PARTITION IDENTITIES II. THE RESULTS OF BATEMAN AND ERDŐS

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Abstract. This paper shows that the natural setting for the Bateman and Erdős study of monotonicity of the kth difference of partition functions $a(n)$ is the class of partition identities

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

with polynomially bounded $p(n)$. The results include a proof of a generalized version of their conjecture for polynomially bounded $p(n)$—their conjecture was for $p(n) \in \{0,1\}$.

1. Introduction

In Part I [6] the property $a(n-1)/a(n) \ \to \ 1$ of partition identities[1]

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n)x^n \ = \ \prod_{n=1}^{\infty}(1-x^n)^{-p(n)} \tag{1}$$

was the topic of investigation. Unless explicitly stated otherwise, it is assumed that the $p(n)$, and hence the $a(n)$, are nonnegative integers. When a partition identity is mentioned without a specific reference then the reader can assume (1) above is meant, using the two counting functions $p(n)$ and $a(n)$.

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[1]In [7] the models $\mathcal{A} = (\mathsf{A}, \mathsf{P}, +, 0, \| \ \|)$ of partition identities are called *additive number systems*, and the partition identity of an additive number system is called its *fundamental identity*. This paper does not assume the reader is familiar with additive number systems; however there are a couple of places in this paper where the development is completely transparent from the point of view of such systems. Such places have a footnote attached to explain this. For a basic reference on additive (and multiplicative) number systems see [7].

The notation used here follows that of Part I [6]:

$a(n)$	partition (count) function
$p(n)$	component (count) function
$\mathbf{A}(x) := \sum a(n)x^n$	generating function for the partitions
$\mathbf{P}(x) := \sum p(n)x^n$	generating function for the components
$\text{rank}(p) := \sum p(n)$	rank of the partition identity.

There are three main results proved in this paper:

(a) Theorem 3.5 gives a full generalization of the Bateman and Erdős monotonicity results for the kth difference $a^{(k)}(n)$ of a partition function $a(n)$ to the cases where the component function $p(n)$ is polynomially bounded. These results show that $a^{(k)}(n)$ is eventually monotone iff a very simple property called $\mathsf{BE}_k(n)$ holds for $p(n)$; and further properties concerning the rate of growth of $a^{(k)}(n)$ hold if $p(n)$ satisfies $\mathsf{BE}_k(n)$.

(b) Theorem 3.6 gives an enormous class of partition identities for which all the conclusions from Theorem 3.5 about the behavior of the kth difference $a^{(k)}(n)$ of the partition function $a(n)$ hold, even though the premisses are quite different from those of Theorem 3.5.

(c) Theorem 4.2 gives a natural extension of the Bateman and Erdős Conjecture to partition identities with polynomially bounded component function $p(n)$, an extension which is proved to be true.

The possibility of further generalizations is discussed briefly, and there are examples to show that the above theorems are essentially best possible.

2. The Property RT_1

The property

$$\frac{f(n-1)}{f(n)} \to 1\,,$$

where $f(n)$ is eventually positive, was discussed in some detail in Part I [6]. It plays a significant role in the results of Bateman and Erdős and is essential to Compton's approach to proving logical 0–1 laws. When dealing with partition functions $a(n)$ it is convenient to use interchangeably any of the phrases:

(i) $a(n)$ *satisfies* RT_1,
(ii) $\mathbf{A}(x)$ *satisfies* RT_1, or
(iii) *the partition identity satisfies* RT_1.

There are three basic results concerning when a partition function $a(n)$ satisfies RT_1, that is, when $a(n-1)/a(n) \to 1$ as $n \to \infty$. But first some definitions. A partition identity is *reduced* if

$$\gcd\big\{n : p(n) > 0\big\} \;=\; 1\,.$$

It is well known that $a(n)$ is eventually positive iff the partition identity is reduced[2]—see, for example, p. 43 of [7]. Given a partition identity let

$$\begin{aligned} d &:= \gcd\big\{n : p(n) > 0\big\} \\ p^\star(n) &:= p(nd) \\ a^\star(n) &:= a(nd)\,. \end{aligned}$$

Then

$$\mathbf{A}^\star(x) \;:=\; \sum_{n=0}^{\infty} a^\star(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p^\star(n)}\,. \tag{2}$$

This is the *reduced form* of the partition identity (1). The reduced form of a partition identity is reduced; and a reduced partition identity is the same as its reduced form.

Here are the three basic theorems concerning conditions on a partition identity that guarantee $a(n)$ satisfies RT_1:

- **Theorem A.** (Bell [4]) *Given a reduced partition identity, if $p(n)$ is polynomially bounded, that is, $p(n) = \mathrm{O}\big(n^\gamma\big)$ for some $\gamma \in \mathbb{R}$, then $a(n)$ satisfies RT_1.* This generalizes a result of Bateman and Erdős [2] that says if $p(n) \in \{0,1\}$ then RT_1 holds.
- **Theorem B.** (Bell and Burris [5]) *If $p(n-1)/p(n) \to 1$ as $n \to \infty$ then $a(n)$ satisfies RT_1.*
- **Theorem C.** (Stewart's Sum Theorem:[3] (see [7], p. 85) *If*

$$\sum_{n=0}^{\infty} a_j(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p_j(n)} \qquad (j=1,2)$$

and each $a_j^\star(n)$ satisfies RT_1 then $a^\star(n)$ also satisfies RT_1, where

$$\sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

with $p(n) = p_1(n) + p_2(n)$.

Repeated use will be made of (iterations of) the following simple application of Stewart's Sum Theorem:

> *if $\mathbf{A}(x)$ is a generating function satisfying RT_1 then, for d a positive integer, $\mathbf{A}(x)\cdot\big(1-x^d\big)^{-1}$ is again a generating function satisfying RT_1.*

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[2]The property BE_0 in Section 3 says that the partition identity is reduced.

[3]This theorem says that the sum of two additive number systems whose reduced forms satisfy RT_1 is again an additive number system whose reduced form satisfies RT_1.

A key feature of these three results is that they are proved by elementary means—one does not need the traditional methods of "hard asymptotics", namely the Cauchy integral theorem, the saddle-point methods, etc.

We adopt the convention of [7] that upper case bold letters name power series whose coefficients are given by the corresponding lower case italic letters, for example

$$\mathbf{F}(x) \;=\; \sum_{n=0}^{\infty} f(n)x^n\,.$$

By this convention $\mathbf{A}(x)$ is the power series $\sum a(n)x^n$ and $\mathbf{A}_1(x)$ is the power series $\sum a_1(n)x^n$, etc. It will be convenient to define coefficients $f(n)$ of a power series $\mathbf{F}(x)$ to be 0 for negative values of n.

3. The Bateman and Erdős Monotonicity Results Revisited

Given a subset J of the positive integers let $a(n)$ be the number of ways to express n as a sum of integers from J. The function $a(n)$ and the set J are connected by the partition identity

$$\sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{j\in J}^{\infty}(1-x^j)^{-1}\,.$$

The kth difference $a^{(k)}(n)$ of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a^{(k)}(n)x^n \;=\; (1-x)^k\prod_{j\in J}^{\infty}(1-x^j)^{-1}\,.$$

In their 1956 paper [2] Bateman and Erdős showed that $a^{(k)}(n)$ is eventually positive iff the following property, called P_k, holds:[4]

- J has at least $k+1$ elements,
- for $j_1,\ldots,j_k \in J$,

$$\gcd\big(J\setminus\{j_1,\ldots,j_k\}\big) \;=\; 1\,.$$

A key ingredient in their proof was to show that $a(n)$ satisfies RT_1. They go on to conjecture that if P_k holds then

[4]Our choice of notation to describe a partition identity is quite different from that of Bateman and Erdős. They use $p(n)$ to count the number of ways to partition n (as a sum of elements of J), whereas our use of $p(n)$ follows [7] where it is used to denote the number of indecomposables, or 'primes', in an additive number system. So, for the partition identities they consider, their $p(n)$ is our $a(n)$.

$$\frac{a^{(k+1)}(n)}{a^{(k)}(n)} \; = \; \mathrm{O}\Big(\frac{1}{\sqrt{n}}\Big)\,. \tag{3}$$

Two major improvements to the work of Bateman and Erdős have been published by Bell, namely the polynomial bound Theorem A mentioned earlier, and the following result from [3].

