Existentially Closed Structures and Boolean Products

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Foreword

Existentially closed structures are simple to define, yet they have proved to be rather elusive. They have interesting connections with decidability (quantifier elimination and word problems), but my own interest has been in using the tools of universal algebra to clarify and elaborate their nature. Following a conversation with Angus Macintyre in Bressanone, Italy, in the summer of 1975, it seemed that the Boolean product construction would prove to be most useful to study existentially closed structures in certain universal Horn classes, in particular in discriminator varieties.

These notes are an expansion of work with Heinrich Werner in 1975–1976, augmented by recent discoveries, especially those of Francoise Point and Herbert Riedel. Special thanks to the Waterloo Logic seminar participants, especially Ross Willard and David Clark, for making the writing of these notes such a pleasure. This research project has been generously supported by a grant from NSERC.

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Chapter 1

The Basic Theory of Existentially Closed Structures

1.1 Preliminaries

A *language* is a set $\mathcal{L}$ of symbols partitioned into two subsets $\mathcal{F}$ and $\mathcal{R}$, and to each symbol of $\mathcal{L}$ there is an associated non-negative integer called its *arity*. $\mathcal{R}$ is the set of *relation* symbols and $\mathcal{F}$ is the set of *function* symbols in $\mathcal{L}$.

An $\mathcal{L}$-structure $\mathbf{A}$ consists of a non-empty set A (the *universe* of $\mathbf{A}$) and an n-ary function $f : \mathrm{A}^n \to \mathrm{A}$ assigned to each n-ary function symbol $f \in \mathcal{F}$, and an n-ary relation $r \subseteq \mathrm{A}^n$ assigned to each n-ary relation symbol $r \in \mathcal{R}$. The functions f [relations r] so assigned are called the *fundamental operations* [*relations*] of $\mathbf{A}$. We usually write $\mathbf{A} = \langle \mathrm{A}, \mathcal{L} \rangle$. If $\mathcal{R} = \varnothing$ then $\mathcal{L}$ is a *language of algebras*, and $\mathbf{A}$ is called an *algebra*. If $\mathcal{F} = \varnothing$ then $\mathcal{L}$ is a *purely relational language*, and $\mathbf{A}$ is a *relational structure*. A 1-element structure is said to be *trivial*.

Let $\mathbf{A}$ be an $\mathcal{L}$-structure. Then $\mathcal{L}_{\mathrm{A}}$ is the language obtained by adding a *new* constant symbol for each element in A; and the expansion of $\mathbf{A}$ by these new constants is designated $\mathbf{A}_{\mathrm{A}}$.

If $\mathbf{A}$ is an $\mathcal{L}$-structure and $\mathrm{B} \subseteq \mathrm{A}$, then we say B is a *subuniverse* of $\mathbf{A}$ if B is closed under the fundamental operations of $\mathbf{A}$. If B is a nonempty subuniverse of $\mathbf{A}$, then the restriction to $\mathbf{B}$ of the fundamental operations and relations of $\mathbf{A}$ gives an $\mathcal{L}$-structure $\mathbf{B}$ with universe B. We say that $\mathbf{B}$ is a substructure of $\mathbf{A}$, written

$$\mathbf{B} \leq \mathbf{A}.$$

We assume the reader is familiar with basic universal algebra and model theory (as presented, for example, in [9], Chapters I, II and V, and in [10]), and we will use the following:

Notation	Meaning
$\mathbf{A} \cong \mathbf{B}$	$\mathbf{A}$ is isomorphic to $\mathbf{B}$
$\alpha : \mathbf{A} \to \mathbf{B}$	α is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$
$\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$	α is an embedding of $\mathbf{A}$ into $\mathbf{B}$
$\mathbf{A} \preceq \mathbf{B}$	$\mathbf{A}$ is an elementary substructure of $\mathbf{B}$
$\mathbf{A} \equiv \mathbf{B}$	$\mathbf{A}$ is elementarily equivalent to $\mathbf{B}$

Theorem 1.1.1 *Let $\mathbf{A}_n$ be a sequence of $\mathcal{L}$-structures with $\mathbf{A}_n \preceq \mathbf{A}_{n+1}$. Let $\mathbf{B} = \bigcup \mathbf{A}_n$. Then $\mathbf{A}_n \preceq \mathbf{B}$ for all n.*

