Bootstrapping 0-1 Laws

by

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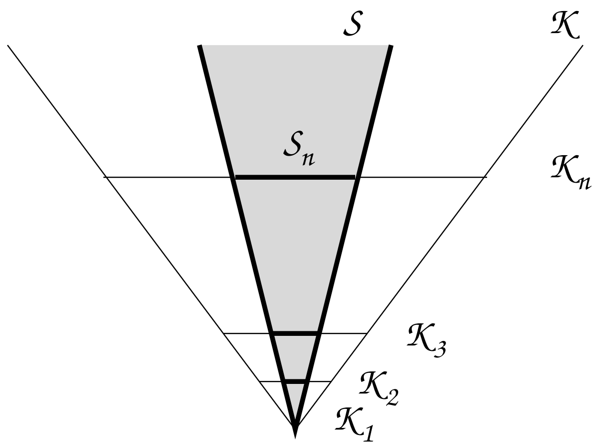
ASYMPTOTIC DENSITY

- $\mathcal{K}$ = a class of finite structures
- $\mathcal{K}_n := \{$structures in $\mathcal{K}$ of size $n\}$
- $f(n) :=$ number of structures in $\mathcal{K}_n$

Asymptotic Density of $\mathcal{S} \subseteq \mathcal{K}$



$$\text{density}(\mathcal{S}) := \lim_{\substack{n \to \infty \\ \mathcal{K}_n \neq \emptyset}} \frac{|\mathcal{S}_n|}{|\mathcal{K}_n|}$$

0-1 LAWS

If φ is a sentence let

$$\mathcal{K}_\varphi := \mathcal{K} \cap \mathsf{Models}(\varphi)$$

$\mathcal{K}$ has a <u>0-1 law</u> means:

$\mathcal{K}_\varphi$ has asymptotic density 0 or 1, for all φ.

EXAMPLES:
(1) Finite Graphs
(2) Finite Linear Forests

SPECTRALLY DETERMINED 0-1 LAWS

A context where the counting function f alone can guarantee a 0-1 law.

ADMISSIBLE CLASSES

- $\mathcal{K}$ is closed under isomorphism
- Members of $\mathcal{K}$ have a unique decomposition into the (relative) indecomposables of $\mathcal{K}$
- $\mathcal{K}$ is closed under disjoint union

These classes are the ones that naturally go with the weighted partition identities. Some are studied in combinatorics.

THE WEIGHTED PARTITION IDENTITY OF AN ADMISSIBLE CLASS $\mathcal{K}$

$f(n)$ is the number of structures in $\mathcal{K}_n$

$g(n)$ is the number of indecomposable structures in $\mathcal{K}_n$

We have the "formal identity"

$$1+\sum_{n=1}^{\infty} f(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-g(n)}.$$

This gives a handy way to remember how to compute $f(n)$ from $g(n)$, namely $f(n)$ is the coefficient of x^n in the expansion of the product of the first n terms on the right.

THE RADIUS OF CONVERGENCE OF $\sum f(n)x^n$

Suppose $\mathcal{K}$ is an admissible class.

For simplicity we will assume that the GCD of the sizes of the indecomposables is 1.

THEOREM [Compton, 1987]

$\mathcal{K}$ has a 0–1 law if $\dfrac{f(n-1)}{f(n)} \to 1.$

One can phrase this as follows:

$\mathcal{K}$ has a 0–1 law if $\sum f(n)x^n$ has a radius of covergence R $= 1$ $\boxed{\textit{by the ratio test}}$.

RESULT #1 [Cayley and Sylvester (1850's), Schur]

$\boxed{\text{Finitely Many Indecomposables}}$

Suppose there are only finitely many indecomposables in an admissible class $\mathcal{K}$.

Let $r = \sum g(n)$, the number of indecomposables.

Using partial fractions in $\mathbf{C}(x)$ one can show

$$f(n) \sim Cn^{r-1}$$

From this it easily follows that

$$\frac{f(n-1)}{f(n)} \to 1$$

so $\mathcal{K}$ has a 0-1 law.

INFINITELY MANY INDECOMPOSABLES

The classic partition identity is

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$$

where $p(n)$ is the numbers of ways to express n as a sum of positive integers.

In 1918 Hardy and Ramanujan used their circle method to show

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Thus $p(n-1)/p(n) \to 1$.

Such asymptotics are quite difficult to obtain.

RESULT #2 [Meinardus (1954)]

$$1+\sum_{n=1}^{\infty} f(n)x^n = \prod_{n=1}^{\infty} (1-x^n)^{-g(n)}$$

with some GRUESOME side conditions

In this rather opaque paper Meinardus also finds explicit asymptotics: $f(n) \sim An^{\alpha}e^{Bn^{\beta}}$ This gives (as $0 < \beta < 1$)

$$\frac{f(n-1)}{f(n)} \to 1.$$

A special case: $\boxed{g(n) = n}$.

Thus an admissible class $\mathcal{K}$ with exactly n indecomposables of size n has a 0-1 law.

RESULT #3 [Bateman and Erdös (1956)]

$$1 + \sum_{n=1}^{\infty} f(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-g(n)}$$

$$\boxed{\text{where } g(n) \leq 1 \text{ for all } n}$$

Using elementary but clever arguments, Bateman and Erdös show that if $g(n) \leq 1$ then one still has

$$\frac{f(n-1)}{f(n)} \to 1.$$

Thus an admissible class $\mathcal{K}$ with at most one indecomposable of each size has a 0-1 law.

And that is about all there is in the literature to guarantee

$$\frac{f(n-1)}{f(n)} \to 1$$

for the counting function of an admissible class $\mathcal{K}$.

