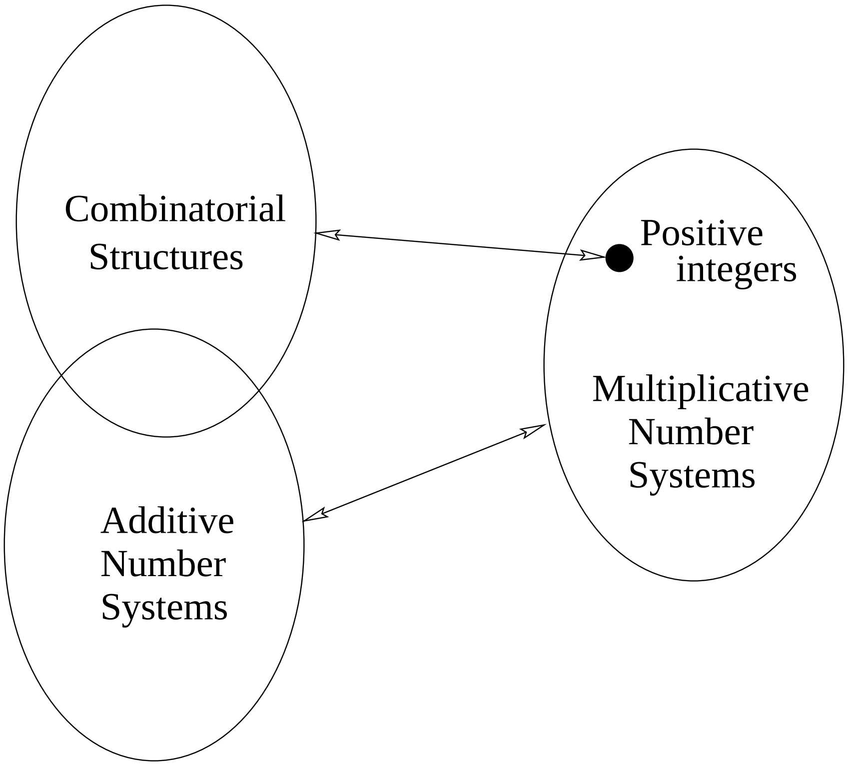
Parallels between Additive Number Systems and Multiplicative Number Systems

Stanley Burris

Pure Mathematics Number Theory Seminar

November, 2002

Parallels



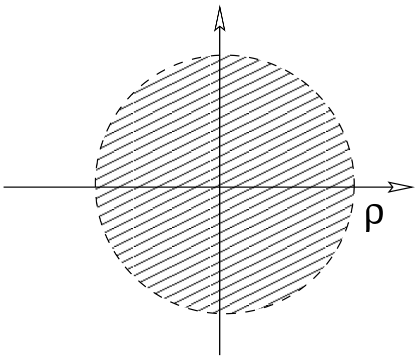
n	– – – – – ––	$\log n$
local	– – – – – ––	global
RT	– – – – – ––	RV

Review

Power Series

$$\mathbf{A}(z) = \sum_{n \geq 0} a(n) z^n$$

Radius of Convergence



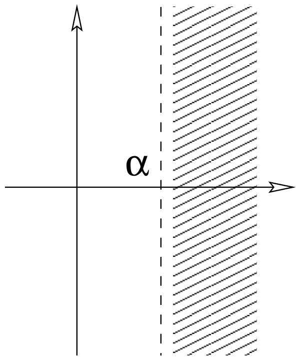
Cauchy Formula

$$a(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \mathbf{A}(z) \frac{dz}{z^{n+1}}$$

Dirichlet Series

$$\mathbf{A}(s) = \sum_{n \geq 1} a(n) n^{-s}$$

Abscissa of Convergence



Perron Formula

$$A(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathbf{A}(s) \frac{x^s}{s} ds$$

if x is not an integer

Number Systems

Additive	Multiplicative
$(A, P, +, 0, \|\ \|)$	$(A, P, \cdot, 1, \|\ \|)$

$$A = \text{the set of "numbers"}$$

$$P = \text{the set of indecomposable "numbers"}$$

$\|\ \|$ is additive	$\|\ \|$ is multiplicative

Basic Requirements:

- only finitely many numbers have a given norm

- each number is uniquely decomposable as a sum of indecomposables.