Theorem 1.1.2 *Given $\mathcal{L}$-structures $\mathbf{A}, \mathbf{B}$ we have $\mathbf{A} \equiv \mathbf{B}$ iff $\mathbf{A}^I/\mathcal{U} \cong \mathbf{B}^I/\mathcal{U}$ holds for some choice of index set I and ultrafilter $\mathcal{U}$ over I.*

The *theory* $\mathrm{Th}(K)$ of a class K of $\mathcal{L}$-structures is the set of $\mathcal{L}$-sentences true of all members of K. K is *complete* if all members of K are elementarily equivalent, that is, have the same theory.

A theory is *universal, inductive*, etc., if the class of its models is universal, inductive, etc.

We will need the notions of *congruence, principal congruence, directly indecomposable algebra, subdirectly irreducible algebra,* and *simple algebra*, and the class operators

I	closure under isomorphism
H	closure under homomorphic images
S	closure under substructures
$\mathrm{S}^{(\preceq)}$	closure under elementary substructures
P	closure under direct products
P_r	closure under reduced products
P_u	closure under ultraproducts
P_s	closure under subdirect products
V	closure under H, S and P
E	closure under I, $\mathrm{S}^{(\preceq)}$ and P_u

We say that

an $\mathcal{L}$-formula φ is	if it is of the form
an identity	$p = q$
a quasi-identity	$\bigwedge_i p_i = q_i \to p = q$
open	*quantifier free*
basic	$\pm$ *atomic*
$\forall$ [or universal]	$\forall$ *open*
$\exists$ [or existential]	$\exists$ *open*
primitive	$\exists \bigwedge \pm$ *atomic*
positive primitive	$\exists \bigwedge$ *atomic*
$\forall\exists$	$\forall\exists$ *open*
basic Horn	$\neg \bigwedge$ *atomic* or $\bigwedge$ *atomic* $\to$ *atomic*
universal Horn	$\forall \bigwedge$ *basic Horn*
Horn	$\vec{\mathbf{Q}} \bigwedge$ *basic Horn*

In the above table, and in the text, we use the convention that $\pm\varphi$ means 'φ or $\neg\varphi$'.

The *universal theory* $\mathrm{Th}_\forall(K)$ of a class K is the set of universal sentences in $\mathrm{Th}(K)$, the $\forall\exists$ theory $\mathrm{Th}_{\forall\exists}(K)$ of K is the set of $\forall\exists$ sentences in $\mathrm{Th}(K)$, etc.

The *diagram* $\mathrm{Diag}(\mathbf{A})$ of a structure $\mathbf{A}$ is the set of basic sentences satisfied by $\mathbf{A}_\mathrm{A}$.

A class K of $\mathcal{L}$-structures is an *elementary class* if it consists of *all* $\mathcal{L}$-structures satisfying some set Σ of $\mathcal{L}$-formulas (we say Σ *defines* K). An elementary class K is

	if it can be defined by sentences of the form
a variety	*identities*
a quasi-variety	*quasi-identities*
universal	$\forall$
existential	$\exists$
$\forall\exists$	$\forall\exists$
universal Horn	*universal Horn*
Horn	*Horn*

The basic results concerning closure and preservation are the following, where K is a class of $\mathcal{L}$-structures.

Theorem 1.1.3 *K is an elementary class iff K is closed under* I, $\mathrm{S}^{(\preceq)}$ *and* P_u. *The smallest elementary class containing K is* $\mathrm{E}(K) = \mathrm{IS}^{(\preceq)}\mathrm{P}_u(K)$.

Theorem 1.1.4 *K is a universal class iff K is closed under* I, S *and* P_u. *The smallest universal class containing K is* $\mathrm{ISP}_u(K)$.

Theorem 1.1.5 *An elementary class K is an $\forall\exists$ class iff K is inductive, that is, K is closed under unions of chains.*

Theorem 1.1.6 *K is a Horn class iff K is closed under* I, $\mathrm{S}^{(\preceq)}$ *and* P_r.

Theorem 1.1.7 *K is universal Horn iff K is closed under* I, S *and* P_r. *The smallest universal Horn class containing K is* $\mathrm{ISP}_r(K)[= \mathrm{ISPP}_u(K)]$.

