmic binarization, "branch-and-bound are trees are at least as large as for original problem."

> "... we conclude that the binary reformulations are unlikely to be of practical (computational) value and should be discouraged in practice unless special techniques are used to handle these variables."

Times on 14 MIPLIB 3.0 instances with CPLEX 6.6:

$P < L(P), F(P), \quad L(P) < F(P)$ on 10 instances (up to 194X)

$F(P) < L(P)$ on 4 instances (up to 3.7X)

Angulo, Van Vyve '17: fixed-charge transportation problems:Up to 100X speedup with $F(P)$ + strengthening

Fixed charge example

Angulo, Van Vyve '17:

$$
\begin{aligned}
\min \quad & \sum_{ij} q_{ij} y_{ij} \\
& \sum_{i} x_{ij} = d_j \quad \forall j \\
& \sum_{j} x_{ij} \leq c_i \quad \forall i \\
& y_{ij} \leq x_{ij} \leq a_{ij} y_{ij} \quad \forall i < j \\
& x_{ij} \in \mathbb{Z}, y_{ij} \in \{0, 1\} \quad \forall i < j
\end{aligned}
$$

Here $a_{ij} = \min\{c_i, d_j\}$.

Binarized fixed charge example

$$
\begin{aligned}
\min \quad & \sum_{ij} q_{ij} y_{ij} \\
& x_{ij} = \sum_{k=1}^{a_{ij}} k z_{ij}^k, \quad \sum_{k=1}^{a_{ij}} z_{ij}^i \leq 1 \;\; \forall i, j \\
& \sum_i \sum_{k=1}^{a_{ij}} k z_{ij}^k = d_j \;\; \forall j \\
& \sum_j \sum_{k=1}^{a_{ij}} k z_{ij}^k \leq c_i \;\; \forall i \\
& y_{ij} \leq x_{ij} \leq a_{ij} y_{ij} \;\; \forall i < j \\
& x_{ij} \in \mathbb{Z}, y_{ij}, z_{ij}^k \in \{0, 1\} \;\; \forall i < j, k
\end{aligned}
$$

Angulo, Van Vyve '17 use $y_{ij} \leq x_{ij} \leq a_{ij} y_{ij}$ to add $y_{ij} = \sum_{k=1}^{a_{ij}} z_{ij}^i$

Fixed charge running times

Geometric mean of running times for 5 instances with CPLEX 12.6.1 (600 sec time limit)

Problem $n, \max\{c_i\}$	P	F(P)	U(P)	AvV
30, 10	4.9	2.5	600	0.9
40, 10	75.2	7.1	600	2.4
40, 20	469.0	23.8	600	6.3

Bonami, Lodi, Tramontani: computational analysis of CPLEX behaviour.

Running times after small changes

Geometric mean of running times for 5 modified instances with CPLEX 12.6.1 (600 sec time limit)

Problem $n, \max\{c_i\}$	P	F(P)	U(P)	AvV
30, 10	4.9	12.0	23.0	10.1
40, 10	75.2	77.0	261.5	48.1
40, 20	469.0	471.1	600	434.1

Keep the following constraints unchanged when defining $F(P), U(P)$:

$$\sum_i x_{ij} = d_j \quad \forall j$$

$$\sum_j x_{ij} \leq c_i \quad \forall i$$

Split cuts

Cook, Kannan, Schrijver '90:
Let $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. A **split cut** for $P \subseteq \mathbb{R}^n$ is an inequality valid for

$$P \setminus S(a,b) = P \setminus \{x \in \mathbb{R}^n : b < a \cdot x < b+1\}$$

$$= P \cap \{a \cdot x \leq b\} \bigcup P \cap \{a \cdot x \geq b+1.\}$$

./splitcut-many2.pdf

Split closure

Let $\mathcal{S}^*$ stand for all split sets, and let $L \subseteq \mathcal{S}^*$.

Split closure of P **with respect to** L := points in P satisfying all split cuts for P derived from splits in L.

$$\mathrm{SC}(P, L) = \bigcap_{S \in L} \mathrm{conv}(P \setminus S)$$

Split closure of $P = \mathrm{SC}(P) = \mathrm{SC}(P, \mathcal{S}^*)$ and $SC^k(P) = SC(SC^{k-1}(P))$.