- **Theorem D.** *The Bateman and Erdős Conjecture* (3) *is correct.*

Clearly Bateman and Erdős are studying partition identities with the restriction that $p(n)$ can only take the values 0 and 1. Our goal (Theorems 3.3, 4.2) is to show that the natural home of the monotonicity results of Bateman and Erdős, and their conjecture, is the collection of partition identities with polynomially bounded $p(n)$.

First the condition P_k is reformulated[5] (and called BE_k) so that it makes sense for any component function $p(n)$:

$$\mathsf{BE}_k \; : \; \left\{ \begin{array}{l} \bullet\ \mathrm{rank}(p) \; > \; k\,, \\[2ex] \bullet\ \text{for any } q(n) \text{ satisfying the two conditions} \\[1ex] \qquad\qquad \left\{ \begin{array}{l} 0 \;\le\; q(n) \;\le\; p(n)\,, \\ \displaystyle\sum_n \big(p(n) - q(n)\big) \;=\; k \end{array} \right. \\[3ex] \quad \text{one has} \\[1ex] \qquad\qquad \gcd\big\{n : q(n) > 0\big\} \;=\; 1\,. \end{array} \right.$$

Of course BE_k holds if $k < 0$ as the conditions on $q(n)$ cannot be satisfied; and clearly BE_{k+1} implies BE_k. If a component function $p(n)$ satisfies BE_k for *all* k then $p(n)$ satisfies BE_∞. A simple and useful criterion for a component function to satisfy BE_∞ is that it be eventually positive. In particular *every component function* $p(n)$ *that satisfies* RT_1 *also satisfies* BE_∞.

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[5]From the point of view of additive number systems, BE_k is the obvious generalization of their P_k. The condition P_k says that the set of indecomposables of the system has at least $k+1$ members, and if one removes any k members of this set then the gcd of the sizes of the remaining indecomposables is 1. This is precisely the definition of BE_k, giving a good example where the language and imagery of additive number systems (say as given in [7]) provides a valuable tool when studying partition identities. Using the language of additive number systems, our goal is to study when the kth difference $a^{(k)}(n)$ of the counting function $a(n)$ for an additive number system is eventually positive. The function $a(n)$ counts the number of ways one can add indecomposables to get a 'number' of size n. The results of Bateman and Erdős apply to precisely the cases where one has at most one indecomposable of each size.

As a particular example of what Theorem 3.5 covers one has that if $p(n)$ is polynomially bounded then: $a(n)$ is eventually strictly increasing iff $p(n)$ satisfies BE_1. Outside of Theorem 3.5 there is one well known result involving BE_k that should be mentioned, namely that *any* partition function $a(n)$ is eventually positive iff its partition identity is reduced. Being reduced is the same as saying $p(n)$ satisfies BE_0.

The next result is for partition identities of finite rank.

Proposition 3.1. *Given a partition identity*

$$\sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

satisfying BE_k *and such that* $r := \operatorname{rank}(p) < \infty$, *let*

$$D \;:=\; \prod_{n\geq 1} n^{p(n)}\,.$$

Then

$$a^{(k)}(n) \;=\; \frac{1}{(r-k-1)!\cdot D}\cdot n^{r-k-1} \;+\; \mathrm{O}\big(n^{r-k-2}\big)\,. \tag{4}$$

$$\frac{a^{(k+1)}(n)}{a^{(k)}(n)} \;=\; 1-\frac{a^{(k)}(n-1)}{a^{(k)}(n)} \;=\; \mathrm{O}\Big(\frac{1}{n}\Big)\,. \tag{5}$$

Proof. The proof follows exactly the same reasoning as that given by Bateman and Erdős (in Lemma 2 and Theorem 3 of [2]) for the case that $p(n) \in \{0,1\}$, namely start with

$$a^{(k)}(n) \;=\; \big[x^n\big](1-x)^k\prod_{n=0}^{\infty}(1-x^n)^{-p(n)}$$

and apply complex partial fractions to the essentially finite product on the right. $\square$

Lemma 3.2. *Given a reduced partition identity*

$$\sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

one has:

(a) *the $a(n)$ are eventually positive;*

(b) *if* $\operatorname{rank}(p)=\infty$ *then $a(n)$ is* super polynomial, *that is,*

$$\lim_{n\to\infty}\frac{a(n)}{n^{\gamma}} \;=\; \infty \qquad \textit{for any } \gamma\in\mathbb{R}\,.$$

Proof. For a sufficiently large positive integer t the partition identity

$$\sum_{n=0}^{\infty} a_t(n)x^n \;=\; \prod_{n=1}^{t}(1-x^n)^{-p(n)}$$

is reduced; so $a_t(n)$ is eventually positive by Proposition 3.1 (with $k=0$). Then noting that $a(n) \geq a_t(n)$ gives (a).

Suppose $\operatorname{rank}(p) = \infty$ and $\gamma \in \mathbb{R}$. Choose t a sufficiently large integer so that $p(1) + \cdots + p(t) - 1 > \gamma$. Then $n^\gamma = \mathrm{o}\big(a_t(n)\big)$ by Proposition 3.1, so $n^\gamma = \mathrm{o}\big(a(n)\big)$, proving (b). □

Proposition 3.3. *Given a partition identity*

$$\sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)},$$

let $r = \operatorname{rank}(p)$. One has

(a) $\displaystyle\lim_{n\to\infty} \frac{a(0) \;+\; \cdots \;+\; a(n)}{n^\gamma} \;\to\; \infty \qquad \textit{for}\;\; \gamma < r\,.$

If $p(n) = \mathrm{O}\big(n^\gamma\big)$ then as $n \to \infty$,

(b) $\displaystyle\frac{a(n)}{a(0)+\cdots+a(n)} \;=\; 1 \;-\; \frac{a(0)+\cdots+a(n-1)}{a(0)+\cdots+a(n)} \;\to\; 0\,.$

Proof. Let $A(n) = a(0) \;+\; \cdots \;+\; a(n)$. The partition identity

$$\sum_{n=0}^{\infty} A(n)x^n \;=\; (1-x)^{-1}\prod_{n=1}^{\infty}(1-x^n)^{-p(n)} \tag{6}$$

is clearly reduced; so Lemma 3.2 (b) gives (a). If $p(n) = \mathrm{O}\big(n^\gamma\big)$ then apply Bell's Theorem A to (6) to show that $A(n)$ satisfies RT_1, which is (b). □

The following gives a compactness result for BE_k.

Lemma 3.4. *$p(n)$ satisfies BE_k iff there is a $p_1(n)$ such that*

(i) $0 \leq p_1(n) \leq p(n)$,

(ii) $\operatorname{rank}(p_1) < \infty$, *and*

(iii) $p_1(n)$ *satisfies* BE_k.

Proof. First note that if there is a $p_1(n)$ satisfying conditions (i)–(iii) then clearly $\mathbf{A}(x)$ must satisfy BE_k. For the converse assume $p(n)$ satisfies BE_k. If $\operatorname{rank}(p) < \infty$ just let $p_1(n) = p(n)$.

Now suppose $\mathrm{rank}(p) = \infty$. For any prime number π,

$$\sum_{\pi \nmid n} p(n) \; > \; k\,.$$

Otherwise there is a $q(n)$ satisfying the hypotheses of BE_k with $q(n) = 0$ whenever $\pi \nmid n$, which leads to $\pi | \gcd \{n : q(n) > 0\}$, contradicting BE_k. Thus for each prime π one can choose a finite subset X_π of positive integers from $\{n : \pi \nmid n\}$ such that

$$\sum_{n \in X_\pi} p(n) \; > \; k\,.$$

Now choose a subset X of $k+1$ positive integers n, each satisfying $p(n) > 0$, say $X = \{n_1, \ldots, n_{k+1}\}$. Then for any set Y of positive integers that contains X and any $Z \subset Y$ with k elements one sees that some member of X is in $Y \setminus Z$; so if a prime π divides the gcd of $Y \setminus Z$ then it must divide one of $n_1, \ldots, n_{k+1}$. Let

$$Y \;=\; X \cup \bigcup_{j=1}^{k+1} \bigcup_{\pi | n_j} X_\pi\,.$$

Then $p_1(n)$ defined to be $p(n)$ for $n \in Y$ and 0 otherwise has the desired properties. $\square$

The next theorem gives our generalization of the main results of Bateman and Erdős.

