PARTITION IDENTITIES I
SANDWICH THEOREMS AND LOGICAL 0–1 LAWS

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Abstract. The *Sandwich Theorems* proved in this paper give a new method to show that the partition function $a(n)$ of a partition identity

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

satisfies the condition RT_1

$$\lim_{n\to\infty} \frac{a(n-1)}{a(n)} \ = \ 1\,.$$

This leads to numerous examples of naturally occuring classes of relational structures whose finite members enjoy a logical 0–1 law.

1. Introduction

Partition identities

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)} \tag{1}$$

have been a staple in combinatorics and additive number theorysince the pioneering work of Hardy and Ramanujan into the number of partitions of a positive integer n, that is, the number of ways to write n as a sum of positive integers. Unless explicitly stated otherwise, it is assumed that the $p(n)$, and hence the $a(n)$, are nonnegative integers. When a partition identity is mentioned without a specific reference then the reader can assume (1) above is meant, using the two counting functions $p(n)$ and $a(n)$.

The nomenclature for the anatomy of a partition identity used here is:

$a(n)$	partition (count) function
$p(n)$	component (count) function
$\mathbf{A}(x) := \sum a(n)x^n$	generating function for the partitions
$\mathbf{P}(x) := \sum p(n)x^n$	generating function for the components
$\mathrm{rank}(p) := \sum p(n)$	rank of the partition identity.

Date: June 15, 2004.

1991 *Mathematics Subject Classification.* Primary 03C13, 05A16, 11P99, 41A60; Secondary 11N45, 11N80, 11U99, 60J20.

Key words and phrases. ratio test, partition identity, Schur's Tauberian theorem, zero-one law.

The second author would like to thank NSERC for support of this research.

Actually any expression for a function that has $\mathbf{A}(x)$ for its power series expansion is also called the generating function for the partitions; likewise for $\mathbf{P}(x)$. When a generating function of a partition identity is mentioned without specifying if it is $\mathbf{A}(x)$ or $\mathbf{P}(x)$, the convention will be that it is $\mathbf{A}(x)$.

In the study of the multiplicative theory of the natural numbers, or of the integers of an algebraic number field, the total count function is readily accessible whereas the prime count function is quite difficult to pin down. Just the opposite tends to be the case in additive number theory, combinatorics and algebra. For example in the partition problems considered by Bateman and Erdős one starts with a set M of natural numbers and asks how many ways one can partition a natural number n into summands from M. In this case $p(n) = \chi_M(n)$, the characteristic function of M; the investigative effort goes into understanding properties of $a(n)$. In graph theory the natural approach to enumeration problems for a class of finite graphs is to start with an enumeration of the components. In algebra, to enumerate the finite Abelian groups one starts with the fact that the indecomposables are the cyclic p-groups, one of size p^k for each prime number p and each positive integer k. The reader should therefore not be surprised that we also like to start with hypotheses on $p(n)$ and deduce information about $a(n)$.

2. The Property RT_1

The property

$$\frac{f(n-1)}{f(n)} \to 1\,,$$

where $f(n)$ is eventually positive, is called RT_1 because it is the condition used in the well known limit form of the *ratio test* for convergence of the power series $\sum f(n)x^n$; if RT_1 holds then the radius of convergence of $\sum f(n)x^n$ is 1 .

When dealing with partition functions $a(n)$ it is convenient to use interchangeably any of the phrases:

(i) $a(n)$ *satisfies* RT_1,
(ii) $\mathbf{A}(x)$ *satisfies* RT_1, or
(iii) *the partition identity satisfies* RT_1.

Clearly $f(n)$ satisfies RT_1 guarantees that 1 is the radius of convergence of $\sum f(n)x^n$; but this does not explain the true value of the RT_1 property. It has much more to do with the fact that the values of $f(n)$ vary slowly as n increases, as expressed by

$$(1-\varepsilon)\cdot f(n-1) \;<\; f(n) \;<\; (1+\varepsilon)\cdot f(n-1)$$

for n sufficiently large.[1] The property RT_1 plays a significant role in the results of Bateman and Erdős and is essential to Compton's approach to proving logical 0–1 laws.

There are three main results concerning when a partition function $a(n)$ satisfies RT_1, that is, when $a(n-1)/a(n) \to 1$ as $n \to \infty$. But first some definitions. A partition identity is *reduced* if

$$\gcd\{n : p(n) > 0\} \ = \ 1\,.$$

It is well known that $a(n)$ is eventually positive iff the partition identity is reduced—see, for example, p. 43 of [7]. Given a partition identity let

$$\begin{aligned} d &:= \gcd\{n : p(n) > 0\} \\ p^\star(n) &:= p(nd) \\ a^\star(n) &:= a(nd)\,. \end{aligned}$$

Then

$$\mathbf{A}^\star(x) \ := \ \sum_{n=0}^{\infty} a^\star(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p^\star(n)}\,. \tag{2}$$

This is the *reduced form* of the partition identity (1). The reduced form of a partition identity is reduced; and a reduced partition identity is the same as its reduced form.

Here are the three principal theorems concerning conditions on a partition identity that guarantee $a(n)$ satisfies RT_1:

- **Theorem A.** (Bell [3]) *Given a reduced partition identity, if $p(n)$ is polynomially bounded, that is, $p(n) = \mathrm{O}(n^\gamma)$ for some $\gamma \in \mathbb{R}$, then $a(n)$ satisfies RT_1.* This generalizes a result of Bateman and Erdős [2] that says if $p(n) \in \{0,1\}$ then RT_1 holds.
- **Theorem B.** (Bell and Burris [5]) *If $p(n-1)/p(n) \to 1$ as $n \to \infty$ then $a(n)$ satisfies RT_1.*
- **Theorem C.** (Stewart's Sum Theorem:[2] (see [7], p. 85) *If*

$$\sum_{n=0}^{\infty} a_j(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p_j(n)} \qquad (j = 1,2)$$

and each $a_j^\star(n)$ satisfies RT_1 then $a^\star(n)$ also satisfies RT_1, where

$$\sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

with $p(n) = p_1(n) + p_2(n)$.

[1] The property RT_ρ, meaning $a(n-1)/a(n) \to \rho$, is called *smoothly growing* by Compton; RT_ρ is the additive number theory analog of the property RV_α, *regular variation of index α*, in multiplicative number theory. RT_1 is the analog of RV_0, *slowly varying at infinity.*

[2] This theorem says that the sum of two additive number systems whose reduced forms satisfy RT_1 is again an additive number system whose reduced form satisfies RT_1.

Repeated use will be made of (iterations of) the following simple application of Stewart's Sum Theorem:

> *if* $\mathbf{A}(x)$ *is a generating function satisfying* RT_1 *then, for* d *a positive integer,* $\mathbf{A}(x) \cdot \left(1 - x^d\right)^{-1}$ *is again a generating function satisfying* RT_1.

We adopt the convention of [7] that upper case bold letters name power series whose coefficients are given by the corresponding lower case italic letters, for example

$$\mathbf{F}(x) \;=\; \sum_{n=0}^{\infty} f(n)x^n\,.$$

By this convention $\mathbf{A}(x)$ is the power series $\sum a(n)x^n$ and $\mathbf{A}_1(x)$ is the power series $\sum a_1(n)x^n$, etc. It will be convenient to define coefficients $f(n)$ of a power series $\mathbf{F}(x)$ to be 0 for negative values of n.

