A FRAGMENT OF BOOLE'S ALGEBRAIC LOGIC SUITABLE FOR TRADITIONAL SYLLOGISTIC LOGIC

STANLEY BURRIS

1. Introduction

Boole introduced his agebraic approach to logic in 1847 in an 82 page monograph, *The Mathematical Analysis of Logic*. He later said that it had been written too hastily. In subsequent years the material was thoroughly reworked, leading to his 1854 classic, *The Laws of Thought*.

Boole's system has long been regarded as mysterious. In [3] Hailperin shows why the system works, by modelling it with **signed multisets**. One objection to Hailperin's analysis of Boole's system is that he adopts the modern semantics for simple class names, allowing them to be any class from the universe, including the empty class and universe. With such semantics one loses several of the arguments that were accepted in Aristotelian logic, and accepted by Boole, such as the argument 'All A is B $\therefore$ Some A is B'. A second, and somewhat related, objection is that Hailperin is not able to vindicate Boole's use of equations to handle the particular statements of Aristotelian logic ([3], p. 154):

> In summary, then, if the Aristotelian particular sentence forms are ignored, Boole's use of the indefinite class symbol v is justifiable and sound, merely amounting to an understood existentially quantified variable.

This note gives a slight refinement of a portion of Boole's equational algebra of logic that can be proved to perform as claimed, for both universal and particular categorical statements, and is faithful to Boole's methods of using **common algebra** as the basis on which to build an equational system that encompasses the traditional logic of Aristotle.

2. The Builiding Blocks

- A class is **proper** if it is neither empty nor the universe.
- **Simple names** A,B,C,..., with various subscripts and superscripts when needed, to refer to proper classes.
- The **constants** 0 and 1.
  0 will refer to the **empty class**.
  1 will refer to the **universe of discourse**.
- The binary **operation symbols** $+, -, \cdot$.
- **Terms** are built from the simple names, constants and operation symbols $+, -, \cdot$.
- **Literals** are either simple names $A, B, \ldots$ or their complements $1 - A, 1 - B, \ldots$.
- The **numerals** 2, 3, etc. will be used as abbreviations for $1 + 1$, $(1 + 1) + 1$, etc. Given a term s, the expressions $2s$, $3s$, etc. are abbreviations for $s + s$, $(s + s) + s$, etc.
- The **semantics** of the operations $+, -, \cdot$ is as follows (writing AB for $A \cdot B$):
  AB denotes the **intersection** $A \cap B$ of A and B.
  $A + B$ denotes the **union** $A \cup B$ of A and B *provided* $AB = 0$.
  $A - B$ denotes the **class difference** $A \setminus B$ of A and B *provided* $B \subseteq A$. Note that this means $1 - B$ denotes the **complement** of B.

- An **assignment** of the simple names to classes is **proper** if each simple name is assigned a proper class.
- Given an assignment of the simple names, a term $s(A, B, \ldots)$ is **interpretable** under the assignment provided it defines a class. A term interpretable under every assignment is **absolutely interpretable**.

Example 2.1. $A+B$ is interpretable under some assignments, but it is not absolutely interpretable, whereas the term $AB + (1 - A)(1 - B)$ is absolutely interpretable.

Example 2.2. Boole expressed the **union** of A and B by

$$A(1 - B) + (1 - A)B + AB,$$

and the **symmetric difference** by

$$A(1 - B) + (1 - A)B.$$

Both of these terms are absolutely interpretable.

3. Translations

Let s and t be absolutely interpretable terms. The **propositions** considered in this note are in one of the following four forms, have the indicated translations into equations, and the indicated reverse translations back into English:[1]

Quantity	English	Equation	Comments on translation
Universal	All s is t.	$st = s$	use in both directions
Universal	No s is t.	$st = 0$	use in both directions
Particular	Some s is t.	$V = Vst$	use a new simple name V for each translation of a premiss into an equation. For the reverse translation, into English, V can be any literal.
Particular	Some s is not t.	$V = Vs(1 - t)$	[same as previous case].