Theorem 3.5. *Given a partition identity*

$$\sum_{n=0}^{\infty} a(n) x^n \;=\; \prod_{n=1}^{\infty} (1 - x^n)^{-p(n)}$$

let $r = \mathrm{rank}(p)$. *Then*

(a) *if* $p(n)$ *satisfies* BE_k *then* $\dfrac{a^{(k)}(n)}{n^\gamma} \;\to\; \infty$ *for* $\gamma < r - k - 1$.

If $p(n)$ *is polynomially bounded then*

(b) *the* kth *difference function* $a^{(k)}(n)$ *is eventually positive iff* $p(n)$ *satisfies* BE_k *;*

(c) *if* $p(n)$ *satisfies* BE_k *then* $a^{(k)}(n)$ *satisfies* RT_1.

Proof. First note that any reduced partition identity for which $a^{(k)}(n)$ is eventually positive has $p(n)$ satisfying BE_k. For if $q(n)$ is as in the hypotheses of BE_k then, letting $r(n) = p(n) - q(n)$, from

$$\prod_{n=1}^{\infty} (1 - x^n)^{r(n)} \sum_{n=1}^{\infty} a^{(k)}(n) x^n \;=\; (1-x)^k \prod_{n=1}^{\infty} (1 - x^n)^{-q(n)}$$

follows

$$\prod_{r(n)>0} (1 + x + \cdots + x^{n-1}) \sum_{n=1}^{\infty} a^{(k)}(n)x^n \;=\; \prod_{n=1}^{\infty} (1-x^n)^{-q(n)}\,.$$

As the coefficients of the power series obtained by multiplying out the left side are eventually positive, it follows that the gcd of the set of n for which $q(n) > 0$ must be 1. This is the condition BE_k.

To show that '$p(n)$ satisfies BE_k implies that $a^{(k)}(n)$ is eventually positive' first note that (4) gives this for the case that $\mathrm{rank}(p) < \infty$. Furthermore, for the case that $\mathrm{rank}(p) < \infty$, (4) shows

$$a^{(k)}(n) \to \infty \qquad \text{as} \;\; n \to \infty \tag{7}$$

holds iff $k + 1 < \mathrm{rank}(p)$.

So now assume that $\mathrm{rank}(p) = \infty$, and decompose $p(n)$ as $p_1(n) + p_2(n)$ where $\mathrm{rank}(p_1) < \infty$ and the generating function determined by $p_1(n)$ has the property BE_k. For $p_1(n)$ and $p_2(n)$ introduce the partition identities

$$\mathbf{A}_j(x) \;:=\; \sum_{n=0}^{\infty} a_j(n)x^n \;=\; \prod_{n=1}^{\infty} (1-x^n)^{-p_j(n)} \qquad (j = 1, 2)\,.$$

By (7) one knows that there is a positive constant M such that

$$a_1^{(k)}(n) \;\geq\; \begin{cases} -M & \text{for all } n \\ 2 & \text{for } \; n \geq M. \end{cases} \tag{8}$$

Using the notation

$$A_2(n) \;:=\; a_2(0) \;+\; \cdots \;+\; a_2(n)$$

one has

$$\begin{aligned}
a^{(k)}(n) \;&=\; \sum_{j=1}^{n} a_1^{(k)}(n-j)a_2(j) \\
&\geq\; \sum_{j=1}^{n-M} a_1^{(k)}(n-j)a_2(j) \;-\; \sum_{j=n-M+1}^{n} \big|a_1^{(k)}(n-j)\big|a_2(j) \\
&\geq\; 2\sum_{j=1}^{n-M} a_2(j) \;-\; M\cdot \sum_{j=n-M+1}^{n} a_2(j) \\
&=\; 2\sum_{j=1}^{n} a_2(j) \;-\; (M+2)\cdot \sum_{j=n-M+1}^{n} a_2(j) \\
&=\; 2A_2(n) \;-\; (M+2)\cdot\big(A_2(n) \;-\; A_2(n-M+1)\big)
\end{aligned}$$

$$= \; 2A_2(n) \; - \; (M+2)\cdot\frac{A_2(n) \; - \; A_2(n-M+1)}{A_2(n)}\cdot A_2(n)\,.$$

Since the fraction in the last line goes to 0 by Proposition 3.3 (b), it follows that eventually

$$a^{(k)}(n) \; > \; A_2(n)\,,$$

showing that $a^{(k)}(n)$ is eventually positive. This finishes the proof of (a): $a^{(k)}(n)$ is eventually positive is equivalent to $p(n)$ satisfies BE_k, given that $p(n)$ is polynomially bounded.

To show (b), that $p(n)$ satisfies BE_k implies $a^{(k)}(n)$ grows much faster than n^γ for $\gamma < \operatorname{rank}(p)-k-1$, first note that if $\operatorname{rank}(p) < \infty$ then this follows from (4). Now assume that $\operatorname{rank}(p) = \infty$. Given any γ one can choose the $p_1(n)$ above to be such that $\operatorname{rank}(p_1) \; > \; \gamma+k+2$. Then from

$$\begin{aligned} a_1^{(k)}(n) \quad &< \quad a^{(k)}(n) \\ \frac{a_1^{(k)}(n)}{n^\gamma} \quad &\to \quad \infty \qquad \text{by } (4) \end{aligned}$$

one has the conclusion (b).

Item (c) follows from Stewart's Sum Theorem since

$$\mathbf{A}^{(k)}(x) \; = \; \mathbf{A}_1^{(k)}(x)\cdot\mathbf{A}_2(x)\,,$$

and using the facts that the reduced form of $a_2(n)$ satisfies RT_1 (by Theorem A as $p_2(n)$ is polynomially bounded) and ${a_1}^{(k)}(n)$ satisfies RT_1 by (5). □

Looking over the proof of Theorem 3.5 (a), we actually showed *every* $p(n)$ for which $a^{(k)}(n)$ is eventually positive has the property BE_k. The last step of the proof that '$p(n)$ satisfies BE_k implies $a^{(k)}(n)$ is eventually positive', where Bell's Theorem A is used to show that $a_2(n)$ satisfies RT_1, is the only obstacle to generalizing the Bateman and Erdős monotonicity proof to any generating function satisfying BE_k.

Examining the proof of (a) it is clear that one can take *any* $p(n)$ satisfying BE_k and decompose it into $p_1(n)+p_2(n)$ such that $p_1(n)$ has all the properties needed in the proof. The only thing missing is that one does not know if $a_2(n)$ satisfies RT_1. If indeed $a_2(n)$ satisfies RT_1, then by Stewart's Sum Theorem one sees that $a(n)$ must also satisfy RT_1. This leads to the following:

Question 1. *Suppose $p(n)$ satisfies BE_k, where $k \geq 1$, and $a(n)$ satisfies RT_1. Does it follow that $a^{(k)}(n)$ is eventually positive?*

We are not able, under these hypotheses, to prove that decomposing $p(n)$ into $p_1(n)+p_2(n)$ as in the proof gives an $a_2(n)$ that satisfies RT_1. Determining whether or not this must always be the case is equivalent to the following:

Question 2. *Given a generating function* $\mathbf{A}(x)$, *if*

- *the coefficients of* $(1-x)^{-2}\cdot\mathbf{A}(x)$ *satisfy* RT_1

does it follow that

- *the coefficients of* $(1-x)^{-1}\cdot\mathbf{A}(x)$ *satisfy* RT_1*?*

This does not seem overtly plausible. However, using Theorem 4.3 (the Eventual Sandwich Theorem from Part I [6] on partition identities), there is clearly an enormous range of partition identities for which all the conclusions from Theorem 3.5 concerning the behavior of $a^{(k)}(n)$ hold, even though $p(n)$ need not be polynomially bounded.

Theorem 3.6. *Suppose the partition identity*

$$\sum_{n=0}^{\infty}\dot{a}(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-\dot{p}(n)} \tag{9}$$

is such that $\dot{p}(n)$ *satisfies* RT_1. *Then for any partition identity*

$$\sum_{n=0}^{\infty}a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)} \tag{10}$$

with

$$\dot{p}(n) \;\le\; p(n) \;=\; \mathrm{O}\big(\dot{a}(n)\big)\,,$$

for n *sufficiently large, one has:*

(a) $a^{(k)}(n)$ *is eventually positive;*

(b) $\dfrac{a^{(k)}(n)}{n^{\gamma}} \;\to\; \infty \qquad$ *for* $\gamma \,<\, \mathrm{rank}(p)-k-1$*;*

(c) $a^{(k)}(n)$ *satisfies* RT_1.

In particular the hypotheses hold for functions $\dot{p}(n)$ *satisfying* RT_1.

