The Laws of Boole's Thought

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Many colleagues in Algebra and Logic think that Boole developed either

- Boolean Algebra, or
- Boolean Rings.

He did neither.

In this talk we look at the issue of how to explain, simply, the nature of Boole's algebra of logic.

The Algebra of Logic was originally developed by Boole (in 1847) to handle **reasoning about classes** as expressed in Aristotelian Logic, for example:

$$\begin{array}{c}\text{All B is C}\\ \text{All A is B}\\ \hline \text{All A is C.}\end{array}$$

In the modern development of the Logic of Classes we start with the semantics of **union**, **intersection**, and **complement**, and deduce the laws that must hold.

$$\boxed{\text{Semantics}} \longrightarrow \boxed{\text{Syntatic aspects}}$$

This would be Jevons approach in the early 1860s.

The Main Question

What mental set could have led Boole to develop an algebra of logic, based on the operations

$$+, -, \times, \div$$

with the operations of $+$ and $-$ only partially defined?

- Gregory's laws for proving the binomial theorem, etc.:

$$\begin{aligned} a(u+v) &= au+av \\ ab &= ba \\ a^m a^n &= a^{m+n} \end{aligned}$$

- Boole's work on differential operators.

- Peacocks Symbolical Algebra

The High School Algebra Hypothesis:
Boole approached the logic of classes from quite a different point of view. He had two completed subjects given to him, namely logic and ordinary algebra.

He accepted the syntatic aspects for $+, -, \times$ as given by ordinary algebra, and only needed enough semantics to make the connection.

$$\boxed{\text{Semantics}} \longleftarrow \boxed{\text{Syntatic aspects}}$$

Aristotelian Logic	**Algebra** (High School)
Categorical logic deals with statements about classes	has procedures for working with the operations $+, -, \times, 0, 1$

Boole's Achievement:

A translation between logic and ordinary algebra such that:

$$\begin{tikzcd}
\text{Premisses} \arrow[r] & \text{Equations} \arrow[d, "\begin{array}{l}\text{Ordinary}\\ \text{Algebraic}\\ \text{Processes}\\ \text{for } +,-,\times,0,1\end{array}"] \\
\text{Conclusions} & \text{Equations} \arrow[l] \arrow[d, "\begin{array}{l}\text{Special}\\ \text{Treatment of}\\ \text{Division}\end{array}"] \\
\text{Conclusions} & \text{Equations} \arrow[l]
\end{tikzcd}$$

The Meaning of Multiplication

It is claimed that Boole came up with the choice:

multiplication means **intersection**

because of his work with differential operators.

However it is curious that in 1847 both he and De Morgan introduced the multiplicative notation AB to denote the intersection of the classes A and B.

Once this decision was made Boole observed that the following equations hold:

$$AA = A \quad \text{and} \quad AB{=}BA.$$

The Meaning of $+, -, 0, 1$

We claim that the meanings of the constants 0,1 and the operations $+, -$ were essentially dictated by the desire to use ordinary algebra.

Note that 0 and 1 are the only idempotent numbers.

So they are the only numbers that could name classes.

What class could 0 be?

From $A0 = 0$ it follows that the intersection of A and 0 must be 0, for any A. thus

0 is the empty class (if it is a class).

What class could 1 be?

From $A1 = A$ it follows that the intersection of A and 1 must be A, for any A. Thus

1 is the universe if it is a class.

We will assume that indeed 0 is the empty class and 1 is the universe.

Some Restrictions

Some parts of ordinary algebra cannot be used in Boole's algebra of logic.

The main example is that from $AB = 0$ we cannot conclude that $A = 0$ or $B = 0$.

On the other hand the deduction

$$2A = 0 \text{ implies } A = 0$$

does not lead to any problems, and is used by Boole.

Indeed it seems that any **purely equational** deduction is acceptable.

What is the Meaning of A+B ?

If A+B is a class then the idempotent law says that

$$(A + B)^2 = A + B.$$

But the idempotence of A and B, with ordinary algebra, leads to:

$$(A + B)^2 = A + B + 2AB.$$

This means that for A+B to be a class we need $2AB = 0$, or, in view of the previous remarks, $AB = 0$.