The goals of this paper are:

- To considerably extend the collection of partition identities for which it is known that $a(n)$ satisfies RT_1; and to show that this extension is, in a natural sense, best possible.
- To give a new proof of Bell's Theorem A: if $p(n)$ is polynomially bounded then $a(n)$ satisfies RT_1.
- To show that the new techniques for proving $a(n)$ satisfies RT_1 lead to new examples of natural classes of finite structures which have a logical 0–1 law.

3. Background requirements

In addition to the results on RT_1 already mentioned we will need the following two well known results:

- **D** The polynomial growth of $a(n)$ when $\operatorname{rank}(p) < \infty$.
- **E** The superpolynomail growth of $a(n)$ when $\operatorname{rank}(p) = \infty$.

Also the following Tauberian theorem is needed:

Theorem 3.1 (Schur)**.** *With* $0 \le \rho < \infty$, *suppose that*

(i) $a(n)$ *satisfies* RT_ρ,
(ii) $\mathbf{B}(x)$ *has radius of convergence greater than* ρ, *and*
(iii) $\mathbf{B}(\rho) > 0$.

Let $\mathbf{C}(x) \;=\; \mathbf{A}(x)\cdot\mathbf{B}(x)$. *Then*

$$c(n) \;\sim\; \mathbf{B}(\rho)\cdot a(n)\,.$$

Proof. (See [7], p. 62.) □

The notation $f(n) \preceq g(n)$ means that $f(n)$ is eventually less or equal to $g(n)$.

4. The Sandwich Theorem

There has long been interest in studying the partial sums $\sum_{j\leq n} a(j)$ of the coefficients of a power series, but here the fixed length tails of these partial sums are of particular interest. For L a nonnegative integer let

$$a^L(n) \;:=\; a(n) \;+\; \cdots \;+\; a(n-L)\,.$$

For a generating function $\mathbf{A}(x)$ whose coefficients are eventually positive, the least nonnegative integer L such that $a(n) > 0$ for $n \geq L$ is called the *conductor*[3] of $\mathbf{A}(x)$; designate it by $L_{\mathbf{A}}$. As the following lemma shows, the coefficients of such a generating function enjoy a weak form of monotonicity that leads to monotonicity for $a^L(n)$ for $L \geq L_{\mathbf{A}}$. Furthermore, the study of $a^L(x)$ leads to powerful methods for showing that $a(n)$ satisfies RT_1.

Lemma 4.1. *Let $\mathbf{A}(x)$ be a generating function whose coefficients are eventually positive. Then for any $L \geq L_{\mathbf{A}}$,*

(a) $a(n) \geq a(m)$ *if* $n - m \geq L$;
(b) $a^L(n)$ *is nondecreasing for all* n;
(c) $a^L(n)$ *is positive for* $n \geq L_{\mathbf{A}}$;
(d) $a^{mL}(n) \;\leq\; m \cdot a^L(n)$ *for* $m = 1, 2, \cdots; n \geq 0$.

Proof. If $a(j) > 0$ then for any m, $a(m) \leq a(m+j)$ (this does not require that the coefficients be eventually positive).[4] For $n - m \geq L_{\mathbf{A}}$ one has $a(n-m) > 0$; so $a(m) \leq a\big(m + (n-m)\big) = a(n)$. This proves (a). For (b) note that

$$a^L(n+1) \;-\; a^L(n) \;=\; a(n+1) \;-\; a(n-L) \;\geq\; 0$$

by part (a). For $n \geq L_{\mathbf{A}}$ clearly $a^L(n) \geq a(n) > 0$; this is (c). Finally for (d) one has

$$\begin{aligned} a^{mL}(n) \;&\leq\; \sum_{j=0}^{m-1}\sum_{i=0}^{L} a(n - jL - i) \;=\; \sum_{j=0}^{m-1} a^L(n - jL) \\ &\leq\; \sum_{j=0}^{m-1} a^L(n) \;=\; m \cdot a^L(mL)\,. \end{aligned}$$

□

Lemma 4.2. *Let $\mathbf{A}(x)$ be a generating function with $a(n)$ eventually positive, and suppose $L \geq L_{\mathbf{A}}$ is an integer such that*

$$\big|a(n) - a(n-1)\big| \;=\; \mathrm{o}\big(a^L(n)\big)\,.$$

[3] Wilf [12], p. 97, uses this name in the case that $p(n) \in \{0,1\}$. He mentions that given such a $p(n)$ that is eventually 0, computing $L_{\mathbf{A}}$ seems to be a difficult problem.

[4] The proof using additive number systems is trivial. Let $\mathcal{A}$ be a model of the partition identity and choose an element $\mathsf{a} \in \mathsf{A}$ with $\|\mathsf{a}\| = j$. Then the cancellation property shows that the mapping $\mathsf{b} \mapsto \mathsf{a} + \mathsf{b}$ is an injective mapping from the elements of norm m into the elements of norm $m + j$.

Then

$$\frac{a(n-1)}{a(n)} \;\to\; 1 \qquad \textit{as} \;\; n \to \infty\,.$$

Proof. Let $\varepsilon > 0$ be given and choose a positive integer M such that for $n \geq M$

$$\big|a(n) - a(n-1)\big| \;\leq\; \frac{\varepsilon}{L(L+1)} a^L(n).$$

For $n \geq M + L$ choose $\widetilde{n}$ and $\widehat{n}$ from $\{n - L, \ldots, n\}$ such that

$$a(\widetilde{n}) \;\leq\; a(j) \;\leq\; a(\widehat{n}) \qquad \text{for} \;\; n - L \leq j \leq n\,.$$

Then for $n \geq M + L$

$$\begin{aligned} 0 \;\leq\; a(\widehat{n}) - a(\widetilde{n}) \;&\leq\; \sum_{j=n-L+1}^{n} \big|a(j) \;-\; a(j-1)\big| \\ &\leq\; \frac{\varepsilon}{L(L+1)} \sum_{j=n-L+1}^{n} a^L(j) \\ &\leq\; \frac{\varepsilon}{L+1} a^L(n) \\ &\leq\; \varepsilon a(\widehat{n})\,, \end{aligned}$$

and thus, as $0 < a(\widetilde{n}) \leq a(n) \leq a(\widehat{n})$ for $n \geq M + L$,

$$1 - \varepsilon \;\leq\; \frac{a(\widetilde{n})}{a(\widehat{n})} \;\leq\; \frac{a(n-1)}{a(n)} \;\leq\; \frac{a(\widehat{n})}{a(\widetilde{n})} \;\leq\; (1-\varepsilon)^{-1}\,.$$

From this it follows that $a(n-1)/a(n) \to 1$ as $n \to \infty$. □

Lemma 4.3. *Suppose* $\mathbf{A}_1(x)$ *and* $\mathbf{A}_2(x)$ *are two generating functions and* $L \geq L_{\mathbf{A}}$ *a positive integer such that, with* $\mathbf{A}(x) = \mathbf{A}_1(x) \cdot \mathbf{A}_2(x)$,