Proof. Since $\dot{p}(n)$ satisfies RT_1 one can split it into $\dot{p}_1(n)+\dot{p}_2(n)$ such that

(i) $\mathrm{rank}(\dot{p}_1)<\infty$ and satisfies BE_k,

(ii) $\dot{p}_2(n)$ satisfies RT_1.

Let

$$\begin{aligned} p_1(n) &= \dot{p}_1(n)\,,\\ p_2(n) &= p(n)-\dot{p}_1(n)\,. \end{aligned}$$

Then using this decomposition one can carry through the part of the proof of (a) in Theorem 3.5 for '$p(n)$ satisfies BE_k implies $a^{(k)}(n)$ is eventually positive', but using Theorem 4.3 of Part I [6] (instead of Bell's Theorem A) at the end to show $\dot{a}_2(n)$ satisfies RT_1. The proofs of (b) and (c) are as in Theorem 3.5. $\square$

Examining the proof it is clear that the property of $\dot{p}(n)$ that is really needed is that it can be decomposed as $\dot{p}_1(n) + \dot{p}_2(n)$ where

(i) $\operatorname{rank}(\dot{p}_1) < \infty$ and satisfies BE_k,
(ii) the coefficients of $(1-x)^{-1} \cdot \dot{\mathbf{A}}_2(x)$ satisfy RT_1.

To show how this theorem can be used several examples are given—one can easily verify that the $\dot{p}(n)$ (the lower bounds) satisfy RT_1. However to show that the upper bounds are $\mathrm{O}\big(\dot{a}(n)\big)$ uses substantial results from the literature (see §5 of Part I [6] for details).

Examples: If a partition identity satisfies one of the following conditions on $p(n)$, where $C_1 > 0$, $\varepsilon > 0$, $s \geq 1$, and $\alpha \geq 1$:

(1) $1 \ \leq \ p(n) \ = \ \mathrm{O}\Big(e^{\pi\sqrt{\frac{2}{3}n}} \Big/ n\Big)$
(2) $C_1 \ \leq \ p(n) \ = \ \mathrm{O}\Big(e^{\left(\pi\sqrt{\frac{2}{3}C_1} - \varepsilon\right)\sqrt{n}} \Big/ n\Big)$
(3) $C_1 n^{\alpha-1} \ \leq \ p(n) \ = \ \mathrm{O}\Big(e^{C_2 n^{\alpha/(\alpha+1)}}\Big)$,
where $C_2 \ = \ \Big(1 + \dfrac{1}{\alpha}\Big)\Big(C_1\zeta(\alpha+1)\Gamma(\alpha+1)\Big)^{1/(\alpha+1)} \ - \ \varepsilon$
(4) $e^{C_1 n^{\alpha/(\alpha+1)}} \ \leq \ p(n) \ = \ \mathrm{O}\Big(e^{\left(C_2 n/(\log n)^{1/\alpha}\right)}\Big)$,
where $C_2 \ = \ \alpha\Big(\dfrac{C_1}{\alpha+1}\Big)^{1+1/\alpha} \ - \ \varepsilon$,
(5) $e^{C_1 n/(\log^{(s)} n)^{1/\alpha}} \ \leq \ p(n) \ = \ \mathrm{O}\Big(e^{\left((C_1-\varepsilon)n/(\log^{(s+1)} n)^{1/\alpha}\right)}\Big)$,

then for *any* k,

(a) $a^{(k)}(n)$ is eventually positive;
(b) $\dfrac{a^{(k)}(n)}{n^\gamma} \ \to \ \infty \qquad$ for $\gamma \in \mathbb{R}$;
(c) $a^{(k)}(n)$ satisfies RT_1.

4. The Bateman and Erdős Conjecture

Since the results of Bateman and Erdős lift so completely to the case that $p(n)$ is polynomially bounded, it is not surprising that their conjecture also has a natural extension.

First a theorem on the asymptotics of $a(n)$ is introduced in the case that $\operatorname{rank}(p) < \infty$, but in the more general context of the $p(n)$ being nonnegative real numbers. We call these *generalized* partition identities. For the case that $p(n)$ has integer values, the next proposition is well known, and is covered by Proposition 3.1 with $k = 0$.)

Proposition 4.1. *Given a 'generalized' partition identity*

$$\mathbf{A}(x) \;=\; \sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

such that $r := \operatorname{rank}(p) < \infty$ *and* $\gcd\big\{n : p(n) > 0\big\} = 1$, *let*

$$D \;=\; \prod_{n\geq 1} n^{p(n)}.$$

Then

$$a(n) \;=\; \frac{n^{r-1}}{\Gamma(r)\cdot D}\Big(1+\mathrm{o}(1)\Big). \tag{11}$$

Proof. The method of partial fractions that works so well when the $p(n)$ are integers does not apply in this generality. Instead we turn to Darboux's Theorem. One can assume $p(n) = 0$ for $n > k$. Then

$$\mathbf{A}(z) \;=\; \prod_{j=1}^{k}(1-z^j)^{-p(j)}. \tag{12}$$

$\mathbf{A}(z)$ has radius of convergence 1. Let

$$\begin{aligned} f(z) &:= \prod_{j=1}^{k}(1-z^j) \\ g(z) &:= (1+z)^{-p(2)}\cdots\big(1+z+\cdots+z^{k-1}\big)^{-p(k)} \\ \beta_\lambda &:= -\sum\{\alpha_j : \lambda^{d_j} = 1\}. \end{aligned}$$

Note that the β_λ are all real numbers, and $g(1) = 1/D$. From

$$\mathbf{A}(z) \;=\; \prod\Big\{\Big(1-\frac{z}{\lambda}\Big)^{\beta_\lambda} : f(\lambda) = 0\Big\}$$

one sees that the only singularities of $\mathbf{A}(z)$ on the circle $|z| = 1$ are algebraic singularities.

Next note that 1 is the only common root of the polynomials $x^{d_1}-1$, $\ldots$, $x^{d_k}-1$, for if λ is a common root then $\lambda^{d_1} = 1$, $\ldots$, $\lambda^{d_k} = 1$ gives $\lambda^d = 1$ where $d = \gcd\big\{d_1,\ldots,d_k\big\}$. But $d = 1$ by assumption. From this one sees that

$$\beta_1 \;=\; -(\alpha_1+\cdots+\alpha_k) \;=\; -r \;<\; \beta_\lambda \qquad (\lambda \neq 1).$$

Thus the unique minimal value among the β_λ occurs when $\lambda = 1$, and in this case $\beta_1 = -r$. As

$$\mathbf{A}(z) \;=\; g(z)\big(1-z\big)^{-r}$$

it follows from Darboux's Theorem (as presented in Odlyzko's survey article [11]) that[6]

$$\begin{aligned} a(n) \;&=\; \frac{g(1)}{\Gamma(r)}\cdot n^{r-1} \;+\; \mathrm{o}\big(n^{r-1}\big) \\ &=\; \frac{1}{\Gamma(r)\cdot D}\cdot n^{r-1} \;+\; \mathrm{o}\big(n^{r-1}\big)\,. \end{aligned}$$

□

Now we are ready for the generalization of the error estimate in the Bateman and Erdős Conjecture, and its proof.

Theorem 4.2. *Given a partition identity*

$$\mathbf{A}(x) \;:=\; \sum_{n=0}^{\infty} a(n)x^n \;=\; \prod_{n=1}^{\infty}(1-x^n)^{-p(n)} \tag{13}$$

with $p(n) = \mathrm{O}\big(n^\gamma\big)$ *and satisfying* BE_k *, where* $\gamma \geq 0$ *, one has*

$$a^{(k+1)}(n) \;=\; \mathrm{O}\Big(\frac{a^{(k)}(n)}{n^{1/(\gamma+2)}}\Big)\,. \tag{14}$$

The conjecture of Bateman and Erdős is the case $p(n) \in \{0,1\}$ (and therefore a special case of our result when $\gamma = 0$). The idea of our proof is as follows:

The case that $\mathrm{rank}(p)$ is finite is taken care of by Proposition 3.1, which actually gives the stronger result

$$a^{(k+1)}(n) \;=\; \mathrm{O}\Big(\frac{a^{(k)}(n)}{n}\Big)\,.$$

So assume that $\mathrm{rank}(p)$ is infinite. From the compactness result Lemma 3.4 for BE_k one can find an arbitrarily large finite set of positive integers J such that

$$p(n) - \chi_J(n)$$

still satisfies BE_k . We will use the notation

$$[x] \;:=\; x - \lfloor x \rfloor\,.$$

[6]Phillipe Flajolet informs us that using integration over Hankel contours (as developed in his paper [9] with Andrew Odlyzko) one can improve the error term to $\mathrm{O}\big(1/n\big)$. However this would not lead to any strengthening of the conclusion in our Theorem 4.2.