4. High School Equational Algebra and Idempotent Simple Names

Simple names denote classes, so they are idempotent: $A^2 = A$, $B^2 = B$, etc. The algebra that will be used is just **high school equational algebra applied to idempotent simple names**, a phrase that will be abbreviated to **HSI**. An equational argument '$s_1 = t_1, \ldots, s_n = t_n \ \therefore s = t$' is **HSI-provable** if it can be justified by HSI. If so, we write

$$s_1 = t_1, \ldots, s_n = t_n \ \vdash_{\text{HSI}} \ s = t,$$

and say that the conclusion **can be derived from** the premisses. If one can derive $s = t$ without using any premisses then we write

$$\vdash_{\text{HSI}} \ s = t.$$

[1]For the **particular** proposition 'Some s is t' Boole originally used $V = st$, and later $Vs = Vt$. The first appeared only in the 1847 book. The use of a simple name V to capture existential content in an equational setting was eventually considered faulty, after sharp criticism by C.S. Peirce (and reiterated by Ernst Schröder). The goal of this note is to put Boole's use of equations to handle existential content on a firm basis.

The translation of particular statements in this note is a slight modification of Boole's approach. It allows one to dispense with having to remember the terms with which V is introduced in the translation. Hailperin ([3], p. 112) regards the feature of Boole's system, of having to keep notes about the terms s for which Vs can be translated as 'some s', sufficiently ad hoc to disqualify Boole's system from being an acceptable formal system. The system presented in this note is based on the restricted (Aristotelian) semantics and allows for a purely equational logic treatment of both the universal and categorical statements of Aristotelian logic. If one adopts **modern semantics**, allowing simple names to be interpreted as the empty class as well as the universe, then one only needs to introduce special simple names $V, W, \ldots$ to denote proper classes.

Example 4.1. The two equational arguments

$$A + A = 0 \ \therefore A = 0$$
$$A + A = A \ \therefore A = 0$$

are HSI-provable arguments, so we can write

$$A + A = 0 \ \vdash_{\mathrm{HSI}} \ A = 0$$
$$A + A = A \ \vdash_{\mathrm{HSI}} \ A = 0.$$

However, the argument '$A^2 = A \ \therefore A = 0$ or $A = 1$' is not HSI-provable. It is not even an equational argument as one can see by writing it more formally: '$A^2 = A \ \therefore A = 0 \ \vee A = 1$'. The conclusion of this argument is a *disjunction* of equations, not an equation.

Example 4.2. By multiplying out the expressions that Boole gives for union and symmetric difference, and collecting terms, we have:

$$\vdash_{\mathrm{HSI}} \ A(1-B) + (1-A)B + AB \ = \ A + B - AB$$
$$\vdash_{\mathrm{HSI}} \ A(1-B) + (1-A)B \ = \ A + B - 2AB.$$

If we take $A + B - AB$ to mean $(A + B) - AB$, then this way of expressing union is not absolutely interpretable. Likewise for symmetric difference. Unfortunately, being absolutely interpretable is a property of terms that is not invariant under 'provably equal'.

Example 4.3. Symmetric difference is associative, that is,

$$\vdash_{\mathrm{HSI}} \ A\triangle(B\triangle C) \ = \ (A\triangle B)\triangle C.$$

To see this we use the expression for symmetric difference from the last example:

$$\begin{aligned}
A\triangle(B\triangle C) &= A + (B\triangle C) - 2A(B\triangle C)\\
&= A + (B + C - 2BC) - 2A(B + C - 2BC)\\
&= A + B + C - 2BC - 2AB - 2AC + 4ABC\\
(A\triangle B)\triangle C &= (A\triangle B) + C - 2(A\triangle B)C\\
&= (A + B - 2AB) + C - 2(A + B - 2AB)C\\
&= A + B + C - 2AB - 2AC - 2BC + 4ABC.
\end{aligned}$$

4.1. **Satisfiability and Completeness.** A set of propositions [equations] is **properly satisfiable** if there is a proper assignment of the simple names such that each of the propositions [equations] in the set is true.