(i) $\dfrac{a_1(n-1)}{a_1(n)} \;\to\; 1\,;$

(ii) $a_2^L(n) \;=\; \mathrm{o}\big(a^L(n)\big)\,.$

Then as $n \to \infty$,

$$\frac{a(n-1)}{a(n)} \;\to\; 1\,.$$

Proof. Given $\varepsilon > 0$ choose a positive integer M that is a multiple of L and such that for $n \geq M$,

$$\big|a_1(n) - a_1(n-1)\big| \;\leq\; \frac{\varepsilon}{2} a_1(n)\,.$$

Then there are positive constants C_1, C_2 such that for $n \geq M$,

$$\begin{aligned} \big|a(n) - a(n-1)\big| \;&=\; \Big|\sum_{j=0}^{n} \big(a_1(j) - a_1(j-1)\big) a_2(n-j)\Big| \\ &\leq\; \sum_{j=M}^{n} \big|a_1(j) - a_1(j-1)\big| a_2(n-j) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j<M} \big|a_1(j) - a_1(j-1)\big| \cdot a_2(n-j) \\
&\leq \frac{\varepsilon}{2} \sum_{j=M}^{n} a_1(j) a_2(n-j) \; + \; C_1 \sum_{j<M} a_2(n-j) \\
&\leq \frac{\varepsilon}{2} \sum_{j=0}^{n} a_1(j) a_2(n-j) \; + \; C_2 \sum_{j\leq M} a_2(n-j) \\
&= \frac{\varepsilon}{2} a(n) \; + \; C_2 a_2^M(n) \\
&\leq \frac{\varepsilon}{2} a(n) \; + \; C_2 \frac{M}{L} a_2^L(n) \,.
\end{aligned}
$$

Now choose $N \geq M$ such that for $n \geq N$

$$C_2 \frac{M}{L} a_2^L(n) \;\leq\; \frac{\varepsilon}{2} a^L(n)\,.$$

Then for $n \geq N$,

$$\big|a(n) - a(n-1)\big| \;\leq\; \varepsilon a^L(n)\,.$$

Thus $\big|a(n) - a(n-1)\big| \;=\; \mathrm{o}\big(a^L(n)\big)$, so by Lemma 4.2 it follows that $a(n)$ satisfies RT_1. $\square$

Theorem 4.4 (Sandwich Theorem). *Suppose*

$$\dot{\mathbf{A}}(x) \;:=\; \sum_{n=0}^{\infty} \dot{a}(n) x^n \;=\; \prod_{n=1}^{\infty} (1 - x^n)^{-\dot{p}(n)}$$

is a reduced partition identity with

$$\frac{\dot{a}(n-1)}{\dot{a}(n)} \;\to\; 1 \qquad \text{as} \;\; n \to \infty\,.$$

Then any partition identity

$$\mathbf{A}(x) \;:=\; \sum_{n=0}^{\infty} a(n) x^n \;=\; \prod_{n=1}^{\infty} (1 - x^n)^{-p(n)} \tag{3}$$

satisfying

$$\dot{p}(n) \;\leq\; p(n) \;=\; \mathrm{O}\big(\dot{a}(n)\big) \tag{4}$$

will be such that

$$\frac{a(n-1)}{a(n)} \;\to\; 1 \qquad \text{as} \;\; n \to \infty\,.$$

Proof. Clearly

$$\mathbf{A}(x) \;=\; \dot{\mathbf{A}}(x) \cdot \ddot{\mathbf{A}}(x)$$

where

$$\ddot{p}(n) \;:=\; p(n) - \dot{p}(n)$$

$$\ddot{\mathbf{A}}(x) \quad = \quad \prod_{n=1}^{\infty}(1-x^n)^{-\ddot{p}(n)}\,.$$

If $\ddot{p}(n)$ is eventually 0, then Stewart's Sum Theorem gives the conclusion, for in this case the reduced form of $\ddot{\mathbf{A}}(x)$ satisfies RT_1 by item D.

So assume $\ddot{p}(n)$ is not eventually 0. Choose positive integers $d_1 > d_2 > 1$ such that $\ddot{p}(d_1)$ and $\ddot{p}(d_2)$ are positive. Let

$$\begin{aligned}
\mathbf{A}_1(x) \quad &:= \quad (1-x^{d_1})^{-1}(1-x^{d_2})^{-1}\dot{\mathbf{A}}(x) && (5)\\
\mathbf{A}_2(x) \quad &:= \quad \big(1-x^{d_1}\big)\big(1-x^{d_2}\big)\ddot{\mathbf{A}}(x)\\
\mathbf{P}_2(x) \quad &:= \quad -x^{d_1} \ - \ x^{d_2} \ + \ \sum_{n=1}^{\infty}\ddot{p}(n)x^n\\
\mathbf{H}_2(x) \quad &:= \quad \mathbf{P}_2(x) \ + \ \mathbf{P}_2(x^2)/2 \ + \ \cdots\\
\mathbf{B}_j(x) \quad &:= \quad (1-x)^{-j}\dot{\mathbf{A}}(x) \qquad \text{for } \ j=1,2\,. && (6)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{A}(x) \quad &= \quad \mathbf{A}_1(x)\mathbf{A}_2(x) && (7)\\
\mathbf{A}_2(x) \quad &= \quad \exp\big(\mathbf{H}_2(x)\big)\,. && (8)
\end{aligned}$$

Our goal is to show that $\mathbf{A}_1(x)$ and $\mathbf{A}_2(x)$ satisfy the conditions of Lemma 4.3. Applying Stewart's Sum Theorem to (5) one has

$$\frac{a_1(n-1)}{a_1(n)} \ \to \ 1\,;$$

so Schur's Tauberian Theorem applied to (5) gives

$$d_1 d_2 \cdot a_1(n) \ \sim \ [x^n](1-x)^{-2}\dot{\mathbf{A}}(x) \ = \ b_2(n)\,. \tag{9}$$

From (6) one readily sees that

$$\begin{aligned}
b_2(n) \quad &= \quad b_1(0) \ + \ \cdots \ + \ b_1(n)\\
\frac{b_1(n-1)}{b_1(n)} \quad &\to \quad 1 \qquad \text{as } \ n\to\infty\,;
\end{aligned}$$

so

$$\frac{b_1(n)}{b_2(n)} \ \to \ 0 \qquad \text{as } \ n\to\infty\,.$$

This, with (9), shows $b_1(n) \ = \ \mathrm{o}\big(a_1(n)\big)$, that is

$$\sum_{j=0}^{n}\dot{a}(j) \ = \ \mathrm{o}\big(a_1(n)\big)\,. \tag{10}$$

Differentiating both sides of (8) with respect to x and equating coefficients gives

$$na_2(n) \ = \ \sum_{j=1}^{n} jh_2(j)\cdot a_2(n-j)\,. \tag{11}$$

By (4), since $p_2(n) \le p(n)$ one has

$$p_2(n) \;=\; \mathrm{O}\big(\dot{a}(n)\big)\,.$$

From this and the fact that $\dot{a}(0) = 1$ it follows that there is a $C > 0$ such that for $n \ge 1$,

$$\sum_{j=1}^{n} p_2(j) \;\le\; C\sum_{j=0}^{n} \dot{a}(j)\,. \tag{12}$$

The definition of $h_2(n)$ and items (10), (12) yield

$$\begin{aligned} nh_2(n) \;&=\; \sum_{j\mid n} jp_2(j) \;\le\; n\sum_{j=1}^{n} p_2(j) \\ &\le\; Cn\sum_{j=0}^{n} \dot{a}(j) \;=\; \mathrm{o}\big(na_1(n)\big)\,. \end{aligned}$$