Since $p(n)$ satisfies BE_k , with

$$r \;:=\; \lceil\gamma\rceil + 2$$

one can choose positive integers

$$1 = d_0 < d_1 < d_2 < \cdots < d_r$$

such that, for $1 \le j \le r$,

$$\prod_{i=1}^{j} \big(1 - x^{d_i}\big)\mathbf{A}(x)$$

has a component function that still satisfies BE_k . Let

$$\begin{aligned}
\beta_j \;&:=\; \begin{cases} [\gamma] + \delta_{[\gamma]=0} & \text{if } \; j = r-1 \\ 1 & \text{otherwise} \end{cases}\\
\mathbf{A}_0(x) \;&:=\; \mathbf{A}^{(k-1)}(x)\\
\mathbf{A}_{j+1}(x) \;&:=\; \big(1 - x^{d_{j-1}}\big)^{\beta_{j-1}}\mathbf{A}_{j-1}(x) \qquad \text{for } \; 0 \le j \le r\\
\mathbf{B}_j(x) \;&:=\; \prod_{i=0}^{j} \big(1 - x^{d_i}\big)^{-\beta_i} \qquad \text{for } \; 0 \le j \le r\,.
\end{aligned}$$

One can picture the $\mathbf{A}_j(x)$ as follows:

$$\mathbf{A}_{r+1} \overset{\left(1-x^{d_r}\right)^{\beta_r}}{\longleftarrow} \mathbf{A}_r \overset{\left(1-x^{d_{r-1}}\right)^{\beta_{r-1}}}{\longleftarrow} \mathbf{A}_{r-1} \;\cdots\; \mathbf{A}_1 \overset{\left(1-x^{d_0}\right)^{\beta_0}}{\longleftarrow} \mathbf{A}_0\,.$$

In particular, if γ is an integer then $r = \gamma + 2 = \lfloor\gamma\rfloor + 2$ and all the β_j are equal to 1. If γ is not an integer then $r = \lfloor\gamma\rfloor + 3$ and all the $\beta_j = 1$ except $\beta_{r-1} = [\gamma]$.

We briefly employ the $\mathbf{A}_j^*(x)$ defined as follows:

$$\begin{aligned}
\mathbf{A}_0^*(x) \;&:=\; \mathbf{A}^{(k-1)}(x)\\
\mathbf{A}_{j+1}^*(x) \;&:=\; \big(1 - x^{d_j}\big)\mathbf{A}_j^*(x) \qquad \text{for } \; 0 \le j \le r\,.
\end{aligned}$$

From our choice of the d_j it follows that each $(1-x)^{-k}\mathbf{A}_j^*(x)$ has a component function that satisfies BE_k , and thus, as $n \to \infty$ one has from Proposition 3.3, for $0 \le j \le r+1$,

$$\begin{aligned}
\frac{a_j^*(n)}{n^\delta} \;&\to\; \infty \qquad \text{for any } \; \delta \in \mathbb{R}\\
\frac{a_j^*(n-1)}{a_j^*(n)} \;&\to\; 1\,,
\end{aligned}$$

and for $0 \le j \le r$

$$a_j^*(n + d_j) \;-\; a_j^*(n) \quad = \quad a_{j+1}^*(n)\,.$$

As

$$\mathbf{A}_j(x) \ := \ \begin{cases} \mathbf{A}_j^*(x) & \text{for } \ 0 \le j \le r-1 \\ \left(1 - x^{d_{r-1}}\right)^{\beta_{r-1}-1} \mathbf{A}_j^*(x) & \text{for } \ j = r,\, r+1\,, \end{cases}$$

one has, for $0 \le j \le r+1$,

$$\frac{a_j(n)}{n^\delta} \ \to \ \infty \qquad \text{for any } \ \delta \in \mathbb{R} \tag{15}$$

$$\frac{a_j(n-1)}{a_j(n)} \ \to \ 1\,, \tag{16}$$

and for $0 \le j \le r$,

$$\big(a_j(n+d_j) \ - \ a_j(n)\big) \ - \ \frac{1}{2}a_{j+1}(n) \ \to \ \infty\,. \tag{17}$$

Property (15) says that not only is $a_j(n)$ eventually positive, but it actually has *super polynomial* growth. Property (16) says $a_j(n)$ satisfies the RT_1 condition. Property (17) says that the difference $a_j(n+d_j) - a_j(n)$ is eventually positive, and indeed exceeds the super polynomially fast growing $a_{j+1}(n)/2$.

Choose a positive integer N such that for $n \ge N$

$$a_j(n) \ > \ 0 \qquad \text{for } \ 0 \le j \le r+1$$

$$a_j(n) \ - \ a_j(n-d_j) \ \ge \ \frac{1}{2}a_{j+1}(n) \qquad \text{for } \ 0 \le j \le r\,. \tag{18}$$

The proof falls into two parts: the *first* part shows that

$$a_r(n) \ = \ \mathrm{O}\big(a_0(n)/n\big)\,. \tag{19}$$

The *second* part uses this to show

$$a_1(n) \ = \ \mathrm{O}\Big(\frac{a_0(n)}{n^{1/(\gamma+2)}}\Big)$$

which is then lifted to

$$a_1{}^{(1)}(n) \ = \ \mathrm{O}\Big(\frac{a_0^{(1)}(n)}{n^{1/(\gamma+2)}}\Big)\,, \tag{20}$$

giving the desired result. Throughout this proof n will only be used to designate *nonnegative* integers.

Proof. **[First part of the proof]:**

This part of the proof concentrates on $\mathbf{A}_0(x)$, $\mathbf{A}_r(x)$ and $\mathbf{A}_{r+1}(x)$ using

$$\begin{aligned} \mathbf{H}(x) \ &:= \ \log \mathbf{A}_r(x) \\ \mathbf{S}(x) \ &:= \ x(1-x^{d_r})^{-\beta_r}\mathbf{H}'(x)\,. \end{aligned}$$

One has

$$
\begin{aligned}
&[x^n]\frac{x\mathbf{A}'(x)}{\mathbf{A}(x)} &=& \sum_{j|n} jp(j)\\
(21)\qquad & x{\mathbf{A}_r}'(x) &=& \mathbf{S}(x)\mathbf{A}_{r+1}(x)\\
(22)\qquad & \mathbf{A}_0(x) &=& \mathbf{B}_r(x)\mathbf{A}_{r+1}(x)\\
(23)\qquad & \frac{1}{C_1}n^{\gamma+2} &<& b_r(n) \;<\; C_2 n^{\gamma+2},
\end{aligned}
$$

for some $C_1,\, C_2 > 0$. Item (23) follows from Proposition 4.1 after observing that

$$\beta_0 + \cdots + \beta_r = \gamma + 3\,.$$

Also

$$x\mathbf{H}'(x) = -\frac{kx}{1-x} - \sum_{j=0}^{r-1}\frac{\beta_j d_j x^{d_j}}{1-x^{d_j}} + \frac{x\mathbf{A}'(x)}{\mathbf{A}(x)}$$

so by (21)

$$nh(n) = -k - \sum_{j=0}^{r-1}\beta_j d_j\cdot\delta_{d_j|n} + \sum_{j|n} jp(j)\,.$$

From this follows

$$(24)\qquad -k - \sum_{j=0}^{r-1}\beta_j d_j\cdot\delta_{d_j|n} \le nh(n) \le \sum_{j|n} jp(j)\,.$$