Theorem 4.4 (Completeness). *Suppose the collection of propositions $\mathcal{P}_1, \ldots, \mathcal{P}_n$ is properly satisfiable.*[2] *Then an argument*

$$\mathcal{P}_1, \ldots, \mathcal{P}_n \ \therefore \ \mathcal{P}$$

is valid iff after translating the propositions $\mathcal{P}_i$ into equations $s_i = t_i$ there is an equation $s = t$ such that

$$s_1 = t_1, \ldots, s_n = t_n \ \vdash_{\mathrm{HSI}} \ s = t$$

holds, and $\mathcal{P}$ is a translation of $s = t$.

5. Application to Aristotelian Logic

Now we want to show that the above Theorem yields the conversions and categorical syllogisms from Aristotelian logic.

[2]This condition can be dropped if we add two inference schemes: $\dfrac{X=0}{1=0}$ and $\dfrac{X=1}{1=0}$, where X can be any simple name.

5.1. **Arguments with a single premiss.** On the left are the propositional arguments, and on the right are the corresponding equational arguments. Details of the equational derivations follow:

	Some A is B		$V = VAB$
$\therefore$	Some B is A.	$\therefore$	$V = VBA$
	No A is B		$AB = 0$
$\therefore$	No B is A.	$\therefore$	$BA = 0$
	All A is B		$A = AB$
$\therefore$	Some A is B.	$\therefore$	$A = AAB$
	All A is B		$A = AB$
$\therefore$	No A is not B.	$\therefore$	$A(1-B) = 0$
	No A is B		$AB = 0$
$\therefore$	Some A is not B.	$\therefore$	$A = AA(1-B)$
	No A is B		$AB = 0$
$\therefore$	All A is not B.	$\therefore$	$A = A(1-B)$

Details of derivations. The following gives (most of) the details of derivations of the above equational arguments. An expression $s =_1 t$ means that one is to use the (first) premiss, whereas just $s = t$ means that one is to use basic high school equational algebra and/or the idempotence of the simple names.

- $V =_1 VAB = VBA$
- $BA = AB =_1 0$
- $A =_1 AB = AAB$
- $A(1-B) = A - AB =_1 AB - AB = 0$
- $AA(1-B) = A(1-B) = A - AB =_1 A - 0 = A$
- $A(1-B) = A - AB =_1 A - 0 = A.$

□

Not only were the above simple propositional arguments stated by Boole, but he also allowed one to substitute for A its contrary not-A, or for B its contrary not-B, or both, in any of the above. (The expression 'not not-A' would be replaced by A, etc.) By making the corresponding substitution into the equational arguments on the right, that is, if not-A has been substituted for A in a propositional argument then substitute $1 - A$ for A in the equational argument, one has an appropriate equational argument. For example,

	No not-A is not-B		$(1-A)(1-B) = 0$
$\therefore$	Some not-A is B.	$\therefore$	$1-A = (1-A)(1-A)B$

with an equational derivation given by

$$(1-A)(1-A)B = (1-A)(1-(1-B)) = (1-A) - (1-A)(1-B) =_1 (1-A) - 0 = (1-A).$$

5.2. **The Syllogisms.** The following gives an equational argument for each of the 19 valid Aristotelian syllogisms, followed by derivations for the equational arguments:

1st Fig AAA			
	All B is C		$B = BC$
	All A is B		$A = AB$
$\therefore$	All A is C.	$\therefore$	$A = AC$

1st Fig AAI

$$\begin{array}{rlrcl} & \text{All } B \text{ is } C & B &=& BC \\ & \text{All } A \text{ is } B & A &=& AB \\ \therefore & \text{Some } A \text{ is } C. & \therefore\ A &=& AAC \end{array}$$

1st Fig AII

$$\begin{array}{rlrcl} & \text{All } B \text{ is } C & B &=& BC \\ & \text{Some } A \text{ is } B & V &=& VAB \\ \therefore & \text{Some } A \text{ is } C. & \therefore\ V &=& VAC \end{array}$$