Theorem 1.1.8 *For K a class of algebras, K is a quasi-variety iff it is a universal Horn class and contains a trivial algebra. The smallest quasi-variety containing K is* $\mathrm{ISP}_r(K_+)$, *where K_+ denotes K augmented by a trivial algebra if one is not already present in K.*

Theorem 1.1.9 *For K a class of algebras, K is a variety iff K is closed under* H, S *and* P. *The smallest variety containing K is* $\mathrm{V}(K) = \mathrm{HSP}(K)$.

Remark If K is an elementary class, then

(i) the smallest universal class containing K is $\mathrm{S}(K)$,

(ii) the smallest universal Horn class containing K is $\mathrm{ISP}(K)$.

1.2 What is an existentially closed structure?

To say that an algebra $\mathbf{A}$ is existentially closed in a class K of algebras simply means that for each finite system of equations and negated equations

$$\begin{array}{c|c} p_1(\vec{a},\vec{x}) = q_1(\vec{a},\vec{x}) & r_1(\vec{a},\vec{x}) \neq s_1(\vec{a},\vec{x}) \\ \vdots & \vdots \\ p_m(\vec{a},\vec{x}) = q_m(\vec{a},\vec{x}) & r_n(\vec{a},\vec{x}) \neq s_n(\vec{a},\vec{x}) \end{array}$$

with parameters $\vec{a}$ from $\mathbf{A}$, if we can solve this system in an extension $\mathbf{B} \in K$ of $\mathbf{A}$, then we can solve it in $\mathbf{A}$. Let us generalize this concept to arbitrary structures, and introduce related definitions.

Definition 1.2.1 Given a first-order language $\mathcal{L}$ let

E_n = the set of *existential* L-formulas $\varphi(v_1, \ldots, v_n)$

$\mathrm{E} = \bigcup\limits_{n<\omega} \mathrm{E}_n$

P_n = the set of *primitive* formulas in E_n

$\mathrm{P} = \bigcup\limits_{n<\omega} \mathrm{P}_n$

P^+_n = the set of *primitive positive* formulas in E_n

$\mathrm{P}^+ = \bigcup\limits_{n<\omega} \mathrm{P}^+_n$.

For $\mathcal{L}$-structures $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A} \leq \mathbf{B}$ we write

$$\mathbf{A} \leq_{ec} \mathbf{B}$$

(read: $\mathbf{A}$ is *existentially closed in* $\mathbf{B}$) if for every $\varphi(\vec{v})$ in E and $\vec{a} \in$ A we have

$$\mathbf{B} \models \varphi(\vec{a}) \Rightarrow \mathbf{A} \models \varphi(\vec{a}).$$

[It is easy to see that $\mathbf{A} \leq_{ec} \mathbf{B}$ holds iff for every $\varphi(\vec{v})$ in P and $\vec{a} \in$ A we have the above implication.]

We say

$$\mathbf{A} \underset{ec}{\hookrightarrow} \mathbf{B}$$

(read: *embeddings of* $\mathbf{A}$ *in* $\mathbf{B}$ *are ec*) if for every embedding $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ we have $\alpha(\mathbf{A}) \leq_{ec} \mathbf{B}$. For $\mathbf{A}$ an $\mathcal{L}$-structure and K a class of $\mathcal{L}$-structures we write

$$\mathbf{A} \underset{ec}{\hookrightarrow} K$$

(read: *embeddings of* $\mathbf{A}$ *in* K *are ec*) if $\mathbf{A} \underset{ec}{\hookrightarrow} \mathbf{B}$ holds for all $\mathbf{B} \in K$. $\mathbf{A}$ is *existentially closed in* K if $\mathbf{A} \in K$ and $\mathbf{A} \underset{ec}{\hookrightarrow} K$. We write K^{ec} for the class of structures which are existentially closed in K.

Remark If K is closed under isomorphism then, for $\mathbf{A} \in K$, we have $\mathbf{A} \in K^{ec}$ iff $\mathbf{A} \leq \mathbf{B} \Rightarrow \mathbf{A} \leq_{ec} \mathbf{B}$ for $\mathbf{B} \in K$.