Furthermore

$$
\begin{aligned}
\mathbf{S}(x) &= x(1-x^{d_r})^{-1}\mathbf{H}'(x) = \sum_{i\ge 0} x^{id_r}\sum_{j\ge 1} jh(j)x^j\\
&= \sum_{n\ge 1}\Big(\sum_{\substack{i\ge 0\\ j\ge 1\\ id_r+j=n}} jh(j)\Big)x^n
\end{aligned}
$$

so by (24)

$$
\begin{aligned}
s(n) &= \sum_{1\le j\le n}\;\sum_{\substack{i\ge 0\\ j\ge 1\\ id_r+j=n}} jh(j) \;\le\; \sum_{1\le j\le n}\sum_{i|j} ip()\\
&\le \sum_{1\le i\le n}\sum_{j\le n/i} ip(i) = n\sum_{1\le i\le n} p(i)\\
&= \mathrm{O}\big(n^{\gamma+2}\big)\,;
\end{aligned}
$$

and again by (24)

$$\begin{aligned} -s(n) &\leq \sum_{1\leq j\leq n} \sum_{\substack{i>0\\ j\geq 1\\ id_r+j=n}} \Big(k + \sum_{j=0}^{r-1} \beta_j d_j \cdot \delta_{d_j|n}\Big) \\ &= \mathrm{O}\big(n^2\big)\,. \end{aligned}$$

Thus there is a constant $C > 0$ such that

$$\big|s(n)\big| \leq C \cdot n^{\gamma+2} \qquad \text{for} \quad n \geq 1\,.$$

From (21)

$$\begin{aligned} na_r(n) &= \sum_{i=0}^{n} s(n-i)a_{r+1}(i) \\ &= \sum_{i=N}^{n} s(n-i)a_{r+1}(i) + \sum_{i<N} s(n-i)a_{r+1}(i) \\ &\leq \underbrace{C\sum_{i=N}^{n}(n-i)^{\gamma+2}a_{r+1}(i)}_{\text{(I)}} + \mathrm{O}\big(n^{\gamma+2}\big)\,. \end{aligned} \tag{25}$$

To estimate (I) one uses:

$$\begin{aligned} &\sum_{i=N}^{n}(n-i)^{\gamma+2}a_{r+1}(i) \\ \leq\ & C_1 \sum_{i=N}^{n} b_r(n-i)a_{r+1}(i) \qquad \text{by} \quad (23) \\ =\ & C_1\Big(\sum_{i=0}^{n} b_r(n-i)a_{r+1}(i) - \sum_{i<N} b_r(n-i)a_{r+1}(i)\Big) \\ \leq\ & C_1\Big(a_0(n) + \mathrm{O}\big(n^{\gamma+2}\big)\Big) \qquad \text{by} \quad (22),\ (23). \end{aligned}$$

This with (25) gives

$$n \cdot a_r(n) \leq CC_1\Big(a_0(n) + \mathrm{O}\big(n^{\gamma+2)}\big)\Big) + \mathrm{O}\big(n^{\gamma+2}\big)\,;$$

consequently

$$a_r(n) = \mathrm{O}\Big(a_0(n)/n\Big)\,.$$

This finishes the first part of the proof, the proof of (19). $\square$

Proof. [**Second part of the proof**]:

For $1 \le j \le r$ let

$$\lambda_j \ := \ \beta_0 \ + \ \cdots \ + \ \beta_{j-1}$$

Then $\lambda_1 = 1$ and $\lambda_r = \gamma + 2$. For $1 \le j \le r$ it is claimed that

$$a_j(n) \ = \ \mathrm{O}\big(a_0(n)/n^{\lambda_j/(\gamma+2)}\big)\,. \tag{26}$$

To prove this start with $j = r$ and work down to $j = 1$. For $j = r$ the claim is true as

$$a_r(n) \ = \ \mathrm{O}\big(a_0(n)/n\big)$$

is the conclusion of the first part of the proof of the theorem. To keep the notation simple the step from $j = 2$ to $j = 1$ is given. All previous steps, from j to $j-1$, use an identical argument[7]. So assume (26) holds for $j = 2$, that is,

$$a_2(n) \ = \ \mathrm{O}\Big(\frac{a_0(n)}{n^{\lambda_2/(\gamma+2)}}\Big)\,; \tag{27}$$

we want to prove

$$a_1(n) \ = \ \mathrm{O}\Big(\frac{a_0(n)}{n^{\lambda_1/(\gamma+2)}}\Big)\,. \tag{28}$$

By (27), for some $C > 0$,

$$a_2(n) \ \le \ C \cdot \frac{a_0(n)}{n^{\lambda_2/(\gamma+2)}} \tag{29}$$

for n sufficiently large—without loss of generality one can assume that (29) **holds for** $n \ge N$.

Assuming $n - jd_1 \ge N$, from (18), for $0 \le i \le j$

$$a_1(n - id_1) \ - \ a_1(n-(i+1)d_1) \ \le \ a_2(n - id_1)/2\,;$$

summing this from $i = 0$ to $j - 1$ gives

$$a_1(n) \ - \ a_1(n - jd_1) \ \le \ \sum_{i=0}^{j-1} a_2(n - id_1)/2\,,$$

or

$$a_1(n - jd_1) \ \ge \ a_1(n) \ - \ \sum_{i=0}^{j-1} a_2(n - id_1)/2\,. \tag{30}$$

Then from (18), (29) and (30),

$$a_1(n - jd_1) \ \ \ge \ \ a_1(n) \ - \ \frac{C}{2}\sum_{i=0}^{j-1} \frac{a_0(n - id_1)}{(n - id_1)^{\lambda_2/(\gamma+2)}}$$

---

[7] A couple of footnotes have been inserted at appropriate spots to indicate the indices to be used in the general j step, that is, in the argument from j to $j-1$.

$$\geq \ a_1(n) \ - \ \frac{jC}{2}\frac{a_0(n)}{(n-jd_1)^{\lambda_2/(\gamma+2)}} \,. \tag{31}$$

From $\mathbf{A}_0(x) = \mathbf{B}_0(x) \cdot \mathbf{A}_1(x)$ follows[8]

$$a_0(n) \ = \ \sum_{j=0}^{n} b_0(j)a_1(n-j) \,.$$

By Proposition 4.1 one can choose a constant $C_1 > 0$ such that[9]

$$b_0(n) \ \geq \ C_1 n^{\lambda_1 - 1} \tag{32}$$

when n is sufficiently large–without loss of generality one can assume (32) **holds for** $n \geq N$.

Since $a_1(n)$ satisfies RT_1 and has super polynomial growth, and $b_0(n)$ has polynomially bounded growth, it follows that

$$\Big(\sum_{0\leq j<N} + \sum_{n-N<j\leq n}\Big)a_1(n-j)b_0(j) \ = \ \mathrm{O}\big(a_1(n)\big) \tag{33}$$

$$\Big(\sum_{0\leq jd<N} + \sum_{(n-N)<jd\leq n}\Big)a_1(n-jd_1)j^{\lambda_1-1} \ = \ \mathrm{O}\big(a_1(n)\big)\,. \tag{34}$$

Noting that $a_1(n-j)$ and $b_0(j)$ are both positive for $N \leq j \leq n-N$ provided $n \geq 2N$ one has, in view of (32), (33) and (34), for $n \geq 2N$

$$\begin{aligned}
a_0(n) \ &= \ \sum_{j=0}^{n} a_1(n-j)b_0(j) \\
&= \ \sum_{j=N}^{n-N} a_1(n-j)b_0(j) \ + \ \mathrm{O}\big(a_1(n)\big) \\
&\geq \ \sum_{N\leq jd\leq(n-N)} a_1(n-jd_1)b_0(jd_1) \ + \ \mathrm{O}\big(a_1(n)\big) \\
&\geq \ C_1 d_1^{\lambda_1-1} \sum_{N\leq jd\leq n-N} a_1(n-jd_1)j^{\lambda_1-1} \ + \ \mathrm{O}\big(a_1(n)\big) \\
&\geq \ C_1 d_1^{\lambda_1-1} \sum_{0\leq jd\leq n} a_1(n-jd_1)j^{\lambda_1-1} \ + \ \mathrm{O}\big(a_1(n)\big)\,. \qquad (35)
\end{aligned}$$

Since $a_1(n)$ satisfies RT_1 and has superpolynomial growth one sees that, as $n \to \infty$,

$$\frac{1}{a_1(n)} \sum_{0\leq jd\leq n} a_1(n-jd_1)j^{\lambda_1-1} \ \to \ \infty\,. \tag{36}$$

---

[8]For the general j step use $\mathbf{A}_0(x) = \mathbf{B}_{j-2}(x) \cdot \mathbf{A}_{j-1}(x)$ at this point in the proof.

[9]In the general j step use $b_{j-2}(n) \ \geq \ C_1 n^{\lambda_{j-1}-1}$.

Thus from (35) and (36)

$$a_0(n) \;\geq\; K \sum_{0\leq jd\leq n} a_1(n-jd_1)j^{\lambda_1-1} \tag{37}$$

for n sufficiently large, where

$$K \;=\; C_1 d_1^{\lambda_1-1}/2.$$

Choose $N_1 \geq 2N$ such that (37) holds for $n \geq N_1$.