Thus for A+B to be a class we need A and B to be disjoint.

If A and B are disjoint, what class can A+B be?

Suppose A+B is a class. Let C be $A \cup B$. Then

$$(A+B)C = AC + BC = A+B.$$

Thus the class A+B is a subclass of $A \cup B$.

Furthermore

$$A(A+B) = AA + AB = A+0 = A,$$

so A is a subclass of A+B. Likewise B is a subclass of A+B. This means $A \cup B$ must be a subclass of A+B.

Thus if A+B is a class it must be $A \cup B$.

We henceforth assume that A+B is the union of A and B when they are disjoint.

What is the Meaning of $A - B$?

If $A - B$ is a class then the idempotent law says that

$$(A - B)^2 = A - B.$$

But the idempotence of A and B, with ordinary algebra, lead to:

$$(A - B)^2 = A - B + 2(B - AB).$$

This means that for $A - B$ to be a class we need $2(B - AB) = 0$, or, in view of previous remarks, $B - AB = 0$.

But this gives $B = AB$.

So for $A - B$ to be a class, B must be a subclass of A.

If B is a subclass of A, what class can $A - B$ **be?**

Let $C = A \setminus B$.

Then $BC=0$, so $B+C = B \cup C = A$.

This means $A - B = C = A \setminus B$.

Let us assume this interpretation holds.

This implies $1 - A$ is the **complement** of the class A.

Now we turn to remarks of Boole and Jevons on the nature of $+$.

- *Boole, Laws of Thought, 1854, p. 66*

  "The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common."

- Jevons, *draft of commentary on Boole's system in a leter to Boole, 1863:*

  "It is surely obvious, however, that $x + x$ is equivalent only to x, ... "

  "Professor Boole's notation [process of subtraction] is inconsistent with a self evident law."

  "If my view be right, his system will come to be regarded as a most remarkable combination of truth and error."

- *Boole, in response:*

  "Thus the equation $x + x = 0$ is equivalent to the equation $x = 0$; but the expression $x + x$ is not equivalent to the expression x."

- Jevons responded by asking if Boole could deny the truth of $x + x = x$.

- Boole replies: "To be explicit, I now, however, reply that it is not true that in Logic $x + x = x$, though it is true that $x + x = 0$ is equivalent to $x = 0$."

  "If I do not write more it is not from any unwillingness to discuss the subject with you, but simply because if we differ on this fundamental point it is impossible that we should agree in others."

- Jevon's: "I do not doubt that it is open to you to hold ... [that $x+x=x$ is not true] according to the laws of your system, and with this explanation your system probably is perfectly consistent with itself ... But the question then becomes a wider one—does your system correspond to the Logic of common thought?"

These quotes strongly suggest that Boole did not feel free to choose the semantics of $+$. The obvious constraint: High School Algebra.

Conclusions

It was Boole's commitment to using ordinary algebra to the greatest extent possible, and his brilliant success in doing so, that led him to partial operations.

Boole was aware of both the **symmetric difference** and the **union** of classes (see Laws, p. 56), but they did not fit in with his desire to use ordinary algebra.

Unlike Jevons, the possibility of developing the algebra of logic from scratch, based on clear semantics for the operations, did not occur to Boole.

Unfortunately Jevon's sharp criticism of Boole's system did nothing to win Boole over to the possibility of alternate approaches.

It has been known since 1976 that Boole's system was not like anything that we use today for the algebra of logic.

In 1976 Theodore Hailperin put Boole's system on a solid foundation in his book *Boole's Logic and Probability*.

In 1981 Hailperin published a paper in the Mathematics Magazine:

Boole's Algebra isn't Boolean Algebra

In the summary he says:

"While he never made his algebra fully explicit, we inferred that what he did use was, if clarified, a commutative ring with unit, without nilpotents, and having idempotents which stand for classes."

Hailperin's main model was SM (signed multisets).