Remark We obtain the notion of an *algebraically closed* structure if we replace existential formulas by positive existential formulas in the above definitions.

The following test for $\leq_{ec}$ will be employed in §1.7.

Proposition 1.2.2 *For* $\mathbf{A} \leq \mathbf{B}$ *we have* $\mathbf{A} \leq_{ec} \mathbf{B}$ *iff for some* $\mathbf{C}$,

$$\mathbf{A} \preceq \mathbf{C} \quad \text{and} \quad \mathbf{B} \leq \mathbf{C}.$$

Proof ($\Leftarrow$) Let $\mathbf{B} \models \varphi(\vec{a})$ where $\varphi \in \mathrm{E}$ and $\vec{a} \in \mathrm{A}$. Then

$$\begin{array}{rl|l} \mathbf{B} \models \varphi(\vec{a}) & \Rightarrow \ \mathbf{C} \models \varphi(\vec{a}) & \varphi \in \mathrm{E} \\ & \Rightarrow \ \mathbf{A} \models \varphi(\vec{a}) & \mathbf{A} \preceq \mathbf{C}, \end{array}$$

thus $\mathbf{A} \leq_{ec} \mathbf{B}$.

($\Rightarrow$) It suffices to show that $\mathrm{Th}(\mathbf{A}_{\mathrm{A}}) \cup \mathrm{Diag}(\mathbf{B})$ is consistent. Let $\mathrm{F}(\vec{a}, \vec{b})$ be a finite fragment of $\mathrm{Diag}(\mathbf{B})$ with $\vec{a} \in \mathrm{A}$, $\vec{b} \in \mathrm{B} - \mathrm{A}$. Then $\mathbf{B} \models \exists \vec{v} \bigwedge \mathrm{F}(\vec{a}, \vec{v})$, so $\mathbf{A} \models \exists \vec{v} \bigwedge \mathrm{F}(\vec{a}, \vec{v})$ as $\mathbf{A} \leq_{ec} \mathbf{B}$. Thus $\mathrm{Th}(\mathbf{A}_{\mathrm{A}}) \cup \mathrm{F}(\vec{a}, \vec{b})$ is consistent; hence $\mathrm{Th}(\mathbf{A}_{\mathrm{A}}) \cup \mathrm{Diag}(\mathbf{B})$ is consistent. $\square$

W. Scott introduced algebraically closed structures in the setting of the class of groups, and in this case the notions of algebraically closed and existentially closed are the same. He proved that every group could be embedded into an algebraically closed group. His proof was presented in the general setting of inductive theories by Eklof and Sabbagh.

Proposition 1.2.3 *Let* K *be an inductive class of* $\mathcal{L}$*-structures. Then every member of* K *can be embedded in a member of* K^{ec}*, that is,* $K \subseteq \mathrm{IS}(K^{ec})$*, and thus*

$$\mathrm{IS}(K) = \mathrm{IS}(K^{ec}).$$

Proof Let $\mathbf{A} \in K$. We can assume that K is closed under isomorphism.

Claim: There is an $\mathbf{A}^* \in K$ with $\mathbf{A} \leq \mathbf{A}^*$ such that for any $\varphi \in \mathrm{E}$ and any $\vec{a} \in \mathrm{A}$ we have

$$\mathbf{A}^* \leq \mathbf{C} \in K \quad \text{and} \quad \mathbf{C} \models \varphi(\vec{a}) \quad \text{implies} \quad \mathbf{A}^* \models \varphi(\vec{a}).$$

To see this, define, for each $\mathbf{B} \in K$ with $\mathbf{A} \leq \mathbf{B}$, the set

$$\Sigma_{\mathbf{B}} = \{\varphi \in \mathrm{E}_{\mathrm{A}} : \mathbf{B} \not\models \varphi, \text{ but some extension of } \mathbf{B} \text{ in } K \text{ satisfies } \varphi\},$$

where E_{A} is the set of existential sentences with parameters from $\mathbf{A}$. Now it is easy to check that for $\mathbf{B}_1, \mathbf{B}_2 \in K$

$$\mathbf{A} \leq \mathbf{B}_1 \leq \mathbf{B}_2 \ \Rightarrow \ \Sigma_{\mathbf{B}_1} \supseteq \Sigma_{\mathbf{B}_2};$$

and that if $\mathbf{A} \leq \mathbf{B}_1 \in K$ with $\Sigma_{\mathbf{B}_1} \neq \varnothing$ then we can choose $\mathbf{B}_2 \in K$ with $\mathbf{B}_1 \leq \mathbf{B}_2$ and $\Sigma_{\mathbf{B}_1} \neq \Sigma_{\mathbf{B}_2}$.