Suppose (28) fails. Given any n we claim that there is a positive integer m such that the following two inequalities hold:

$$m \;\geq\; N + d\cdot m^{1/(\gamma+2)} \tag{38}$$

$$a_1(m) \;\geq\; \frac{C}{2} n \cdot \frac{a_0(m)}{m^{\lambda_1/(\gamma+2)}}. \tag{39}$$

Item (38) holds for m sufficiently large, and (39) holds for infinitely many m as (28) fails. Thus for each n there are infinitely many solutions m to (38) and (39). For the rest of the proof fix $\mathfrak{m} = \mathfrak{m}(n)$ to provide one solution for each n. Let

$$\mathfrak{q} \;=\; \mathfrak{q}(n) \;:=\; \left\lfloor \mathfrak{m}^{1/(\gamma+2)} \right\rfloor.$$

Clearly

$$\mathfrak{m} \;\to\; \infty \qquad \text{and} \;\; \mathfrak{q} \to \infty$$

as $n \to \infty$. Item (38) yields

$$\mathfrak{m} - \mathfrak{q}d \;\geq\; N,$$

and this with (18) gives

$$a_1(\mathfrak{m} - \mathfrak{q}d_1) \;\leq\; a_1(\mathfrak{m} - jd_1) \qquad \text{for} \;\; 0 \leq j \leq \mathfrak{q}. \tag{40}$$

From (31), (39), (40), for $0 \leq j \leq \mathfrak{q}$,

$$\begin{aligned} a_1(\mathfrak{m} - jd_1) \;&\geq\; a_1(\mathfrak{m}) \;-\; \frac{C}{2}\frac{j a_0(\mathfrak{m})}{(\mathfrak{m}-jd_1)^{\lambda_2/(\gamma+2)}} \\ &\geq\; \frac{C}{2} a_0(\mathfrak{m})\Big(\frac{n}{\mathfrak{m}^{\lambda_1/(\gamma+2)}} \;-\; \frac{j}{(\mathfrak{m}-\mathfrak{q}d_1)^{\lambda_2/(\gamma+2)}}\Big). \end{aligned} \tag{41}$$

Choose $N_2 > 0$ such that $\mathfrak{m} \geq N_1$ for $n \geq N_2$. Then, from (40) and (37), for $n \geq N_2$

$$a_0(\mathfrak{m}) \;\geq\; K \sum_{j=0}^{\mathfrak{q}-1} a_1(\mathfrak{m} - jd_1) j^{\lambda_1-1}. \tag{42}$$

From (41) and (42) one has, for $n \geq N_2$,

$$a_0(\mathfrak{m}) \;\geq\; \frac{KC}{2} a_0(\mathfrak{m}) \Big(n \sum_{j=0}^{\mathfrak{q}-1} \frac{j^{\lambda_1 - 1}}{\mathfrak{m}^{\lambda_1/(\gamma+2)}} \;-\; \sum_{j=0}^{\mathfrak{q}-1} \frac{j^{\lambda_1}}{(\mathfrak{m} - \mathfrak{q} d_1)^{\lambda_2/(\gamma+2)}} \Big),$$

so

$$1 \;\geq\; \frac{KC}{2} \cdot \Big(n \cdot \sum_{j=0}^{\mathfrak{q}-1} \frac{j^{\lambda_1 - 1}}{\mathfrak{m}^{\lambda_1/(\gamma+2)}} \;-\; \sum_{j=0}^{\mathfrak{q}-1} \frac{j^{\lambda_1}}{(\mathfrak{m} - \mathfrak{q} d_1)^{\lambda_2/(\gamma+2)}} \Big). \tag{43}$$

For estimates of these sums, as functions of n, one has, as $\lambda_2 \geq \lambda_1 \geq 1$,

$$\begin{aligned}
\sum_{j=0}^{\mathfrak{q}-1} \frac{j^{\lambda_1 - 1}}{\mathfrak{m}^{\lambda_1/(\gamma+2)}} \;&=\; \frac{1}{\mathfrak{m}^{\lambda_1/(\gamma+2)}} \Big(\frac{\mathfrak{q}^{\lambda_1}}{\lambda_1} \;+\; \mathrm{o}\big(\mathfrak{q}^{\lambda_1}\big) \Big) \\
&=\; \frac{1}{\mathfrak{m}^{\lambda_1/(\gamma+2)}} \Big(\frac{\mathfrak{m}^{\lambda_1/(\gamma+2)}}{\lambda_1} \;+\; \mathrm{o}\big(\mathfrak{m}^{\lambda_1/(\gamma+2)}\big) \Big) \\
&=\; \frac{1}{\lambda_1} \;+\; \mathrm{o}\big(1\big). \qquad (44)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{\mathfrak{q}-1} \frac{j^{\lambda_1}}{(\mathfrak{m} - \mathfrak{q} d_1)^{\lambda_2/(\gamma+2)}} \;&=\; \frac{1}{(\mathfrak{m} - \mathfrak{q} d_1)^{\lambda_2/(\gamma+2)}} \Big(\frac{\mathfrak{q}^{\lambda_2}}{\lambda_2} \;+\; \mathrm{o}\big(\mathfrak{q}^{\lambda_2}\big) \Big) \\
&=\; \frac{1}{(\mathfrak{m} - \mathfrak{q} d)^{\lambda_2/(\gamma+2)}} \Big(\frac{\mathfrak{m}^{\lambda_2/(\gamma+2)}}{\lambda_2} \;+\; \mathrm{o}\big(\mathfrak{m}^{\lambda_2/(\gamma+2)}\big) \Big) \\
&=\; \frac{1}{\lambda_2} \;+\; \mathrm{o}\big(1\big). \qquad (45)
\end{aligned}$$

Substituting (44) and (45) into (43) gives

$$1 \;\geq\; KC \cdot \Big(n \cdot \Big(\frac{1}{\lambda_1} \;+\; \mathrm{o}\big(1\big) \Big) \;-\; \Big(\frac{1}{\lambda_2} \;+\; \mathrm{o}\big(1\big) \Big) \Big). \tag{46}$$

Taking the limit of both sides of (46) as $n \to \infty$ gives $1 \geq \infty$, a contradiction.

Item (26), with $j = 1$, gives

$$a_1(n) \;=\; \mathrm{O}\Big(\frac{a_0(n)}{n^{1/(\gamma+2)}} \Big),$$

that is

$$a^{(k)}(n) \;=\; \mathrm{O}\Big(\frac{a^{(k-1)}(n)}{n^{1/(\gamma+2)}} \Big).$$

To obtain the desired result

$$a^{(k+1)}(n) \;=\; \mathrm{O}\Big(\frac{a^{(k)}(n)}{n^{1/(\gamma+2)}}\Big)$$

simply copy the last part of Bell's proof of the original Bateman and Erdős conjecture in [3], pp. 151–152, replacing the exponent 1/2 with $1/(\gamma+2)$. □

5. (Counter)Examples

In this section examples are given to show that the results are, to a great extent, the best possible.

5.1. **Polynomially bounded is best possible for BE_k.** Let $f(n)$ be a nonnegative super polynomial function. Now it is shown that if $p(n) \le f(n)$ and satisfies BE_k then $a(n)$ need not be eventually strictly increasing; indeed it need not even be eventually nondecreasing.