Let $L = L_{\mathbf{A}}$. Given $\varepsilon > 0$ choose M to be a multiple of L such that

$$nh_2(n) \;<\; \frac{\varepsilon}{2}\cdot na_1(n) \qquad \text{for } \; n \ge M\,. \tag{13}$$

By (7) one has, for all n,

$$a_2(n) \;\le\; a(n)\,. \tag{14}$$

There is a positive constant K such that, for $n \ge M$,

$$\begin{aligned} na_2(n) \;&=\; \sum_{j=M}^{n} jh_2(j)a_2(n-j) \;+\; \sum_{j<M} jh_2(j)a_2(n-j) \qquad \text{by (11)} \\ &\le\; \sum_{j=M}^{n} \frac{\varepsilon}{2}na_1(j)a_2(n-j) \;+\; \sum_{j<M} a_2(n-j)jh_2(j) \qquad \text{by (13)} \\ &\le\; n\frac{\varepsilon}{2}a(n) \;+\; Ka_2^M(n) \qquad \text{by (7)} \\ &\le\; n\frac{\varepsilon}{2}a(n) \;+\; Ka^M(n) \qquad \text{by (14)}\,; \end{aligned}$$

so

$$a_2(n) \;\le\; \frac{\varepsilon}{2}a(n) \;+\; \frac{KM}{L}\frac{a^L(n)}{n}\,.$$

Thus for $n \ge M$, using Lemma 4.1 (b), (d),

$$\begin{aligned} a_2^L(n) \;&\le\; \frac{\varepsilon}{2}a^L(n) \;+\; \frac{KM}{L}\sum_{j=0}^{L} \frac{a^L(n-j)}{n-j} \\ &\le\; \frac{\varepsilon}{2}a^L(n) \;+\; \frac{KM}{L(n-L)}\sum_{j=0}^{L} a^L(n) \\ &=\; \frac{\varepsilon}{2}a^L(n) \;+\; \frac{KM(L+1)}{L(n-L)}a^L(n)\,; \end{aligned}$$

so by choosing $N \geq M$ such that for $n \geq N$,

$$\frac{KM(L+1)}{L(n-L)} \leq \frac{\varepsilon}{2},$$

one has, for $n \geq N$,

$$a_2^L(n) \leq \varepsilon a^L(n).$$

Thus $a_2^L(n) = \mathrm{o}\big(a^L(n)\big)$, so $a(n-1)/a(n) \to 1$ as $n \to \infty$, by Lemma 4.3. □

4.1. **A New Proof of Bell's Polynomial Bound Theorem.** The following theorem is one of our favorites for proving the RT_1 property, and is at the very heart of our considerable generalization of the Bateman and Erdős results in *Partition Identities II.*

Theorem 4.5 (Bell [3]). *For a reduced partition identity*

$$\sum_{n=0}^{\infty} a(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

with $p(n) = \mathrm{O}\big(n^\gamma\big)$ *one has*

$$\frac{a(n-1)}{a(n)} \to 1 \qquad as \quad n \to \infty.$$

Proof. For the case that $\mathrm{rank}(p) < \infty$ simply appeal to Item D. So now suppose that $\mathrm{rank}(p) = \infty$.

Choose $\dot{p}(n)$ with $0 \leq \dot{p}(n) \leq p(n)$ satisfying the three conditions:

(i) $\dot{p}(n)$ is eventually 0,
(ii) $\dot{p}(n)$ is equal to $p(n)$ on at least $\gamma + 2$ values of n for which $p(n)$ does not vanish, and
(iii) the gcd of the n for which $\dot{p}(n)$ does not vanish is 1.

(One can find such a $p(n)$.) Condition (iii) says that the partition identity determined by $\dot{p}(n)$, namely

$$\sum_{n=0}^{\infty} \dot{a}(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-\dot{p}(n)}$$

is reduced.

Clearly $\dot{p}(n) \leq p(n)$, and by Item D one has $p(n) = \mathrm{O}\big(\dot{a}(n)\big)$. Again from Item D it is known that $\dot{a}(n)$ satisfies RT_1 . Thus by Theorem 4.4 the conclusion follows. □

4.2. **Showing** $p(n) = \mathrm{O}\big(\dot{a}(n)\big)$ **is best possible.** An example is given to show that the upper bound condition on the Sandwich Theorem does not allow for any obvious improvement such as $p(n) = \mathrm{O}\big(n^k \cdot \dot{a}(n)\big)$.

Let $f(n) \geq 1$ be a positive nondecreasing unbounded function. An example is constructed of a generating function $\dot{\mathbf{A}}(x)$ satisfying RT_1 for which one can find a $p(n)$ satisfying

$$\dot{p}(n) \;\leq\; p(n) \;=\; \mathrm{O}\big(f(n)\dot{a}(n)\big)$$

but $a(n)$ fails to satisfy RT_1. This shows that Theorem 4.4 is, in an important sense, the best possible. $f(n)$ can be replaced by a function which is unbounded (but not necessarily nondecreasing) and the result will still be true, but it requires a little more detail.

The construction of $\dot{\mathbf{A}}(x)$ proceeds by recursion, essentially by defining $\dot{p}(n)$ on longer and longer initial segments of the natural numbers. Let

$$\begin{aligned} m_1 &:= 1 \\ \alpha_j &:= 1 + \frac{1}{1+j} \qquad \text{for } j \geq 0 \\ \dot{p}_0(n) &:= 1 \qquad \text{for } n \geq 1 \\ \dot{p}_1(n) &:= 2^{n-1} \qquad \text{for } n \geq 1\,. \end{aligned}$$

Given a $\dot{p}_k(n)$, of define $\dot{a}_k(n)$ by

$$\dot{\mathbf{A}}_k(x) \;:=\; \sum_{n\geq 0} \dot{a}_k(n)x^n \;=\; \prod_{n\geq 1} \big(1-x^n\big)^{-\dot{p}_k(n)}.$$

Let $\Phi(k)$ be the conjunction of the following three assertions:

(a) $\dot{p}_{k-1}(m_k) \;>\; m_k$

(b) $[x^n](1-x)^{-k}\cdot\dot{\mathbf{A}}_{k-1}(x) \;<\; \alpha_{k-1}\cdot f(n) \qquad \text{for } n > m_k$

(c) $\dot{p}_k(n) \;=\; \begin{cases} \dot{p}_{k-1}(n) & \text{if } 1 \leq n \leq m_k \\ \lfloor \alpha_{k-1}^{n-m_k}\cdot\dot{p}_{k-1}(m_k)\rfloor & \text{if } n > m_k\,. \end{cases}$

It is easy to check that $\Phi(1)$ holds. We claim:

> Given m_j, $\dot{p}_j(n)$, and $\dot{a}_j(n)$ for $1 \leq j \leq k$, such that each of the conditions
>
> $$\Phi(1),\;\cdots,\;\Phi(k)$$
>
> holds, one can find m_{k+1}, $\dot{p}_{k+1}(n)$, and $\dot{a}_{k+1}(n)$ such that $\Phi(k+1)$ holds.