Let $\mathrm{E}_{\mathrm{A}} = \{\varphi_\kappa\}_{\kappa<\nu}$, where ν is a limit ordinal. Define $\mathbf{A}_0 \leq \mathbf{A}_1 \leq \cdots$ as follows:

$$\mathbf{A}_0 = \mathbf{A}$$

$$\mathbf{A}_\lambda = \bigcup_{\kappa<\lambda} \mathbf{A}_\kappa \text{ if } \lambda \text{ is a limit ordinal } \leq \nu,$$

and for $\kappa + 1 < \nu$ let

$$\mathbf{A}_{\kappa+1} = \mathbf{A}_\kappa \text{ if } \varphi_\kappa \notin \Sigma_{\mathbf{A}_\kappa}$$

$\mathbf{A}_{\kappa+1} = \mathbf{B}$ where $\mathbf{B}$ is any structure such that $\mathbf{A}_\kappa \leq \mathbf{B} \in K$ and $\varphi_\kappa \notin \Sigma_{\mathbf{B}}$, if $\varphi_\kappa \in \Sigma_{\mathbf{A}_\kappa}$.

Then $\mathbf{A}^* = \mathbf{A}_\nu$ is easily seen to satisfy the claim.

Now that the claim is established we define a sequence $\{\mathbf{B}_n\}_{n<\omega}$ by

$$\mathbf{B}_0 = \mathbf{A}^*$$

$$\mathbf{B}_{n+1} = \mathbf{B}_n^*$$

and let $\mathbf{B} = \bigcup_{n<\omega} \mathbf{B}_n^*$. Then $\mathbf{A} \leq \mathbf{B} \in K^{ec}$, for if $\mathbf{B} \leq \mathbf{C} \in K$ and $\mathbf{C} \models \varphi(\vec{b})$, $\varphi \in \mathrm{E}$, $\vec{b} \in \mathrm{B}$ then $\vec{b} \in \mathrm{B}_n$ for some $n < \omega$, and as $\mathbf{B}_n \leq \mathbf{B}_n^* \leq \mathbf{C}$ we have $\mathbf{B}_n^* \models \varphi(\vec{b})$, so $\mathbf{B} \models \varphi(\vec{b})$. $\square$

Proposition 1.2.4 *If K is an elementary class then K^{ec} is closed under elementary substructures, that is,*

$$\mathrm{S}^{(\preceq)}(K^{ec}) \subseteq K^{ec}.$$

Proof Given $\mathbf{A} \preceq \mathbf{B} \in K^{ec}$, suppose $\mathbf{A} \leq \mathbf{C} \models \varphi(\vec{a})$ for some $\mathbf{C} \in K$, $\vec{a} \in \mathrm{A}$ and $\varphi \in \mathrm{E}$. We can assume that $\mathrm{B} \cap \mathrm{C} = \mathrm{A}$. Note that $\mathrm{Th}(K) \cup \mathrm{Diag}(\mathbf{B}) \cup \mathrm{Diag}(\mathbf{C})$ is consistent, for otherwise some finite fragment of the diagram of $\mathbf{B}$ would be inconsistent with the diagram of $\mathbf{C}$, and hence there would be an existential $\mathcal{L}_{\mathrm{A}}$-sentence ψ with $\mathbf{B} \models \psi$ but $\mathbf{C} \not\models \psi$. This is impossible as $\mathbf{A} \models \psi$ as well. So let $\mathbf{D} \in K$ be such that there are embeddings $\beta : \mathbf{B} \hookrightarrow \mathbf{D}$ and $\gamma : \mathbf{C} \hookrightarrow \mathbf{D}$ which agree on $\mathbf{A}$. Then

$$\begin{array}{rl|l} \mathbf{C} \models \varphi(\vec{a}) & \Rightarrow\ \mathbf{D} \models \varphi(\gamma\vec{a}) & \varphi \in \mathrm{E} \\ & \Rightarrow\ \mathbf{B} \models \varphi(\vec{a}) & \mathbf{B} \in K^{ec} \\ & \Rightarrow\ \mathbf{A} \models \varphi(\vec{a}) & \mathbf{A} \preceq \mathbf{B}. \quad \square \end{array}$$