1st Fig EAE

$$\begin{array}{rlrcl} & \text{No } B \text{ is } C & BC &=& 0 \\ & \text{All } A \text{ is } B & A &=& AB \\ \therefore & \text{No } A \text{ is } C. & \therefore\ AC &=& 0 \end{array}$$

1st Fig EA0

$$\begin{array}{rlrcl} & \text{No } B \text{ is } C & BC &=& 0 \\ & \text{All } A \text{ is } B & A &=& AB \\ \therefore & \text{Some } A \text{ is not } C. & \therefore\ A &=& AA(1-C) \end{array}$$

1st Fig EIO

$$\begin{array}{rlrcl} & \text{No } B \text{ is } C & BC &=& 0 \\ & \text{Some } A \text{ is } B & V &=& VAB \\ \therefore & \text{Some } A \text{ is not } C. & \therefore\ V &=& VA(1-C) \end{array}$$

2nd Fig AEE

$$\begin{array}{rlrcl} & \text{All } C \text{ is } B & C &=& BC \\ & \text{No } A \text{ is } B & AB &=& 0 \\ \therefore & \text{No } A \text{ is } C. & \therefore\ AC &=& 0 \end{array}$$

2nd Fig AEO

$$\begin{array}{rlrcl} & \text{All } C \text{ is } B & C &=& BC \\ & \text{No } A \text{ is } B & AB &=& 0 \\ \therefore & \text{Some } A \text{ is not } C. & \therefore\ A &=& AA(1-C) \end{array}$$

2nd Fig AOO

$$\begin{array}{rlrcl} & \text{All } C \text{ is } B & C &=& BC \\ & \text{Some } A \text{ is not } B & V &=& VA(1-B) \\ \therefore & \text{Some } A \text{ is not } C. & \therefore\ V &=& VA(1-C) \end{array}$$

2nd Fig EAE

$$\begin{array}{rlrcl} & \text{No } C \text{ is } B & CB &=& 0 \\ & \text{All } A \text{ is } B & A &=& AB \\ \therefore & \text{No } A \text{ is } C. & \therefore\ AC &=& 0 \end{array}$$

2nd Fig EAO

$$\begin{array}{rlrcl} & \text{No } C \text{ is } B & CB &=& 0 \\ & \text{All } A \text{ is } B & A &=& AB \\ \therefore & \text{Some } A \text{ is not } C. & \therefore\ A &=& AA(1-C) \end{array}$$

2nd Fig EIO

$$\begin{array}{rlrcl} & \text{No } C \text{ is } B & CB &=& 0 \\ & \text{Some } A \text{ is } B & V &=& VAB \\ \therefore & \text{Some } A \text{ is not } C. & \therefore\ V &=& VA(1-C) \end{array}$$

3rd Fig AAI	
All B is C	$B = BC$
All B is A	$B = BA$
$\therefore$ Some A is C.	$\therefore\ B = BAC$

3rd Fig AII	
All B is C	$B = BC$
Some B is A	$V = VBA$
$\therefore$ Some A is C.	$\therefore\ V = VAC$

3rd Fig EAO	
No B is C	$BC = 0$
All B is A	$B = BA$
$\therefore$ Some A is not C.	$\therefore\ B = BA(1-C)$

3rd Fig EIO	
No B is C	$BC = 0$
Some B is A	$V = VBA$
$\therefore$ Some A is not C.	$\therefore\ V = VA(1-C)$

3rd Fig IAI	
Some B is C	$V = VBC$
All B is A	$B = BA$
$\therefore$ Some A is C.	$\therefore\ V = VAC$

3rd Fig OAO	
Some B is not C	$V = VB(1-C)$
All B is A	$B = BA$
$\therefore$ Some A is not C.	$\therefore\ V = VA(1-C)$

4th Fig AAI	
All C is B	$C = CB$
All B is A	$B = BA$
$\therefore$ Some A is C.	$\therefore\ C = CAC$

