Understanding Boole's Algebra of Logic

ASL Special Session

Algebraic Logic and Universal Algebra

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Stanley Burris
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario Canada
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www.thoralf.uwaterloo.ca

The major portion of Boole's work on logic was properly understood*, and justified, for the first time by

Theodore Hailperin, **Boole's Logic and Probability** 1976/1986.

However, like others before him, he failed to make sense of Boole's use of **equations to handle Particular Propositions** (those with existential import) in Aristotelian arguments.

This last gap will be filled in this talk.

*Modulo the universal failure to understand the semantics of simple names used by Boole.

Boole's Approach to the Algebra of Logic
or
The Laws of Boole's Thought

We have (in 1847)

Common Algebra	and	Aristotelian Logic

Boole embraces both of these!

Boole's Goal: Find the Missing Link between the two subjects.

Boole's Key Step:
Multiplication corresponds to Intersection

This leads to the **idempotent law**: $\boxed{A^2 = A}$

Boole says **all** of common algebra applies to his algebra of logic.

But in practice he avoids obvious pitfalls like

$$A^2 = A \;\rightarrow\; A = 0 \text{ or } A = 1.$$

In practice Boole is using $\boxed{\text{quasi-identities}}$ from common algebra.

For example, he has

$$A + A = 0 \;\rightarrow\; A = 0,$$

and

$$A + A = A \;\rightarrow\; A = 0.$$

This rules out '+' corresponding to either *symmetric difference* or *union*.

For his algebra of logic, Boole wants to know:

1. If a quasi-identity

$$\bigwedge_i t_i(\vec{A}) = 0 \ \rightarrow \ t(\vec{A}) = 0$$

   is valid.

2. The result $t(\vec{B}\ ) = 0$ of **eliminating** $\vec{A}$ from $s(\vec{A}, \vec{B}\ ) = 0$.

3. **The most general solution** $A = t(\vec{B}\ )$ to $s(A, \vec{B}\ ) = 0$.

More Than A Century of Confusion

The literature from the late 1800s through the entire 20th century is filled with the wreckage of muffled circumlocutions and failed attempts to explain what Boole was doing.

The Universal Error has been to assume that Boole was using **Modern Semantics** for simple class names.

These Modern Semantics were introduced in 1880 by C. S. Peirce.

Peirce's definition of when 'All A is B' is true agrees precisely with our modern usage of $A \subseteq B$—the case $A = \emptyset$ belongs to the true cases.

Boole used **Aristotelian Semantics**!!

For Boole the **Conversion by Limitation** argument

‘All A is B’, therefore ‘Some B is A’

was correct (and proved in both his 1847 and 1854 texts).

The **next most common error** is to misunderstand Boole's operation '$+$'.

For Boole this was a partial operation (Laws of Thought, 1854, p. 66):

> The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common.

Boole was quite aware of the notion of *union* and of *symmetric difference*, and showed how to express them in terms of his operation '$+$'. But these operations were not his '$+$'.

Boole's Translation of Categorical Statements into Equations

Affirmative Propositions

		1847	1848/1854
A	All X is Y	$X = XY$	$X = VY$
I	Some X is Y	$V = XY$	$VX = VY$

For the Negative Propositions just replace Y by $1 - Y$ in the above.

		1847	1848/1854
E	No X is Y	$X = X(1 - Y)$	$X = V(1 - Y)$
O	Some X is not Y	$V = X(1 - Y)$	$VX = V(1 - Y)$

Boole Freed From Every Flaw (BF_3)

A	All X is Y	$X = XY$
I	Some X is Y	$V = VXY$

For the negative propositions replace Y by $1 - Y$.

The choice for **A** is the same as Boole 1847.

The choice for **I** lies between Boole 1847 and Boole 1848/1854 since:

$$V = XY \quad \text{implies} \quad V = VXY$$
$$\text{implies} \quad VX = VY.$$

Kneale and Kneale, *The Development of Logic*, 1962, p. 411

Hailperin, *Boole's Logic and Probability*, 1986, p. 112

OBJECTION: Boole has special rules for interpreting V as 'some', for example:

Some X is not Y	$VX = V(1-Y)$
	algebraically leads to
Some Y is not X	$VY = V(1-X)$

VY is not to be interpreted as 'some Y'.

ANSWER: In BF_3 there are no special rules limiting when one can interpret the symbols as 'some'.

In BF_3 one has

Some X is not Y $\quad V = VX(1-Y)$

does not lead to

Some Y is not X $\quad V = VY(1-X)$

Furthermore we note that in BF_3 any letter can play the role of V.

EXAMPLE: (Limitation)

All X is Y $\quad X = XY$

leads to

Some X is Y $\quad X = X(XY)$

C.S. Peirce, *On the agebra of logic*, 1880, [p. 22]

OBJECTION: ‘Some X is not Y’ yields, by Boole’s algebraic system, ‘Some Y is not X’.

ANSWER: (See previous two slides):

(1) Peirce does not realize that Boole has restrictions on when one can interpret V as ‘some’.

In Peirce’s defence: In *Laws* Boole does not formulate his restrictions on using V as ‘some’ until Chapter XV, the last chapter on logic.

(2) Cannot make this inference in BF_3 .

E. Schröder, *Algebra der LogiK Vol. II*, 1891, [p. 92]

OBJECTION: He gives a rigorous proof that one cannot use a system of equations to express 'Some X is Y'.

ANSWER:

(1): He is using modern semantics. His proof uses 0,1 as possible values for X,Y,...

In a footnote he says that further details are needed if the value 0 is excluded.

His proof uses substitution—he actually proves no single equation $E(X,Y)=0$ can uniformly express 'Some X is Y' using modern semantics.

(2) A minor modification of BF_3 works for modern semantics.

T. Hailperin, *Boole's Logic and Probability*, 1986

He notes (p. 110) that Boole always converts $X = VY$ into $X(1 - Y) = 0$, but he goes on to say that

OBJECTION: "in the case of the particular categorical there is no easy way out."

ANSWER:

(1) He is using the modern semantics.

(2) There is an easy way out: $V = VXY$

(3) Actually Boole does not always eliminate the V in the translations of universal statements — see Chap. XV of *Laws of Thought* where Boole gives a uniform treatment of syllogisms.