Corollary 1.2.5 *If K is an elementary class, then:*

$$K^{ec} \textit{ is an elementary class iff } \mathrm{P}_u(K^{ec}) \subseteq K^{ec}.$$

Proof Use 1.1.3. $\square$

Lemma 1.2.6 *For any K,*

$$K^{ec} \;=\; K \cap \mathrm{IS}(K)^{ec}.$$

Proof Certainly ($\supseteq$) holds. For ($\subseteq$) let $\mathbf{A} \in K^{ec}$, and suppose $\mathbf{A} \leq \mathbf{B} \in \mathrm{IS}(K)$ with $\mathbf{B} \models \varphi(\vec{a})$, $\varphi \in \mathrm{E}$, $\vec{a} \in \mathrm{A}$. We can assume K is closed under isomorphism. Choose $\mathbf{C} \in K$ such that $\mathbf{B} \leq \mathbf{C}$. Then

$$\begin{array}{rl|l} \mathbf{B} \models \varphi(\vec{a}) & \Rightarrow\; \mathbf{C} \models \varphi(\vec{a}) & \varphi \text{ is } \exists \\ & \Rightarrow\; \mathbf{A} \models \varphi(\vec{a}) & \mathbf{A} \in K^{ec}. \end{array}$$

Thus $\mathbf{A} \in \mathrm{IS}(K)^{ec}$. $\square$

1.3 K is elementary implies K^{ec} has an infinitary axiomatization

This section is based on the 1972 work of Simmons. The following lemma shows that if an existential formula is not satisfied in an existentially closed structure by certain elements, then there is a global reason.

Lemma 1.3.1 *Let K be an elementary class and let $\mathbf{A} \in K^{ec}$. Then for $\varphi(\vec{x}) \in \mathrm{P}_n$ and for $\vec{a} \in A$ the following are equivalent:*

(1) $\mathbf{A} \not\models \varphi(\vec{a})$

(2) *there is an $\eta(\vec{x}) \in \mathrm{P}_n$ such that*

  (a) $K \models \eta \to \neg\,\varphi$, *and*

  (b) $\mathbf{A} \models \eta(\vec{a})$.

Proof (1) $\Rightarrow$ (2): As $\mathbf{A} \in K^{ec}$ it follows that for all $\mathbf{B} \in K$ with $\mathbf{A} \leq \mathbf{B}$ we have $\mathbf{B} \not\models \varphi(\vec{a})$. As K is elementary it follows that $\mathrm{Th}(K) \cup \mathrm{Diag}(\mathbf{A}) \cup \{\varphi(\vec{a})\}$ is inconsistent, so there must be a finite fragment $\mathrm{F} \subseteq \mathrm{Diag}(\mathbf{A})$ such that $\mathrm{Th}(K) \cup \mathrm{F} \cup \{\varphi\}$ is inconsistent, and then

$$K \models \bigwedge \mathrm{F} \;\to\; \neg\,\varphi(\vec{a}).$$

Let $\bigwedge \mathrm{F}$ be $\psi(\vec{a}, \vec{b})$, where no b_j is equal to any a_i. Then there is an $\eta \in \mathrm{P}_n$, namely $\exists \vec{y} \psi(\vec{x}, \vec{y})$, such that

$$\begin{array}{rl} K & \models \eta(\vec{x}) \to \neg\,\varphi(\vec{x}) \\ \mathbf{A} & \models \eta(\vec{a}). \end{array}$$

(2) $\Rightarrow$ (1): Since $\mathbf{A} \models \eta(\vec{a})$, by (2a) we have $\mathbf{A} \not\models \varphi(\vec{a})$. $\square$

This easily leads to the desired axiomatization of K^{ec}.