4th Fig AEE	
All C is B	$C = CB$
No B is A	$BA = 0$
$\therefore$ No A is C.	$\therefore\ AC = 0$

4th Fig AEO	
All C is B	$C = CB$
No B is A	$BA = 0$
$\therefore$ Some A is not C.	$\therefore\ A = AA(1-C)$

4th Fig EAO	
No C is B	$CB = 0$
All B is A	$B = BA$
$\therefore$ Some A is not C.	$\therefore\ B = BA(1-C)$

4th Fig EIO	
No C is B	$CB = 0$
Some B is A	$V = VBA$
$\therefore$ Some A is not C.	$\therefore\ V = VA(1-C)$

4th Fig IAI				
	Some C is B		V	$= VCB$
	All B is A		B	$= BA$
$\therefore$	Some A is C.	$\therefore$	V	$= VAC$

Details of derivations. An expression $s =_1 t$ means that one is to use the first premiss, $s =_2 t$ means that one is to use the second premiss, whereas just $s = t$ means that one is to use basic high school equational algebra and/or the idempotence of the simple names.

1st Fig AAA	$A =_2 AB =_1 ABC =_2 AC$
1st Fig AAI	$A =_2 AB =_1 ABC =_2 AC = AAC$
1st Fig AII	$V =_2 VAB =_1 VABC =_2 VAC$
1st Fig EAE	$AC =_2 ABC =_1 A0 = 0$
1st Fig EA0	$A(1-C) =_2 AB(1-C) = AB - ABC =_1 AB - 0 = AB =_2 A$
1st Fig EIO	$VA(1-C) =_2 VAB(1-C) = VAB - VAC =_1 VAB - VA0 = VAB =_2 V$
2nd Fig AEE	$AC =_1 ABC =_2 0C = 0$
2nd Fig AEO	$AA(1-C) = A(1-C) = A - AC =_1 A - ABC =_2 A - 0C = A$
2nd Fig AOO	$VA(1-C) =_2 VA(1-B)A(1-C) = VA(1-B-C+BC) =_1 VA(1-B) =_2 V$
2nd Fig EAE	$AC =_2 ABC =_1 0$
2nd Fig EAO	$AA(1-C) = A(1-C) =_2 AB(1-C) = AB - ABC =_1 AB =_2 A$
2nd Fig EIO	$VA(1-C) =_2 VABA(1-C) = VAB(1-C) = VAB - VABC =_1 VAB =_2 V$
3rd Fig AAI	$BAC =_2 BC =_1 B$
3rd Fig AII	$VAC =_2 VBAAC = VABC =_1 VAB =_2 V$
3rd Fig EAO	$BA(1-C) = BA - BAC =_1 BA =_2 B$
3rd Fig EIO	$VA(1-C) =_2 VBAA(1-C) = VAB(1-C) = VAB - VABC =_1 VAB =_2 V$
3rd Fig IAI	$VAC =_1 VBCAC = VABC =_1 VAB =_2 V$
3rd Fig OAO	$VA(1-C) =_1 VB(1-C)A(1-C) = VAB(1-C) =_2 VB(1-C) =_1 V$
3rd Fig AAI	$CAC = AC =_1 ABC =_2 BC =_1 C$
4th Fig AEE	$AC =_1 ABC =_2 0C = 0$
4th Fig AEO	$AA(1-C) = A(1-C) = A - AC =_1 A - ABC =_2 A - 0 = A$
4th Fig EAO	$BA(1-C) = AB - ABC =_1 AB - 0 =_2 B$
4th Fig EIO	$VA(1-C) =_2 VBAA(1-C) = VAB(1-C) = VAB - VABC =_1 VAB =_2 V$
4th Fig IAI	$VAC =_1 VCBAC = VABC =_2 VBC =_1 V.$

□

As with simple arguments, one can substitute contrary names for any of A, B, C in each of the above syllogisms and have a valid syllogism in Boole's system. The corresponding equational arguments are obtained as before.