A **0-1 split cut** for $P \subseteq \mathbb{R}^n$ is an inequality valid for

$$P \cap \{x_i \leq 0\} \bigcup P \cap \{x_i \geq 1\}$$

$$= P \setminus \{x \in \mathbb{R}^n : 0 < x_i < 1\} = P \setminus S_i \ \ (S_i \text{ is a 0-1 split set}).$$

Literature

▷ Roy '07: For the Owen-Mehrotra instances, rank-2+ 0-1 split cuts for $U(P)$ "stronger" than rank-2+ simple split cuts for P.

▷ Bonami, Margot '15: rank-2 0-1 split cuts for $U(P)$ are stronger than rank-2 simple split cuts for P (for a small example and for 22 MIPLIB instances).

Definition

Informally, we want linear constraints that define a bijection between $\{0, \ldots, u\}$ and some points of $\{0,1\}^q$.

./mapping.pdf

Full, unary, logarithmic "binarization matrices" (bijection) for $u = 3$:

$$\left[\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline z_1 & 0 & 1 & 0 & 0 \\ z_2 & 0 & 0 & 1 & 0 \\ z_3 & 0 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline z_1 & 0 & 1 & 0 & 0 \\ z_2 & 0 & 1 & 1 & 0 \\ z_3 & 0 & 1 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline z_1 & 0 & 1 & 0 & 1 \\ z_2 & 0 & 0 & 1 & 1 \end{array}\right]$$

Binarization polytopes

Let Γ_u^q be the set of all rational polytopes $B \subseteq [0, u] \times [0, 1]^q$ such that

$$B \cap (\mathbb{R} \times \{0, 1\}^q) = \left\{ \begin{pmatrix} 0 \\ w^0 \end{pmatrix}, \ldots, \begin{pmatrix} u \\ w^u \end{pmatrix} \right\}$$

where $w^0 \ldots, w^u$ are distinct vectors in $\{0, 1\}^q$.

$B \in \Gamma_u^q$ is *perfect* if $B = \text{conv}(B \cap (\mathbb{R} \times \{0, 1\}^q))$.

▷ A perfect binarization polytope is fully defined by its binarization matrix.

▷ The full and unary binarizations are perfect.

▷ The log binarization is not perfect if $u + 1$ is not a power of 2, but can be made perfect with $O(\log u)$ inequalities (call this $L^+(P)$).

Proposition Let $u > 0$, and let $c_i = 2^{i-1}$. Then $\text{conv}(x \in \{0, 1\}^n : c \cdot x \leq u)$ can be described by $3n - 1$ inequalities.

(Follows from Laurent and Sassano '92)

Binary extended formulation

Let u be a positive integer, $0 \leq \ell \leq n$ and $I = \{1, \dots, \ell\}$,
$P = \Big\{x \in \mathbb{R}^n \;:\; Ax \leq b,\; x_i \in [0, u] \text{ for } i \in I\Big\}$,
$P^I = \{x \in P : x_i \in \mathbb{Z},\; \text{for } i \in I\}$.

Let $\mathcal{B} = (B^1, \dots, B^l)$ where $B^i \in \Gamma_u^{q_i}$, and $q = \sum_{i \in I} q_i$.
Defn

$$P_{\mathcal{B}} = \Big\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^q \;:\; Ax \leq b,\; (x_i, z_i) \in B^i \text{ for } i \in I\Big\}.$$

Defn Let $I_{\mathcal{B}} = \{1, \dots, l, n{+}1, \dots, n{+}q\}$. $P_{\mathcal{B}}^{I_{\mathcal{B}}}$ is a *binary extended formulation* (or, binarization) of P.

Fact: $\text{proj}_x(P_{\mathcal{B}}) = P$ and $\text{proj}_x(P_{\mathcal{B}}^{I_{\mathcal{B}}}) = P^I$.

Comparing binarizations

Defn If $B_1(P)$ and $B_2(P)$ are two binary extended formulations of P, then $B_1(P)$ is stronger than $B_2(P)$ if

$$\mathrm{proj}_x \mathrm{SC}(B_1(P)) \subseteq \mathrm{proj}_x \mathrm{SC}(B_1(P))$$

We also want to compare $\mathrm{proj}_x \mathrm{SC}(B_1(P))$ with $\mathrm{SC}(P)$.