To do this one only needs to find an m_{k+1} that satisfies (a) and (b) as one can use (c) to define $\dot{p}_{k+1}(n)$. One can find such an m_{k+1} because Theorem B leads to

$$\frac{\dot{a}_k(n-1)}{\dot{a}_k(n)} \;\to\; \frac{1}{\alpha_k}\,,$$

and this in turn allows us to invoke Schur's Tauberian Theorem to obtain

$$\frac{[x^n](1-x)^{-k}\dot{\mathbf{A}}_k(x)}{\dot{a}_k(n)} \;\to\; \Big(\frac{\alpha_k}{\alpha_k - 1}\Big)^k \;<\; \infty.$$

Note that for k any positive integer one has $\dot{p}_{k-1}(n)$ agreeing with $\dot{p}_k(n)$ on the interval $1 \le n \le m_k$. One arrives at $\dot{p}(n)$ by letting

$$\dot{p}(n) \ := \ \dot{p}_k(n) \qquad \text{for any } k \text{ such that } n \le m_{k+1}\,.$$

Then $\dot{p}(n)$ satisfies RT_1 since for $n \ge m_k$ one has

$$\begin{aligned} \dot{p}(n) &\le \alpha_k \cdot \big(1 + \dot{p}(n-1)\big) \\ \dot{p}(n) &\to \infty \\ \alpha_k &\to 1. \end{aligned}$$

Thus by Theorem B, $\dot{a}(n)$ satisfies RT_1.

Now define a $p(n)$ that lies between $\dot{p}(n)$ and $3f(n)\dot{a}(n)$ for which $a(n)$ does not satisfy RT_1. Put $n_1 = 1$ and let $\Psi(k)$ be the conjunction of the following two assertions:

(a) $p_k(n) \ = \ \begin{cases} 2\lfloor f(n_k)\dot{a}(n_k)\rfloor + 1 & \text{if } n = n_k \\ \dot{p}(n) & \text{otherwise}, \end{cases}$

(b) $a_k(n_k) \ < \ f_k(n_k)\dot{a}(n_k)\,.$

As before, given a $p_k(n)$ define the corresponding $a_k(n)$ by

$$\mathbf{A}_k(x) \ := \ \sum_{n\ge 0} a_k(n)x^n \ = \ \prod_{n\ge 1} \big(1 - x^n\big)^{-p_k(n)}$$

Clearly $\Psi(1)$ holds, and we claim:

> Given m_j, $p_j(n)$, and $a_j(n)$ for $1 \le j \le k$, such that each of the conditions
>
> $$\Psi(1),\ \cdots,\ \Psi(k)$$
>
> holds, one can find m_{k+1}, $p_{k+1}(n)$ and $a_{k+1}(n)$ such that $\Psi(k+1)$ holds.

One only needs to find an n_{k+1} that satisfies (b), and this is possible since

$$a(n) \ \le \ [x^n](1-x)^{-\dot{p}(n_1) - \cdots - \dot{p}(n_k)} \cdot \dot{\mathbf{A}}(x)$$

holds, by construction, for infinitely many n.

Note that for k any positive integer one has $p_{k-1}(n)$ agreeing with $p_k(n)$ on the interval $1 \le n < n_k$. One arrives at $p(n)$ by letting

$$p(n) \ := \ p_k(n) \qquad \text{for any } k \text{ such that } n \le n_k\,.$$

Now we want to show that $a(n)$ does not satisfy RT_1. Notice that $a(n)$ is nondecreasing (as $p(1) > 0$) and

$$a(n_k) \ \ge \ p(n_k) \ = \ 2\lfloor f(n_k)\dot{a}(n_k)\rfloor \ + \ 1\,.$$

Let $n \in [n_k, n_{k+1}]$. Then

$$a_{k-1}(n) \ < \ f(n)\dot{a}(n)\,,$$

so

$$a(n_k - 1) \ < \ f(n_k)\dot{a}(n_k)$$

and thus

$$\frac{a(n_k)}{a(n_k - 1)} \geq 2 .$$

One has

$$\dot{p}(n) \ \leq \ p(n) \ = \ \mathrm{O}(f(n)\dot{a}(n))$$

and an infinite sequence n_k with

$$a(n_k)/a(n_k - 1) \geq 2 ,$$

so $a(n)$ certainly does not satisfy RT_1 .

5. The Eventual Sandwich Theorem

Partition functions $a(n)$, defined by partition identities

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty} (1 - x^n)^{-p(n)} ,$$

can be notoriously sensitive to changes in $p(n)$. However if $p(n)$ satisfies RT_1 then the situation is much more stable. The following says that if one removes any finite number of factors from the product expression in a partition identity and put back the same number of factors, but possibly with different powers of x involved, then the resulting partition function is asymptotic to a positive constant times the original partition function.

Lemma 5.1 (The Exchange Lemma). *Let $p_1(n)$ satisfy RT_1. If $p_1(d_1) > 0$ and d_2 is a positive integer let*

$$p_2(n) \ := \ p_1(n) + \delta_{n=d_2} - \delta_{n=d_1} .$$

Then

$$\frac{a_1(n)}{a_2(n)} \ \sim \ \frac{d_2}{d_1} .$$

Proof. The connection between the two partition functions is described by

$$\mathbf{A}_2(x) \ = \ \big(1 - x^{d_2}\big)^{-1} \cdot \big(1 - x^{d_1}\big) \cdot \mathbf{A}_1(x) . \tag{15}$$

Now, using the fact that both $p_1(n)$ and $p_2(n)$ satisfy RT_1, for $j = 1, 2$ one has

$$\begin{aligned} [x^n]\big(1 - x^{d_j}\big) \cdot \mathbf{A}_j(x) \ &= \ [x^n]\big(1 + x + \cdots + x^{d_j - 1}\big) \cdot \mathbf{A}_j^{(1)}(x) \\ &= \ a_j^{(1)}(n) + \cdots + a_j^{(1)}(n - d_j + 1) \\ &\sim \ d_j \cdot a_j^{(1)}(n) . \end{aligned}$$

From (15)

$$[x^n]\big(1 - x^{d_j}\big) \cdot \mathbf{A}_j(x) \ = \ [x^n]\big(1 - x^{d_j}\big) \cdot \mathbf{A}_j(x) .$$

Thus

$$\frac{a_1^{(1)}(n)}{a_2^{(1)}(n)} \ \sim \ \frac{d_2}{d_1} .$$

Given $\varepsilon > 0$ choose N such that

$$\left|\frac{a_1^{(1)}(n)}{a_2^{(1)}(n)} - \frac{d_2}{d_1}\right| \leq \varepsilon .$$