The **Fundamental Identity** of a number system:

Additive Case

$$\begin{aligned}\mathbf{A}(x) \;:=\; \sum_{n\geq 0} a(n)x^n \;&=\; \prod_{n\geq 1}(1-x^n)^{-p(n)}\\ &=\; \exp\Big(\sum_{m\geq 1}\mathbf{P}(x^m)/m\Big)\end{aligned}$$

where

$$\mathbf{P}(x) \;:=\; \sum_{n\geq 1} p(n)x^n.$$

Multiplicative Case

$$\begin{aligned}\mathbf{A}(x) \;:=\; \sum_{n\geq 1} a(n)n^{-x} \;&=\; \prod_{n\geq 2}(1-n^{-x})^{-p(n)}\\ &=\; \exp\Big(\sum_{m\geq 1}\mathbf{P}(mx)/m\Big)\end{aligned}$$

where

$$\mathbf{P}(x) \;:=\; \sum_{n\geq 2} p(n)n^{-x}.$$

Radius/Abscissa of convergence

We have

$0 \leq \rho \leq 1$	$0 \leq \alpha \leq \infty$
$\rho = 0$ fast growth	$\alpha = \infty$ fast growth
$\rho > 0$ slow growth	$\alpha < \infty$ slow growth

Generating Functions of subsets of A

For B $\subseteq$ A

Additive	Multiplicative
$\mathbf{B}(x) = \sum_{n \geq 0} b(n) x^n$	$\mathbf{B}(x) = \sum_{n \geq 1} b(n) n^{-x}$

where $b(n)$ counts the number of elements of B of size n.

Finitely Generated Number Systems
show parallel behaviour

Let $r = \sum p(n)$.

Additive

$$a(n) \ \sim \ \left(\prod m^{-p(m)}\right) \cdot \frac{n^{r-1}}{(r-1)!}$$

Multiplicative

$$A(n) \ \sim \ \left(\prod (\log m)^{-p(m)}\right) \cdot \frac{(\log n)^{r}}{r!}$$

Another Parallel Result

Knopfmacher$_2$ + Warlimont

$$p(n) \;=\; a\beta^n \;+\; \mathrm{O}(\gamma^n)$$

implies

$$a(n) \;\sim\; C\beta^n \frac{e^{2\sqrt{an}}}{n^{3/4}}$$

Oppenheim +

$$p(n) \;=\; an^{\beta} + \mathrm{O}(n^{\gamma})$$

implies

$$A(x) \;\sim\; Cx^{\beta+1} \frac{e^{2\sqrt{a \log x}}}{(\log x)^{3/4}}$$

Asymptotic Density
of a subset B of A

Local Density

$$\delta(\mathsf{B}) \;=\; \lim_{n\to\infty} \frac{b(n)}{a(n)}$$

Global Density

$$\Delta(\mathsf{B}) \;=\; \lim_{x\to\infty} \frac{B(x)}{A(x)}$$

Dirichlet Density
of a subset B of A

$$\partial(\mathsf{B}) \;=\; \lim_{x\to\rho-} \frac{\mathbf{B}(x)}{\mathbf{A}(x)}$$

need $\rho > 0$

$$\partial(\mathsf{B}) \;=\; \lim_{x\to\alpha+} \frac{\mathbf{B}(x)}{\mathbf{A}(x)}$$

need $\alpha < \infty$

Results that look as though they should have been proved long ago, but are actually very recent

Additive	Multiplicative
[Bell] **[Bell,. . . ,Richmond]**	**[Warlimont]**
$\delta(\mathsf{P}) = 0$ implies	$\Delta(\mathsf{P}) = 0$ implies
$\begin{cases} \rho > 0 \text{ and} \\ \mathbf{A}(\rho) = \infty \end{cases}$	$\begin{cases} \alpha < \infty \\ \mathbf{A}(\alpha) = \infty \end{cases}$
	[Ruzsa] has a new proof.

Ratio Test and Regular Variation at Infinity

Definition A sequence $s(n)$ is in RT_ρ if it is eventually positive and

$$\lim_{n\to\infty} \frac{s(n-1)}{s(n)} \;=\; \rho.$$

Definition A function $S(x)$ is in RV_α if it is eventually defined, eventually positive and

$$\lim_{t\to\infty} \frac{S(tx)}{S(t)} \;=\; x^\alpha$$

for $x > 0$. We say $S(x)$ has **regular variation at infinity of index** α.

A function in RV_0 is **slowly varying at infinity**.

The parallels in the concepts are clearer if we write:

$$\frac{s(n-k)}{s(n)} \to \rho^k \quad \text{and} \quad \frac{S(t/x)}{S(t)} \to x^{-\alpha}$$

One Reason we like RT and RV

All sets of the form

$$b + A$$

have asymptotic density iff RT_ρ holds.