NEW RESULT: During the Special Year on Logic and Algorithms I discovered a simple bootstrap theorem (with the help of Cam Stewart)

Is this really the first new general result on $f(n-1)/f(n) \to 1$ in 40 years?

Is this really new?

BOOTSTRAP THEOREM

Suppose $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ are admissible classes with

$$g = g_1 + g_2$$

If $\mathcal{K}_1$ and $\mathcal{K}_2$ both have a 0-1 law by the ratio test then $\mathcal{K}$ also has a 0-1 law by the ratio test.

COROLLARY
An admissible class $\mathcal{K}$ with $g(n) \leq B$ for all n has a 0-1 law.

SUMMARY

For $\mathcal{K}$ an admissible class,

$\dfrac{f(n-1)}{f(n)} \to 1$ implies the following:

- $\mathcal{K}$ has a 0-1 law
- f is subexponential, i.e., $f(n) = O(c^n)$ for all $c > 1$

$\dfrac{f(n-1)}{f(n)} \to 1$ follows from any of:

- $\sum g(n) < \infty$
- $g(n) = n$
- $g \leq B$

And we have the Bootstrap Theorem.

OPEN: Does $g(n) = O(n^c)$ imply $\frac{f(n-1)}{f(n)} \to 1$

THE KEY

Let $\mathcal{K}$ be an admissible class.

For φ a first-order sentence one can find

- a finite partition $\mathcal{P}_1, \ldots, \mathcal{P}_s$ of the indecomposables of $\mathcal{K}$ and
- a positive integer c such that

$$\text{Models}(\varphi) = \bigcup \mathcal{S}_i$$

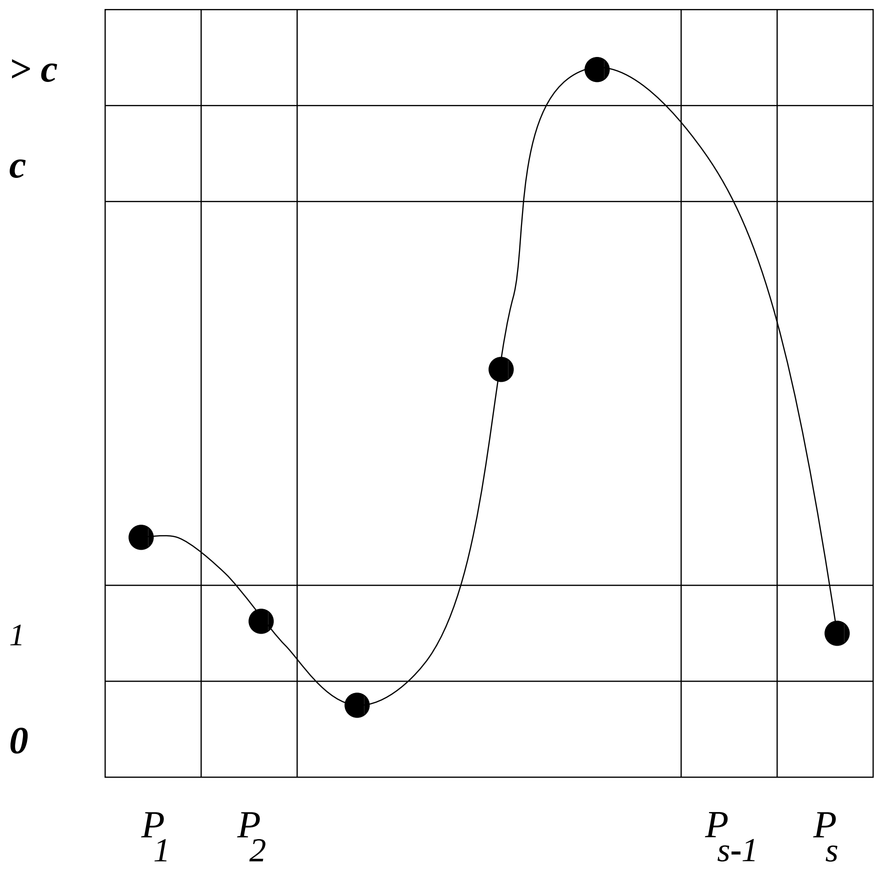
where each $\mathcal{S}_i$ is of the form

$$\lambda_1 \mathcal{P}_1 \cup \cdots \cup \lambda_s \mathcal{P}_s$$

[READ: the collection of structures that have λ_1 indecomposables from $\mathcal{P}_1$, etc.]

with each λ_i in $\{0, 1, \cdots, c, > c\}$.

“VISUALIZING” $\mathcal{S}_j = \sum \lambda_i \mathcal{P}_i$



Thus if one can show each

$$(\geq n_1)\mathcal{P}_1 \cup \cdots \cup (\geq n_s)\mathcal{P}_s$$

has a 0 or 1 asymptotic density, then $\mathcal{K}$ has a 0-1 law.

THE DENSITY OF $\mathbf{A} \cup \mathcal{K}$

Let $\mathcal{K}$ be an admissible class with counting function f. Let $\mathbf{A} \in \mathcal{K}$.

Then the density of $\mathbf{A} \cup \mathcal{K}$ is $\lim_{n\to\infty} \frac{f(n-|A|)}{f(n)}$

So $\frac{f(n-1)}{f(n)} \to 1$ implies the density of $\mathbf{A} \cup \mathcal{K}$

is 1 for every $\mathbf{A}$ in $\mathcal{K}$,

And then, for every partition $\mathcal{P}_1, \ldots, \mathcal{P}_s$ of the indecomposables and for every sequence $n_1, \ldots, n_s$ of nonnegative integers we have:

the density of $(\geq n_1)\mathcal{P}_1 \cup \cdots \cup (\geq n_s)\mathcal{P}_s$ is 1.

Thus $\mathcal{K}$ has a 0-1 law.