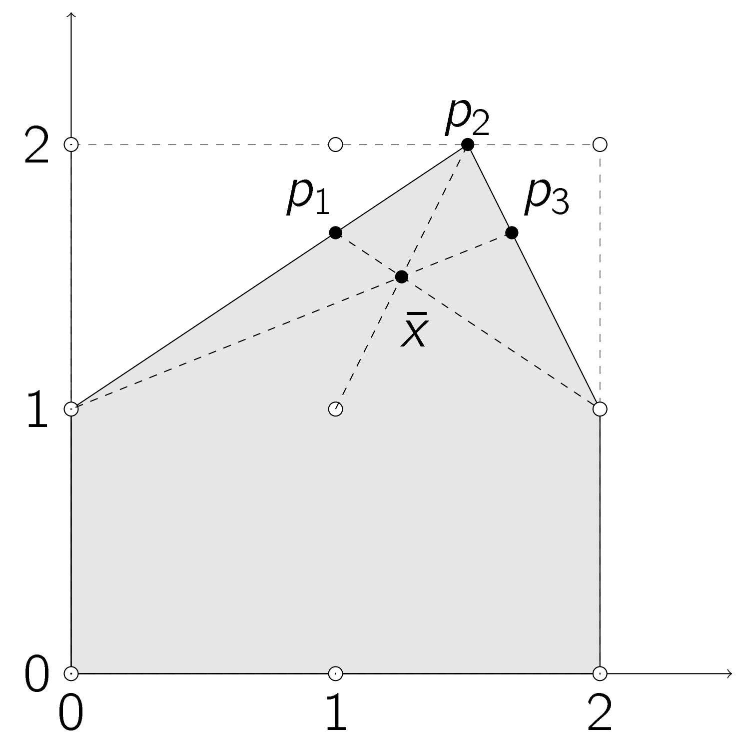
Thm (Dash, Günlük, Hildebrand '17)

$\mathrm{proj}_x \mathrm{SC}(F(P)) = \mathrm{proj}_x \mathrm{SC}(U(P)) \subseteq \mathrm{proj}_x \mathrm{SC}(L^+(P)) \subseteq \mathrm{proj}_x \mathrm{SC}(L(P)) \subseteq \mathrm{SC}(P)$
All inclusions can be strict.

Example 1

$$P = \Big\{ x \in [0,2]^2 \; : \; 2x_1 + x_2 \le 5, \; -2x_1 + 3x_2 \le 3 \Big\}$$

$$L(P) = \Big\{ (x,z) \; : \; x \in P, \; z \in [0,1]^4 \; : \; x_i = z_{i1} + 2z_{i2}, \quad i = 1,2 \Big\}$$

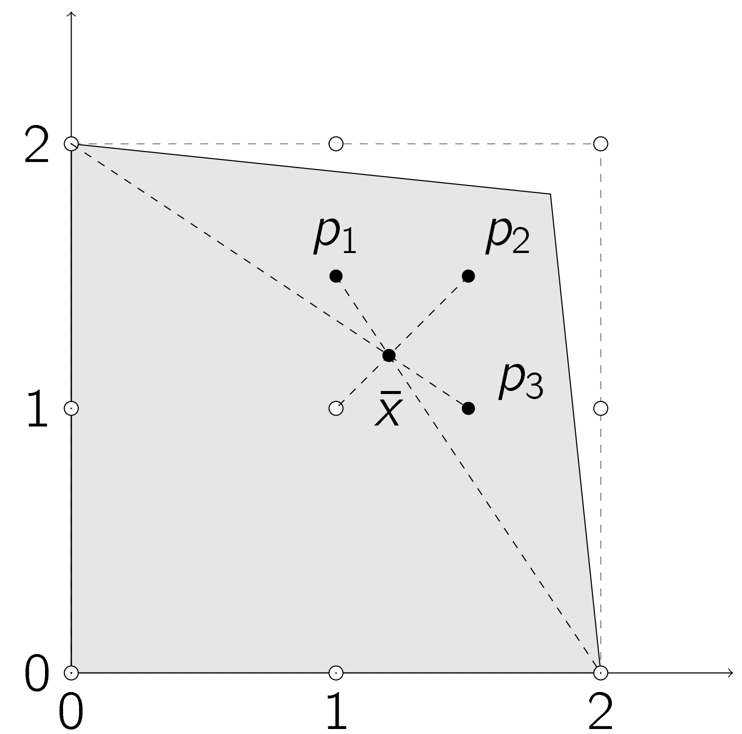


Thm $\text{proj}_x SC(L(P)) \subsetneq SC(P)$.