Then for $n \geq N$,

$$\left(\frac{d_2}{d_1} - \varepsilon\right) \cdot a_2^{(1)}(n) \leq a_1^{(1)}(n) \leq \left(\frac{d_2}{d_1} + \varepsilon\right) \cdot a_2^{(1)}(n) .$$

Summing this gives

$$\begin{aligned}\left(\frac{d_2}{d_1} - \varepsilon\right) \cdot \big(a_2(n) - a_2(M)\big) &\leq a_1(n) - a_1(M) \\ &\leq \left(\frac{d_2}{d_1} + \varepsilon\right) \cdot \big(a_2(n) - a_2(M)\big) ,\end{aligned}$$

and this yields the result since $a_2(n) \to \infty$ as $n \to \infty$. ☐

Lemma 5.2 (Shuffle Lemma). *Let $p_1(n)$ satisfy RT_1 and suppose $p_2(n)$ differs from $p_1(n)$ for only finitely many n. If*

$$\sum_n \big(p_1(n) - p_2(n)\big) = 0$$

we say $p_2(n)$ is a shuffle *of $p_1(n)$. If $p_2(n)$ is a shuffle of $p_1(n)$ then, for some constant $c > 0$,*

$$a_2(n) \sim c \cdot a_1(n) .$$

Proof. One can transform $p_1(n)$ into $p_2(n)$ by a finite sequence of exchanges as in Lemma 5.1. ☐

One of the difficulties with working with the Sandwich Theorem is that one needs to have $\dot{p}(n) \leq p(n)$ for *all* $n \geq 1$, and often one only has the 'eventually' less than result $\dot{p}(n) \preceq p(n)$. The next theorem shows that if $\dot{p}(n)$ satisfies RT_1 then one has some much appreciated leeway.

Theorem 5.3 (The Eventual Sandwich Theorem). *Suppose*

(i) $\dot{p}(n)$ *satisfies* RT_1

(ii) $\dot{p}(n) \preceq p(n) = \mathrm{O}\big(\dot{a}(n)\big)$

(iii) $\sum_n \big(p(n) - \dot{p}(n)\big) \geq 0 .$

Then

$$\frac{a(n-1)}{a(n)} \to 1 \qquad \text{as } n \to \infty .$$

Proof. Condition (iii) says precisely that one can find a finite shuffle $\widehat{p}(n)$ of $\dot{p}(n)$ such that for all $n \geq 1$,

$$\widehat{p}(n) \leq p(n) .$$

Then

$$\widehat{p}(n) \leq p(n) = \mathrm{O}\big(\widehat{a}(n)\big)$$

since Lemma 5.2 gives

$$a(n) \;=\; \mathrm{O}\big(\widehat{a}(n)\big)\,.$$

Consequently the Sandwich Theorem applies. □

6. The Classical Partition Function Heirarchy

Thanks to Theorem B that shows RT_1 is preserved in the passage from $p(n)$ to $a(n)$, one can start with a favorite function satisfying RT_1 and, by iterating this procedure, create an infinite heirarchy of "intervals" $[p(n), \mathrm{O}\big(a(n)\big)]$ to use to prove that partition functions satisfy RT_1; and thus to prove logical 0–1 laws.

Our favorite heirarchy we call the *Classical Partition Function* heirarchy, and it is defined recursively as follows:

$$\begin{aligned} \mathsf{part}_0(n) \;&:=\; 1 \qquad \text{for } \; n \geq 1 \\ \sum_{n=0}^{\infty} \mathsf{part}_{k+1}(n) x^n \;&=\; \prod_{n=1}^{\infty} \big(1 - x^n\big)^{-\mathsf{part}_k(n)}\,. \end{aligned}$$

Clearly the original partition function $\mathsf{part}(n)$ is $\mathsf{part}_1(n)$ in this heirarchy. Fortunately the asymptotics of this heirarchy have been well-studied using the tools of analytic number theory. One could use these results to see that each of the functions $\mathsf{part}_k(n)$ indeed satisfies RT_1; but invoking Theorem B seems much simpler. However these asymptotics allow us to draw other conclusions that strengthen our use of Sandwich Theorems. Thus they are given here in detail, following Petrogradsky's presentation.

Theorem 6.1 (See Petrogradsky [11], Theorem 2.1)**.** *In the following the 'input' $p(n)$ to a partition identity is in the left column, the 'output' $a(n)$ is in the right column, where $\alpha \geq 1$ and $k \geq 1$, and the constants θ and κ are defined after the table:*

$p(n)$	$a(n)$
$\big(\sigma + \mathrm{o}(1)\big) \cdot n^{\alpha-1}$	$\exp\Big(\big(\theta + \mathrm{o}(1)\big) \cdot n^{\alpha/(\alpha+1)}\Big)$
$\exp\Big(\big(\sigma + \mathrm{o}(1)\big) \cdot n^{\alpha/(\alpha+1)}\Big)$	$\exp\Big(\big(\kappa + \mathrm{o}(1)\big) \cdot \dfrac{n}{(\log n)^{1/\alpha}}\Big)$
$\exp\Big(\big(\sigma + \mathrm{o}(1)\big) \cdot \dfrac{n}{(\log^{(k)} n)^{1/\alpha}}\Big)$	$\exp\Big(\big(\sigma + \mathrm{o}(1)\big) \cdot \dfrac{n}{(\log^{(k+1)} n)^{1/\alpha}}\Big)$

where

$$\begin{aligned} \theta \;&=\; \big(1 + 1/\alpha\big) \cdot \Big(\sigma\zeta(\alpha+1) \cdot \Gamma(\alpha+1)\Big)^{1/(\alpha+1)} \\ \kappa \;&=\; \alpha \cdot \Big(\frac{\sigma}{\alpha+1}\Big)^{1+(1/\alpha)}\,. \end{aligned}$$

It is easy to verify that the $p(n)$ discussed in Theorem 6.1 do indeed satisfy RT_1. The usual Hardy-Ramanujan asymptotics for $\mathsf{part}(n)$ do indeed fit into the above table:

$$
\begin{aligned}
\mathsf{part}(n) &\sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4\sqrt{3}n}\\
&= \exp\Big(\left(\pi\sqrt{2n/3}\right) - \left(\log\left(4\sqrt{3}n\right)\right)\Big)\\
&= \exp\left(\left(\pi\sqrt{2/3} - \frac{\log\left(4\sqrt{3}n\right)}{\sqrt{n}}\right)\cdot\sqrt{n}\right)\\
&= \exp\Big(\left(\pi\sqrt{2/3} + \mathrm{o}(1)\right)\cdot\sqrt{n}\Big)\\
&= \exp\Big(\left(\sigma + \mathrm{o}(1)\right)\cdot\sqrt{n}\Big),
\end{aligned}
$$

where

$$\sigma \;=\; \pi\sqrt{2/3}\,.$$

Starting with this one can apply Theorem 6.1 to find the asymptotics for the classical partition heirarchy.