If all sets of the form

$$b \cdot A$$

have asymptotic density then either RV_α holds or the system is discrete.

RV_α implies all sets of the form bA have asymptotic density.

The cases RT_1 and RV_0 were studied first (they yield 0–1 laws).

[Bell] $p(n) = \mathsf{O}(n^c)$ implies RT_1	**[Bell]** $P(x) = \mathsf{O}((\log x)^c)$ implies RV_0
[Bell] $p(n) \in \mathsf{RT}_1$ implies $a(n) \in \mathsf{RT}_1$	**[Bell]** (Conj II) $P(x) \in \mathsf{RV}_0$ implies $A(x) \in \mathsf{RV}_0$
[Stewart] $\mathcal{A}_i \in \mathsf{RT}_1$ implies $\mathcal{A}_1 + \mathcal{A}_2 \in \mathsf{RT}_1$	**[Odlyzko]** $\mathcal{A}_i \in \mathsf{RV}_0$ implies $\mathcal{A}_1 + \mathcal{A}_2 \in \mathsf{RV}_0$

[Karen Yeats] has a common generalization that gives the precise conditions for the sum/product of two number systems to preserve the existence of asymptotic density for all partition sets.

Schur's Theorem

[Schur 1918]

If

- $\mathbf{R}(x) = \mathbf{S}(x)\mathbf{T}(x)$
- $t(n) \in \mathsf{RT}_\rho$
- $\rho_{\mathbf{S}} > \rho$

then

$$\lim_{n\to\infty} \frac{r(n)}{t(n)} = \lim_{x\to\rho} \frac{\mathbf{R}(x)}{\mathbf{T}(x)}$$

[Burris/Yeats 2001]

If

- $\mathbf{R}(x) = \mathbf{S}(x)\mathbf{T}(x)$
- $T(x) \in \mathsf{RV}_\alpha$
- $\alpha_{\mathbf{S}} < \alpha$

then

$$\lim_{x\to\infty} \frac{R(x)}{T(x)} = \lim_{x\to\alpha} \frac{\mathbf{R}(x)}{\mathbf{T}(x)}$$

For example these can be used on the alternate form of the fundamental identities

$$\mathbf{A}(x) = e^{\sum_{m\geq 1} \mathbf{P}(x^m)/m}$$

$$\mathbf{A}(x) = e^{\sum_{m\geq 1} \mathbf{P}(mx)/m}$$

to focus attention on the asymptotics of the count function of $\boxed{e^{\mathbf{P}(x)}}$.

Applying Schur to get the asymptotics for the count function of a number system.

Using Schur:

$$\begin{aligned} &\mathbf{A}(x) \\ =\ & e^{\sum_{m\geq 1}\mathbf{P}(x^m)/m} \\ =\ & e^{\mathbf{P}(x)}\cdot e^{\sum_{m\geq 2}\mathbf{P}(x^m)/m} \end{aligned}$$

to get

$$a(n)\ \sim\ C\cdot [x^n]e^{\mathbf{P}(x)}$$

where

$$C\ =\ e^{\sum_{m\geq 2}\mathbf{P}(\rho^m)/m}$$

Using Schur Analog:

$$\begin{aligned} &\mathbf{A}(x) \\ =\ & e^{\sum_{m\geq 1}\mathbf{P}(mx)/m} \\ =\ & e^{\mathbf{P}(x)}\cdot e^{\sum_{m\geq 2}\mathbf{P}(mx)/m} \end{aligned}$$

to get

$$A(x)\ \sim\ C\cdot \sum_{n\leq x}[n^{-u}]e^{\mathbf{P}(u)}$$

where

$$C\ =\ e^{\sum_{m\geq 2}\mathbf{P}(m\alpha)/m}$$

What is a Partition Set?

Let $\mathsf{P}_1, \ldots, \mathsf{P}_k$ be a partition of the set P of indecomposables into finitely may sets.

Let γ_i be in one of the forms

$$\gamma_i \;=\; m_i \qquad \gamma_i \;=\; (\leq m_i) \qquad \gamma_i \;=\; (\geq m_i).$$

Then a subset B of the form

$$\mathsf{P}_1^{\gamma_1} \cdots \mathsf{P}_k^{\gamma_k}$$

is a partition set.

In other words, B consists of all elements of A which have exactly

γ_1 indecomposables from P_1,
...
γ_k indecomposables from P_k.