The point $\bar{x} = (1.25, 1.5)$ belongs to the split closure of P but the inequality $x_2 \le 1.4$ is valid for the split closure of $L(P)$.

Example 2

$$P = \Big\{x \in [0,2]^2 \ : \ x_1 + 10x_2 \le 20, \ 10x_1 + x_2 \le 20\Big\}$$



$$L(P) = \Big\{(x,z) \ : \ x \in P, \ z \in [0,1]^4, \ x_i = z_{i1} + 2z_{i2}, \quad i = 1,2\Big\}$$

$L^+(P)$ also has $z_{i1} + z_{i2} \le 1$

Thm $\text{proj}_x SC(L^+(P)) \subsetneq \text{proj}_x SC(L(P))$.

The point $\bar{x} = (6/5, 6/5)$ belongs to $\text{proj}_x(SC(L(P)))$ but the inequality $x_1 + x_2 \le 2$ is valid for the split closure of $L^+(P)$.

Example 3

$$
\begin{aligned}
P \quad = \quad & \Big\{(x, y) \in \mathbb{R}^3 \times [0, 1]^3 \; : \\
& x_1 + x_2 + x_3 = 4 \\
& x_i \le 4y_i, \;\; 0 \le x_i \le 3 \;\; \forall i\Big\}
\end{aligned}
$$

$$
\begin{aligned}
F(P) \quad = \quad & \Big\{(x, y, z) \; : \; (x, y) \in P, \quad z \in [0, 1]^9, \\
& x_i = z_{i1} + 2z_{i2} + 3z_{i3} \;\; \forall i, \\
& z_{i1} + z_{i2} + z_{i3} \le 1 \;\; \forall i\Big\}
\end{aligned}
$$

Thm $\mathrm{proj}_x SC(F(P)) \subsetneq \mathrm{proj}_x SC(L^+(P))$.

Strongest binarizations

A binarization polytope $B \in \Gamma_u^u$ is *unimodular* if the first column of its binarization matrix is zero, and the remaining columns form a unimodular matrix.

Full, unary, logarithmic binarization matrices for $u = 3$:

$$\left[\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline z_1 & 0 & 1 & 0 & 0 \\ z_2 & 0 & 0 & 1 & 0 \\ z_3 & 0 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline z_1 & 0 & 1 & 0 & 0 \\ z_2 & 0 & 1 & 1 & 0 \\ z_3 & 0 & 1 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline z_1 & 0 & 1 & 0 & 1 \\ z_2 & 0 & 0 & 1 & 1 \end{array}\right]$$

Dash, Günlük, Hildebrand '17:

Thm Let $B_1(P)$ and $B_2(P)$ be any two perfect binary extended formulations of P defined by unimodular binarization polytopes. Then $\text{proj}_x SC(B_1(P)) = \text{proj}_x SC(B_2(P))$.

Thm Let $B(P)$ be any perfect binary extended formulation of P. Then $\text{proj}_x SC(F(P)) \subseteq \text{proj}_x SC(B_2(P))$.

Branching

Thm (Dash, Günlük, Hildebrand '17): Let $u \geq 4$ and $n \geq 2$. There is a polytope P_n such that any branch-and-bound tree that solves $\max\{|x - \mathbf{u}/2|_1 : x \in P \cap \mathbb{Z}^n\}$ requires $2^{n+1} - 1$ nodes, but there exists a branch-and-bound tree with $2^n + n$ nodes solving $\max\{|x - \mathbf{u}/2|_1 : (x, z) \in P' \cap \mathbb{Z}^n\}$ for a binary extended formulation P' of P using the binarization:

$$\{(x, z) \in \mathbb{R} \times [0, 1]^u :$$

$$x = uz_u + \sum_{i=1}^{u-1} z_i$$

$$0 \leq z_{u-1} \leq \ldots \leq z_1 \leq 1$$

$$z_1 + z_u \leq 1\}.$$