Corollary 6.2. *Defining* $\log^{(0)}(n) = 1$ *one has*

$$\mathsf{part}_k(n) \;=\; \exp\left(\left(C_k + \mathrm{o}(1)\right)\cdot\frac{n}{(\log^{(k-1)} n)^{1/2}}\right),$$

for $k \geq 1$ *and for suitable positive constants* C_k *.*

Corollary 6.3. *For* $k, r \geq 1$ *and* $\varepsilon > 0$ *one has*

$$
\begin{aligned}
n^{n^{1-\varepsilon}}\cdot\mathsf{part}_k(n) &= \mathrm{o}\big(\mathsf{part}_{k+1}(n)\big)\\
\mathsf{part}_k(n)^r &= \mathrm{o}\big(\mathsf{part}_{k+1}(n)\big)\,.
\end{aligned}
$$

Proof. By Corollary 6.2. □

Corollary 6.4. *Given* $\varepsilon > 0$ *and* $k, r \geq 1$*, suppose* $p(n)$ *is a component function that satisfies one of the conditions:* $p(n)$ *such that one of the following holds*

$$
\begin{aligned}
\mathsf{part}_k(n) &\preceq p(n) = \mathrm{O}\big(\mathsf{part}_k(n)^r\big)\\
\frac{\mathsf{part}_m(n)}{n^{n^{1-\varepsilon}}} &\preceq p(n) = \mathrm{O}\big(\mathsf{part}_m(n)\big)\,.
\end{aligned}
$$

Then $a(n)$ *satisfies* RT_1*.*

Proof. Apply the Eventual Sandwich Theorem to Corollary 6.3. □

Theorem 6.1 offers further concrete examples of function intervals which we can use to prove $a(n)$ satisfies RT_1. These will be featured in the examples in *Partition Identities II.*

Corollary 6.5. *If a partition identity satisfies one of the following conditions on* $p(n)$*, where* $C_1 > 0$*,* $\varepsilon > 0$*,* $k \geq 1$*, and* $\alpha \geq 1$*:*

(1) $1 \;\preceq\; p(n) \;=\; \mathrm{O}\Big(e^{\pi\sqrt{\frac{2}{3}n}}\big/n\Big)$

(2) $C_1 \preceq p(n) = \mathrm{O}\Big(e^{\big(\pi\sqrt{\frac{2}{3}C_1} - \varepsilon\big)\sqrt{n}}\Big/ n\Big)$

(3) $C_1 n^{\alpha-1} \preceq p(n) = \mathrm{O}\Big(e^{C_2 n^{\alpha/(\alpha+1)}}\Big)$,

where $C_2 = \Big(1+\dfrac{1}{\alpha}\Big)\cdot\Big(C_1\zeta(\alpha+1)\Gamma(\alpha+1)\Big)^{1/(\alpha+1)} - \varepsilon$

(4) $e^{C_1 n^{\alpha/(\alpha+1)}} \preceq p(n) = \mathrm{O}\Big(e^{C_2 n/(\log n)^{1/\alpha}}\Big)$,

where $C_2 = \alpha\cdot\Big(\dfrac{C_1}{\alpha+1}\Big)^{1+1/\alpha} - \varepsilon$,

(5) $e^{C_1 n/(\log^{(k)} n)^{1/\alpha}} \preceq p(n) = \mathrm{O}\Big(e^{(C_1-\varepsilon)\cdot n/\big(\log^{(k+1)} n\big)^{1/\alpha}}\Big)$,

then

$$\frac{a(n-1)}{a(n)} \to 1 \qquad \text{as } n \to \infty .$$

Proof. Apply the Eventual Sandwich Theorem to Theorem 6.1. □

7. Logical 0–1 Laws

A class $\mathcal{K}$ of finite relational structures is *adequate* if it is closed under disjoint union and extracting components. A perfectly general way to construct an adequate class is to take a collection $\mathcal{P}$ of finite connected structures and let $\mathcal{K}$ be the class of finite structures with components from $\mathcal{P}$. $\mathcal{K}$ has a MSO 0–1 law means that for every monadic second order sentence φ the probability that φ holds in a randomly chosen member of $\mathcal{K}$ is either 0 or 1. Define the count functions $a_{\mathcal{K}}(n)$ and $p_{\mathcal{K}}(n)$ as follows (counting up to isomorphism):

- $a_{\mathcal{K}}(n)$ is the number of members of $\mathcal{K}$ that have exactly n elements in their universe.
- $p_{\mathcal{K}}(n)$ is the number of members of $\mathcal{P}$ that have exactly n elements in their universe.

Compton ([8], [9]) showed that if $\mathcal{K}$ is an adequate class and $a_{\mathcal{K}}(n)$ satisfies RT_1 then $\mathcal{K}$ has a monadic second-order 0–1 law.

Given *any* partition function $a(n)$ satisfying RT_1 there is a simple way to create an adequate class $\mathcal{K}$ with $a_{\mathcal{K}}(n) = a(n)$. Start with the class of graphs $\mathcal{G}$. The number $p_{\mathcal{G}}(n)$ of connected graphs of size n grows exponentially, certainly faster than $p(n)$ as the radius of convergence of $\sum p_{\mathcal{G}}(n)x^n$ is 0. Of course $p(n)$ may exceed $p_{\mathcal{G}}(n)$ for a finite number of values, so add enough coloring predicates $Red(x)$, $Blue(x)$ etc., to the language of graphs so that the number of connected colored graphs of size n exceeds $p(n)$ for all $n \geq 1$. Now let $\mathcal{P}$ be a subclass of this class $\mathcal{G}_c$ of connected colored graphs with exactly $p(n)$ members of size n. Let $\mathcal{K}$ be the class of all finite colored graphs whose components come from $\mathcal{P}$.

Theorem 4.4 shows us how to find a vast array of partition identities satisfying RT_1, and thus one has a correspondingly vast array of classes of relational structures with a monadic second-order 0–1 law. Such examples are of course custom made. It is more satisfying to prove MSO 0–1 laws for *naturally* occurring classes of structures. We will conclude this section with the examples

- Forests of bounded height
- Varieties of MonoUnary Algebras
- Acyclic Graphs of bounded diameter

to illustrate the power of our new results. Before discussing these examples let it be mentioned that the previous techniques for proving a logical 0–1 law for adequate classes $\mathcal{K}$ (based solely on knowledge of $a_{\mathcal{K}}(n)$) relied on the following results:

(A1) Finding *explicit asymptotics for* $a_{\mathcal{K}}(n)$. This is highly desirable, but rarely possible.
(A2) Applying Bell's *polynomially bounded* result [3]: if $p_{\mathcal{K}}(n) = \mathrm{O}(n^k)$ then $a_{\mathcal{K}}(n)$ satisfies RT_1.
(A3) Applying the 'p is RT_1' result from Bell and Burris [5]: if $p_{\mathcal{K}}(n)$ satisfies RT_1 then $a_{\mathcal{K}}(n)$ satisfies RT_1.
(A4) Applying *Stewart's Sum Theorem* to combine adequate classes $\mathcal{K}_i$ whose count functions $a_{\mathcal{K}_i}(n)$ satisfy RT_1 (see [7] Chap. IV).