6. The General Theory

In 1847 Boole gave a leisurely and detailed treatment of Aristotelian logic, by elementary methods, before proceeding to develop a general theory for his algebraic system. This is reversed in 1854—the discussion of Aristotelian logic is deferred to the very end of his treatment of logic, and then he employs his most powerful tools, namely reduction, elimination, and division, in an outline of a uniform approach to the syllogism.

6.1. **The Rule of 0 and 1.** Boole based his 1847 system on a most inadequate set of axioms, but he deserves credit for including a rule of inference that is now called the **replacement rule** in equational logic. In 1854 he started out with an expanded version of this equational proof system, but suddenly switched to a rule of 0 and 1 for his foundation. The next theorem says that this new foundation agrees with using $\vdash_{\mathrm{HSI}}$.

Theorem 6.1 (Rule of 0 and 1).

$$s_1 = t_1, \ldots, s_n = t_n \ \vdash_{\mathrm{HSI}} \ s = t$$

holds iff

$$s_1 = t_1, \ldots, s_n = t_n \ \therefore \ s = t$$

is a correct argument about integers when the simple names are restricted to the values 0 and 1.

6.2. **Constituents and Term Expansions.** Given a finite list of simple names $A_1, \ldots, A_n$, the **constituents** for these names are the 2^n products

$$\begin{array}{l} A_1 A_2 \cdots A_n \\ (1 - A_1) A_2 \cdots A_n \\ A_1 (1 - A_2) \cdots A_n \\ \quad\vdots \\ (1 - A_1)(1 - A_2) \cdots (1 - A_n). \end{array}$$

Every term $s(A_1, \ldots, A_n)$ has an **expansion**[3] as a linear combination of $A_1, \ldots, A_n$ constituents (with integer coefficients):

$$\vdash_{\mathrm{HSI}} \ s(A_1, \ldots, A_n) \ = \ s(1,1,\ldots,1) A_1 \cdots A_n \ + \ \cdots \ + \ s(0,0,\ldots,0)(1 - A_1) \cdots (1 - A_n).$$

The coefficients of the expansion are called the **indices** of $s(A_1, \ldots, A_n)$. Any constituent whose coefficient is not 0 is said to **belong to** $s(A_1, \ldots, A_n)$.

Thus, for example,

$$\vdash_{\mathrm{HSI}} \ A + B \ = \ (1 - A)B \ + \ 2AB \ + \ A(1 - B),$$

so the only A, B constituent not belonging to $A + B$ is $(1 - A)(1 - B)$.

Proposition 6.2. *A term s has an absolutely interpretable expansion iff all indices are idempotent, that is, 0 or 1, and this holds iff s is idempotent.*

Boole explains this as saying that idempotence characterizes interpretability. Given a term s let $\overline{s}$ be the sum of the constituents belonging to s.

Proposition 6.3. *For any term s,*

(a) *$\overline{s}$ is absolutely interpretable, and*
(b) *$s = 0$ iff $\overline{s} = 0$, that is, each is HSI-provable from the other.*

The last item is the basis for Boole's claim that equations are always interpretable, even though terms may not be.

6.3. **The Reduction Theorem.** Boole introduced a powerful elimination theorem in 1854 that applied to single equations. To apply it to a system of equations, such as one finds in the translation of premisses, he needed to reduce the system to a single equation.

Theorem 6.4 (The Reduction Theorem). *A finite system of equations $s_1 = 0, \ldots, s_n = 0$ is (HSI-provably) equivalent to the single equation*

$$s_1^2 + \cdots + s_n^2 \ = \ 0.$$

[3]Boole preferred the word **development**.