First, select a sequence of positive integers $4 \le m_1, \ldots$ as follows. Having selected $m_1, \ldots, m_d$, find some m_{d+1} such that

$$[x^{m_{d+1}+1}](1-x^2)^{-k-1}(1-x^3)^{-k-1}\prod_{j=1}^{d}(1-x^{m_j})^{-\lfloor f(m_j)\rfloor} \;<\; \lfloor f(m_{d+1})\rfloor.$$

Let $p(n) = 0$ if $n \ge 4$ and $n \notin \{m_1, \ldots\}$. Let $p(1) = 0$, $p(2) = p(3) = k+1$ and $p(m_i) = \lfloor f(m_i)\rfloor$. Observe that with this set up, $a(m_d) \ge \lfloor f(m_d)\rfloor$. Notice however, that

$$\begin{aligned}
a(m_d+1) \;&=\; [x^{m_d+1}]\prod_{j=1}^{\infty}(1-x^j)^{-p(j)}\\
&=\; [x^{m_d+1}](1-x^2)^{-k-1}(1-x^3)^{-k-1}\prod_{j=1}^{d}(1-x^{m_j})^{-\lfloor f(m_j)\rfloor}\\
&=\; [x^{m_d+1}](1-x^2)^{-k-1}(1-x^3)^{-k-1}\\
&\qquad\cdot\prod_{j=1}^{d-1}(1-x^{m_j})^{-\lfloor f(m_j)\rfloor} + [x](1-x^2)^{-k-1}(1-x^3)^{-k-1}\\
&\qquad\cdot\prod_{j=1}^{d-1}(1-x^{m_j})^{-\lfloor f(m_j)\rfloor}\\
&=\; [x^{m_d+1}](1-x^2)^{-k-1}(1-x^3)^{-k-1}\prod_{j=1}^{d-1}(1-x^{m_j})^{-\lfloor f(m_j)\rfloor}\\
&<\; \lfloor f(m_d)\rfloor
\end{aligned}$$

$$< \quad a(m_d).$$

5.2. **Showing $\mathrm{O}(n^{-1/(\gamma+2)})$ is best possible.** Bateman and Erdős showed that their result

$$\frac{a^{(k+1)}a(n)}{a^{(k)}(n)} \;=\; \mathrm{O}\big(n^{-1/2}\big)$$

for partition identities with $p(n) \in \{0,1\}$ satisfying their condition P_k was best possible by examining the asymptotics for the classical partition function. In the same spirit it will be shown that Theorem 3.5

$$\frac{a^{(k+1)}(n)}{a^{(k)}(n)} \;=\; \mathrm{O}\Big(n^{-1/(\gamma+2)}\Big)$$

for partition identities with $p(n) = \mathrm{O}\big(n^{\gamma}\big)$ and satisfying BE_k is best possible.

To see this, for convenience we use real nonnegative values of $p(n)$. Let k be a nonnegative integer and take

$$p(n) = (k+1)\delta_{1,n} + n^{\gamma}.$$

Then

$$a^{(k+1)}(n) \;=\; \prod_{j=1}^{\infty}(1-x^j)^{-n^{\gamma}}.$$

Meinardus' theorem (Theorem 6.2 of Andrews [1]) will be needed. Using the notation of Andrews,

$$D(s) \;=\; \sum_{n=1}^{\infty}\frac{n^{\gamma}}{n^s} \;=\; \zeta(s-\gamma).$$

Classical results (see page 91 of Andrews [1] for results about $\zeta(s)$ and $\Gamma(s)$ and look at Riemann's formula for the Zeta function for negative values of $\mathrm{Re}(s)$ in Conway [8]) show that $D(s)$ satisfies the conditions of the theorem with a pole of order 1 at $s = \gamma + 1$. Thus according to his theorem,

$$a^{(k+1)}(n) \;\sim\; Cn^{\kappa}\exp(Bn^{\frac{\gamma+1}{\gamma+2}}),$$

where C and B are nonzero constants and

$$\kappa \;=\; \frac{\zeta(-\gamma) - 1 - \frac{\gamma+1}{2}}{\gamma+2}.$$

Similarly, using Meinardus' theorem again,

$$a^{(k)}(n) \;\sim\; C'n^{\kappa'}\exp(Bn^{\frac{\gamma+1}{\gamma+2}}),$$

where again C' is a nonzero constant, B is the same constant as appearing in the asymptotic expression for $a^{(k+1)}(n)$ and

$$\kappa' \ = \ \frac{\zeta(-\gamma) - \frac{\gamma+1}{2}}{\gamma + 2}.$$

Hence

$$a^{(k+1)}(n)/a^{(k)}(n) \ \sim \ C_0 n^{\kappa-\kappa'} \ = \ C_0 n^{-1/(\gamma+2)},$$

where $C_0 = C/C'$. This shows that our result is best possible.

6. Generalized Partition Identities

The notion of generalized partition identities was briefly introduced in Proposition 4.1, that is, where the $p(n)$ are allowed to take on nonnegative *real* values.[10] Essentially everything presented goes through in this setting. Our reason for restricting our attention to the case that the $p(n)$ have nonnegative integer values is simply that this is where the applications to combinatorics, additive number theory, and logical limit laws are to be found.

The modification of the previous proofs to apply to generalized partition identities is quite straightforward; however for the Bateman and Erdős results some clarification may be useful. For the general setting

[10] Allowing the $p(n)$ to be nonnegative real numbers is a rather obvious generalization, but it does not seem to have found much favor in the literature, perhaps because there is no clear connection to combinatorial systems.

Another direction to generalize partition identities that seems to have been more resonant is to keep the $p(n)$ as nonnegative integers but allow the exponents of x in the factors of the product to be nonnegative real numbers. One interprets this in number systems by saying that the indecomposables have been assigned real valued norms. If one does this only on the right side of a partition identity, the product side

$$\prod_{n=1}^{\infty} (1 - x^{r_n})^{-p(n)},$$

then one might still expect to obtain a power series on the left side. But this is not the direction pursued in abstract additive number theory (see Knopfmacher [10]). Instead one adds the assumption $\{m : r_m \leq n\}$ is finite for all n and expresses the product as a *generalized* power series

$$\mathbf{A}(x) \ := \ \sum_{n=0}^{\infty} a(n) x^{t_n},$$

where the values t_n come from the additive monoid generated by the r_n. With this approach the study of generalized additive number systems is essentially identical to the study of generalized multiplicative systems where one has generalized Dirichlet series $\sum a(n) \exp\big(-\lambda_n s\big)$ for the generating functions. It is not clear that one even has good and/or interesting analogues of RT_1 and kth difference functions in this setting.

the following notion of BE_κ is introduced, where κ is a real number:

$$\mathsf{BE}_\kappa \ : \ \begin{cases} \bullet\ \mathrm{rank}(p)\ >\ \kappa\,, \\ \bullet\ \text{for any (real-valued) } q(n) \text{ satisfying the two conditions} \\ \qquad \begin{cases} 0 \le q(n) \le p(n) \\ \sum_n \big(p(n) - q(n)\big)\ =\ \kappa\,. \end{cases} \\ \quad \text{one has} \\ \qquad \gcd\big\{n : q(n) > 0\big\}\ =\ 1\,. \end{cases}$$

Then a reduced generalized generating function $\mathbf{A}(x)$ will satisfy BE_κ iff the coefficients of the κ-fold difference function

$$(1-x)^\kappa \cdot \mathbf{A}(x)$$

are eventually positive. The conclusion for the conjecture can be extended to

$$a^{(\kappa+\beta)}(n)\ =\ \mathrm{O}\Big(\frac{a^{(\kappa)}(n)}{n^{\beta/(\gamma+2)}}\Big)$$

where $0 \le \beta \le 1$.

References

[1] George Andrews, *The Theory of Partitions.* Cambridge University Press, 1984.
[2] P. T. Bateman and P. Erdős, *Monotonicity of partition functions*, Mathematika **3** (1956), 1–14.
[3] Jason P. Bell, *A proof of a partition conjecture of Bateman and Erdős.* Journal of Number Theory **87** (2001), 144–153.
[4] Jason P. Bell, *Sufficient conditions for zero-one laws.* Trans. Amer. Math. Soc. **354** (2002), no. 2, 613–630.
[5] Jason P. Bell and Stanley N. Burris, *Asymptotics for logical limit laws: when the growth of the components is in an RT class.* Trans. Amer. Math. Soc. **355** (2003), 3777–3794.
[6] Jason P. Bell and Stanley N. Burris, *Partition identities I. Sandwich theorems and logical 0-1 laws.* (Preprint, 2004)
[7] Stanley N. Burris, *Number Theoretic Density and Logical Limit Laws.* Mathematical Surveys and Monographs, Vol. **86**, Amer. Math. Soc., 2001.
[8] John Conway, *Functions of one Complex Variable. Vol. I* Springer-Verlag, seventh corrected printing 1995.
[9] P. Flajolet and A.M. Odlyzko, *Singularity analysis of generating functions.* SIAM J. Discrete Math. **3** (1990), 216–240.
[10] J. Knopfmacher, *Abstract analytic number theory.* North-Holland Mathematical Library, Vol. 12. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, (1975).
[11] A.M. Odlyzko, *Asymptotic enumeration methods.* Handbook of Combinatorics Vol. II, R.L. Graham, M. Grötschel and L. Lovász, eds., 1063–1230, Elsevier, 1995.

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