Let $(\Diamond)$ be the property

$\boxed{\text{All } \textbf{partition sets} \text{ have asymptotic density.}}$

Then

$(\Diamond)$ implies RT_ρ	$(\Diamond)$ implies discrete or RV_α
$(\Diamond)$ implies $\delta(P) = 0$	$(\Diamond)$ implies $\Delta(P) = 0$
RT_1 implies $(\Diamond)$	RV_0 implies $(\Diamond)$

On the next slide we look at **the most powerful conditions** known to imply that $(\Diamond)$ holds when $\rho < 1$, resp. $\alpha > 0$.

[Compton]

If $a(n) \in \mathsf{RT}_\rho$ and

$$\frac{a(n-m)}{a(n)} \leq C\rho^m$$

for $K \leq m \leq n$
then ($\lozenge$) holds.

[Sárközy]

If $A(x) \in \mathsf{RV}_\alpha$ and

$$\frac{A(x/m)}{A(x)} \leq Cm^{-\alpha}$$

for $K \leq m \leq n$
then ($\lozenge$) holds.

[Bell]

If $p(n) \in \mathsf{RT}_\rho$ and

$$\limsup_{n\to\infty} np(n)\rho^n > 1$$

then Compton's conditions for ($\lozenge$) hold.

[Bell] (Conj III)

If

$$P(x) \sim x^\alpha P_0(x)/\log x,$$

$P_0(x) \in \mathsf{RV}_0$, eventually increasing and

$$\lim_{x\to\infty} P_0(x) \in (1/\alpha, \infty)$$

then Sárközy's conditions for ($\lozenge$) hold.

Admissible Functions

These are functions for which one can use the famous saddle-point method to determine asymptotics for the count function of the series expansion.

[Compton]

If $\mathbf{A}(x)$ is **Hayman-admissible**
then Compton's conditions for ($\Diamond$) hold.

[Burris, Warlimont, Yeats]

If $\mathbf{A}(x)$ is **admissible** then Sárközy's conditions for ($\Diamond$) hold.

A **Tenenbaum admissible** Dirichlet series is admissible.

Admissible Power Series

Suppose

$$\mathbf{A}(z) \;:=\; \sum_{n\geq 0} a(n)z^n \;=\; e^{\mathbf{H}(z)}$$

is Hayman-admissible. Let the Taylor series expansion of $\mathbf{H}(re^{i\theta})$ about $\theta = 0$ be

$$\mathbf{H}(re^{i\theta}) \;=\; \mathbf{H}(r) \;+\; i\mathbf{a}(r)\theta \;-\; \mathbf{b}(r)\theta^2/2 \;+\; \cdots$$

Theorem.

$$a(n) \;=\; \frac{\mathbf{A}(r)}{r^n\sqrt{2\pi\mathbf{b}(r)}}\Big(\exp\big(\frac{-(\mathbf{a}(r)-n)^2}{2\mathbf{b}(r)}\big) \;+\; R(r,n)\Big)$$

where $R(r,n) \to 0$ as $r \to \rho-$, **uniformly** for $n \geq 0$.

The **saddle point** is r_n, the solution r of

$$\mathbf{a}(r) - n \;=\; 0$$

Admissible Dirichlet Series

Suppose

$$\mathbf{A}(s) \ := \ \sum_{n\geq 1} a(n)/n^s \ = \ e^{\mathbf{H}(s)}$$

is an admissible Dirichlet series. Let the Taylor series expansion of $\mathbf{H}(\sigma+it)$ about $t=0$ be

$$\mathbf{H}(\sigma+it) \ = \ \mathbf{H}(\sigma) \ + \ i\mathbf{a}(\sigma)t \ - \ \mathbf{b}(\sigma)t^2/2 \ + \ \cdots$$

Theorem.

$$\widehat{A(x)} \ = \ \frac{x^{\sigma+1}\mathbf{A}(\sigma)}{\sigma(\sigma+1)\sqrt{2\pi\mathbf{b}(\sigma)}}\Big(\exp\big(\frac{-(\mathbf{a}(\sigma)+\log x)^2}{2\mathbf{b}(\sigma)}\big) \ + \ R(\sigma,x)\Big)$$

where $R(\sigma,x)\to 0$ as $\sigma\to\alpha+$, **uniformly** for $x>0$.

The **saddle point** is σ_x, the solution σ of

$$\mathbf{a}(\sigma) \ + \ \log x \ = \ 0.$$