6.4. **The Elimination Theorem.** The goal of an elimination theorem is to find the most general consequence of a set of premisses that does not mention some specified simple name(s) in the premisses.[4]

Theorem 6.5. *The result of eliminating A from an equation $s(A, B_1, \ldots, B_n) = 0$ is*

$$s(1, B_1, \ldots, B_n) \cdot s(0, B_1, \ldots, B_n) \;=\; 0. \tag{6.1}$$

7. Comments on Boole's approach to Aristotelian Logic

The following shows Boole's primary translations of the categorical propositions in 1847 and in 1854, as well as the translations used in this note:

Proposition	1847 Translation	1854 Translation	This Note
All A is B	$A = AB$	$A = VB$	$A = AB$
No A is B	$AB = 0$	$A = V(1-B)$	$AB = 0$
Some A is B	$V = AB$	$VA = VB$	$V = VAB$
Some A is not B	$V = A(1-B)$	$VA = V(1-B)$	$V = VA(1-B)$

The translations into equations of 1854 were also used in the 1847 version, but they were viewed as consequences of the 1847 translations, and not the primary translations. It is easy to see, in the case of the particular statements, that the 1854 translations can be derived from the 1847 translations. In the case of universal statements, in 1847 Boole justified the second translation as the result of solving for A. In his 1854 book Boole makes heavy use of the fact that, for universal statements, he can also derive the 1847 form from the 1854 form. This is easy to show. Indeed, in his 1854 book one gets the impression that Boole sets up his examples according to the 1854 translation above, but then immediately derives the 1847 translations of all the universal statements and makes no further use of their 1854 translation. We will say more about this apparent clumsiness later. Note that, in the case of particular statements, the translations used in this note imply Boole's translations of 1854, and are implied by his translations of 1847.

Boole's favorite tool for tackling syllogisms in 1847 was to use

$$\begin{array}{rrcl} & s_1B & = & t_1 \\ & s_2B & = & t_2 \\ \therefore & s_1t_2 & = & s_2t_1, \end{array}$$

an argument that is easily seen to follow from high school algebra by multiplying the first equation by s_2 and the second by s_1. This can be applied to derive many of the valid syllogisms by putting the premiss equations in the appropriate form. For example,

1st Fig AAI	
All B is C	$(1-C)B \;=\; 0$
All A is B	$AB \;=\; A$
$\therefore$ Some A is C.	$\therefore\; A(1-C) \;=\; 0$

However, there were also a number of syllogisms for which this did not yield the desired conclusion, namely those for which the premisses translated into:

$$\begin{array}{rcl} s_1B & = & 0 \\ s_2B & = & 0 \end{array}$$

[4]If one uses modern semantics (see Hailperin, [3]) then one has, in modern symbolism, a result about **quantifier elimination**:

$$\exists A\Big(s(A, B_1, \ldots, B_n) = 0\Big) \;\leftrightarrow\; s(1, B_1, \ldots, B_n) \cdot s(0, B_1, \ldots, B_n) \;=\; 0.$$

But only the implication from left to right holds for the restricted semantics used here.

This could only happen if both premisses were universal, and in such cases Boole used the second translation on one of the premisses. For example,

3rd Fig AAI	
All B is C	$B = VC$
All B is A	$(1-A)B = 0$
$\therefore$ Some A is C.	$\therefore\ VC(1-A) = 0.$

In this case, by recalling that VC is nonempty, Boole could conclude that 'Some A is C'. This detail, of having to remember that VC is nonempty, was an irritating aspect of Boole's system referred to earlier.

Boole's 1847 treatment of syllogisms is rather ad hoc. Attempts to clean up this presentation must have greatly influenced his choice of translations in 1854, where the V is often conveniently read as 'some'. Hailperin notes ([3], p. 109) that in the 1854 book Boole always eliminates the V introduced in a translation of a universal statement, yielding the 1847 form. This observation is correct if one does not look at Chapter XV, the last chapter on logic. This is the chapter where Boole finally, and somewhat reluctantly, tackles the syllogism ([2], p. 226):

> The logical system of Aristotle, modified in its details, but unchanged in its essential features, occupies so important a place in academical education, that some account of its nature, and some brief discussion of the leading problems which it presents, seem to be called for in the present work. It is, I trust, in no narrow or harshly critical spirit that I approach this task.