7.1. **Forests of Bounded Height.** In this example one can view a forest as either a poset or as graph with rooted trees. In the poset case, the height of a forest is one less than the maximum number of vertices in a chain in the forest. Each of the classes is defined by a finitely many universally quantified sentences. For example in the poset case (where the tree roots are at the top) one can use

$$
\begin{aligned}
&(\forall x)\,\big(x \le x\big)\\
&(\forall x \forall y)\,\big(x \le y \,\&\, y \le z \ \rightarrow x \le z\big)\\
&(\forall x \forall y)\,\Big(\big(x \le y \,\&\, x \le z\big) \ \rightarrow\ \big(y \le z\big) \vee \big(z \le y\big)\Big)\,.
\end{aligned}
$$

Let $\mathcal{F}_m$ be the collection of forests of height at most m, and let $p_m(n)$ and $a_m(n)$ be its counting functions. For $m = 0$ one has $p_0(1) = 1$, and otherwise $p_0(n) = 0$; and $a_0(n) = 1$ for all n. For $m = 1$ clearly $p_1(n) = 1$ for all $n \ge 1$, so $a_1(n) = \mathsf{part}(n)$. For $m \ge 1$ it is easy to see that removing the root from a tree in $\mathcal{F}_m$ gives a forest in $\mathcal{F}_{m-1}$, and indeed this operation is a bijection between the trees of $\mathcal{F}_m$ and all of $\mathcal{F}_{m-1}$. Thus for $m \ge 1$

$$p_m(n) \;=\; a_{m-1}(n-1),$$

and this leads to the formula

$$p_m(n) \;=\; \mathsf{part}_m(n-m)\,.$$

Thus $\mathcal{F}_m$ has a monadic second-order 0–1 law.

7.2. **Varieties of MonoUnary Algebras.** A monounary algebra $\mathbf{S} = (S, f)$ is a set with a unary operation. It has long been known that every variety of monounary algebras can be defined by a single equation, either one of the form $f^m(x) = f^m(y)$ or $f^{m+k}(x) = f^m(x)$. Only the trivial variety defined by $x = y$ has unique factorization. However one can view any class of algebras as relational structures by simply converting n-ary operations into $n+1$-ary relations. (Historically this is how logic developed, with function symbols being added later.) Although this is not the usual practice in algebra, for the purpose of logical properties it can be considered an equivalent formulation. If one treats the operation f of a monounary algebra as a binary relation then one obtains a digraph (directed graph) with the defining characteristics of a function, namely each vertex has a unique outdirected edge (possibly to itself). This formulation does not help with varieties defined by an equation of the form $f^m(x) = f^m(y)$ as such a variety is not closed under disjoint union. However for the variety of monounary algebras $\mathcal{M}_{m,k}$ defined by the equation $f^{m+k}(x) = f^m(x)$, the relational formulation gives an adequate class of relational structures, so one can view its finite members as an additive number system.

The connected digraphs of the models of the identity $f^{m+k}(x) = f^m(x)$ look like a directed cycle (whose number of vertices divides k) with trees attached to some (perhaps all, perhaps none) of its vertices. Let the count functions for $\mathcal{M}_{m,k}$ be $p_{m,k}(n)$ and $a_{m,k}(n)$.

Case $k = 1$**:** The connected digraphs of the models of an identity $f^{m+1}(x) = f^m(x)$ look just like (rooted) trees of height at most m, so by our analysis of trees of bounded height one sees that each variety $\mathcal{M}_{m,1}$ has a monadic second-order 0–1 law.

Case $k > 1$**:** Let $a_{m,1,d}(n)$ count the number of arrangements of forests in $\mathcal{M}_{m,1}$ with at most d components in a directed cycle of d vertices. Then it is quite straightforward to see that

$$\begin{aligned} p_{m,1}(n) &\leq p_{m,k}(n) \\ &= \sum_{d|k} a_{m,1,d}(n-d) \\ &\leq \sum_{d|k} d! \cdot a_{m,1}(n-d) \\ &\leq k \cdot k! \cdot a_{m,1}(n)\,. \end{aligned}$$

By our main theorem, and the previous case $k = 1$, it follows that $a_{m,k}(n)$ satisfies RT_1, and the class $\mathcal{M}_{m,k}$ of monounary algebras has a monadic second-order 0–1 law.

7.3. **Acyclic Graphs of Bounded Diameter.** Let $\mathcal{G}_d$ be the class of acyclic graphs of diameter at most d, meaning here that the length of the longest path in the graph is at most d, where the length of a path with j vertices is $j - 1$.

Given a connected member of this class there is a vertex v such that the distance from v to any other vertex is at most $\lceil d/2\rceil + 1$. (Such a vertex is in the *center* of the graph.) Let

$$\begin{aligned} c(d) &= \lceil d/2\rceil \\ f(d) &= \lfloor d/2\rfloor . \end{aligned}$$

Now we claim that for all $n \geq 1$

$$\frac{p_{\mathcal{F}_{f(d)}}(n)}{n} \leq p_{\mathcal{G}_d}(n) \leq p_{\mathcal{F}_{f(d)}}(n) + n \cdot p_{\mathcal{F}_{f(d)-1}}(n-1)\,. \tag{16}$$

For the lower bound note that any member of $\mathcal{F}_{f(d)}$ becomes a member of $\mathcal{G}_d$ by ignoring the root; and this map is at most n to 1. This gives the first inequality in (16).

For the upper bound note that any member of $\mathcal{G}_d(n)$ turns into a member of $\mathcal{F}(n)$ by simply designating a vertex to be the root. By choosing the root vertex in the center one obtains a member of $\mathcal{F}_{c(d)}(n)$. By snipping at most one leaf (and only if one has to) from the tree one has a member of $\mathcal{F}_{f(d)-1}(n) \cup \mathcal{F}_{f(d)}(n)$. This mapping is an injection for the part that maps into $\mathcal{F}_{f(d)}(n)$, and at most n to one for the part mapping into $\mathcal{F}_{f(d)-1}(n-1)$. Then using Corollary 6.3 this gives

$$\begin{aligned} \frac{p_{\mathcal{F}_{f(d)}}(n)}{n} &\leq p_{\mathcal{G}_d}(n) \\ &\leq p_{\mathcal{F}_{f(d)}}(n) + n \cdot p_{\mathcal{F}_{f(d)-1}}(n-1) \\ &= \mathrm{O}\big(p_{\mathcal{F}_{f(d)}}(n)\big)\,. \end{aligned}$$

The component function for $\mathcal{F}_k$ is $\mathsf{part}_k(n-d)$, so by Corollary 6.4 every $a_{\mathcal{F}_k}(n)$ satisfies RT_1. Thus the class of acyclic graphs of diameter at most d enjoys RT_1 growth and has a monadic second-order 0–1 law.

8. Generalized Partition Identities

Generalized partition identities allow $p(n)$ to take on nonnegative real values. Essentially everything that has been presented goes through in this setting. The reason for restricting attention to the case that the $p(n)$ have nonnegative integer values is simply that this is where the applications to combinatorics, additive number theory, and logical limit laws are to be found. The modification of the previous results to apply to generalized partition identities is quite straightforward.

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