At the end of the chapter he says ([2], p. 241):

> To what final conclusions are we then led respecting the nature and extent of the scholastic logic? I think to the following: that it is not a science, but a collection of scientific truths, too incomplete to form a system of themselves, and not sufficiently fundamental to serve as the foundation upon which a perfect system may rest.

It is only in Chapter XV that one clearly sees why Boole adopted the translation of universal statements that, in all previous chapters, had been immediately replaced by the 1847 translation, as well as the rather weak translation of particular statements.[5] With the 1854 translations the single premiss arguments are easy to handle. For example, the derivation of 'Some A is B' from 'All A is B' follows by multiplying both sides of $A = VB$ by V.

To give a uniform algebraic approach to syllogisms Boole first used the fact that, thanks to the 1854 translations, every categorical statement using the terms A and B can be put in one of the four forms:

$$\begin{aligned} \alpha A &= \beta B \\ \alpha(1-A) &= \beta B \\ \alpha A &= \beta B \\ \alpha(1-A) &= \beta(1-B), \end{aligned}$$

where α and β are either 1 or some simple name V introduced in the translation. Following this observation he turned to an analysis of two cases for the premisses of a syllogism, one case where the middle terms are the same, and another where they are complements:

$$\begin{aligned} \alpha A &= \beta B & \qquad \alpha A &= \beta B \\ \gamma C &= \delta B & \qquad \gamma C &= \delta(1-B). \end{aligned}$$

[5]Particular statements had been completely avoided in the premisses of the examples of arguments analyzed in preceding chapters.

The other possible syllogisms follow from his analysis of these two cases by substituting $1 - A$ for A, etc. The method of analysis is to reduce the two premiss equations to a single equation:

$$\big(\alpha A - \beta B\big)^2 \;+\; \big(\gamma C - \delta B\big)^2 = 0 \qquad\qquad \big(\alpha A - \beta B\big)^2 \;+\; \big(\gamma C - \delta(1 - B)\big)^2 = 0$$

Applying the elimination theorem, to eliminate the middle term B, leads to the two cases:

$$\Big(\big(\alpha A - \beta\big)^2 \;+\; \big(\gamma C - \delta\big)^2\Big) \cdot \Big(\alpha A \;+\; \gamma C\Big) \;=\; 0 \qquad\qquad \Big(\big(\alpha A - \beta\big)^2 \;+\; \gamma C\Big) \cdot \Big(\alpha A \;+\; \big(\gamma C - \delta\big)^2\Big) \;=\; 0.$$

The details of the reduction and elimination steps just given, and all of the grunt work to solve each of these for A, αA, and $1 - A$, are omitted from Boole's book. Then follows his sketchy derivation of three basic rules for finding valid syllogisms from given premisses—the details are again left for the reader to fill in.[6]

Thus Boole gave a reasonably uniform, but quite complicated, algebraic treatment of the syllogism. In the past 150 years no one has managed to do any better.

References

[1] George Boole, *The Mathematical Analysis of Logic,* 1847.
[2] George Boole, *The Laws of Thought,* 1854, Dover Reprint.
[3] Theodore Hailperin, *Boole's Logic and Probability,* 2nd ed., Kluwer Academic Publ., 1986.

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1 Canada
E-mail address: snburris@thoralf.uwaterloo.ca
URL: www.thoralf.uwaterloo.ca

[6]Boole succeeded in deriving all valid syllogisms by reduction and elimination only by using a secondary translation of universal propositions into equations. Using the translations in this note one can derive many, but not all, of these syllogisms by this method. For an example where it fails, consider the premisses of the 3rd Fig. AAI syllogism:

All B is C	$B \;=\; BC$
All B is A	$B \;=\; BA.$

Applying reduction gives

$$B(1 - C) + B(1 - A) \;=\; 0,$$

and then eliminating B gives

$$0 \;=\; 0.$$

One cannot obtain the traditional (and Boole's) conclusion 'Some A is C' from $0 = 0$.