Theorem 1.3.2 *Let K be an elementary class. Then K^{ec} is axiomatized by*

$$\mathrm{Th}(K) \cup \{\varphi(\vec{x}) \vee \Phi(\vec{x}) : \varphi \in \mathrm{P}_n\} \tag{$*$}$$

where

$$\Phi(\vec{x}) \;=\; \bigvee\!\!\!\bigvee \{\eta(\vec{x}) \in \mathrm{P}_n : K \models \eta \to \neg\,\varphi\}.$$

Proof Certainly K^{ec} satisfies $(*)$ by 1.3.1. If now $\mathbf{A}$ satisfies $(*)$ then $\mathbf{A} \in K$. Also for any $\mathbf{B} \in K$ with $\mathbf{A} \leq \mathbf{B}$ and for any $\varphi \in \mathrm{P}_n$ and $\vec{a} \in \mathrm{A}$ we have

$$\begin{array}{rl|l} \mathbf{A} \not\models \neg\,\varphi(\vec{a}) & \Rightarrow\; \mathbf{A} \models \Phi(\vec{a}) & \mathbf{A} \models (*) \\ & \Rightarrow\; \mathbf{B} \models \Phi(\vec{a}) & \Phi \text{ is } \bigvee\!\!\!\bigvee \text{ existential.} \\ & \Rightarrow\; \mathbf{B} \models \neg\,\varphi(\vec{a}) & K \models \Phi \to \neg\,\varphi \end{array}$$

Thus $\mathbf{A} \models (*)$ implies $\mathbf{A} \in K^{ec}$. $\square$

Remark Note that if K is elementary then, with φ and Φ as above, we have

$$K^{ec} \;\models\; \neg\,\varphi(\vec{x}) \leftrightarrow \Phi.$$

Corollary 1.3.3 *Let K be an elementary class. Then K^{ec} is an elementary class implies*

(i) *every universal formula is equivalent modulo K^{ec} to an existential formula;*

(ii) *every formula is equivalent modulo K^{ec} to an existential [or universal] formula.*

Proof Since $K^{ec} \models \neg\,\varphi(\vec{x}) \leftrightarrow \Phi(\vec{x})$, by compactness each Φ is equivalent modulo K^{ec} to a disjunction of only finitely many of its disjuncts. Thus, modulo K^{ec}, each negated primitive formula is equivalent to an existential formula, and (i) follows from this. Clearly (i) implies (ii). $\square$

1.4 Model complete classes

Model complete theories were introduced by Robinson as a natural model-theoretic generalization of the theory of algebraically closed fields.

Definition 1.4.1 A class of $\mathcal{L}$-structures is *model complete* if, whenever $\mathbf{A}, \mathbf{B} \in K$ and $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ we have $\alpha\mathbf{A} \preceq \mathbf{B}$. A theory is model complete if the class of its models is model complete.

Theorem 1.4.2 *An elementary model complete class K is an inductive class, and thus an $\forall\exists$ class.*

Proof Let $\mathbf{A}_1 \leq \mathbf{A}_2 \leq \cdots$ with $\mathbf{A}_n \in K$. Then $\mathbf{A}_1 \preceq \mathbf{A}_2 \preceq \cdots$. Let $\mathbf{B} = \bigcup \mathbf{A}_n$. By 1.1.1 we have $\mathbf{A} \preceq \mathbf{B}$, so $\mathbf{B} \in K$. Thus K is an inductive class, so by 1.1.5 K is an $\forall\exists$ class. $\square$

The next result shows how model completeness is related to existentially closed structures.

Theorem 1.4.3 *Let K be an elementary class of $\mathcal{L}$-structures. Then the following are equivalent:*

(i) *for all* $\mathbf{A}, \mathbf{B} \in K,\ \mathbf{A} \leq \mathbf{B} \Rightarrow \mathbf{A} \leq_{ec} \mathbf{B}$

(ii) *K is model complete*

(iii) $K^{ec} = K$

(iv) *every universal $\mathcal{L}$-formula is equivalent modulo K to an existential formula*

(v) *every $\mathcal{L}$-formula is equivalent modulo K to an existential* [*universal*] *formula.*

Proof Certainly (ii) $\Rightarrow$ (iii), and by 1.3.3 (iii) $\Rightarrow$ (iv). As noted in §1.3, (iv) $\Rightarrow$ (v). It is easy to see that (v) $\Rightarrow$ (ii), for if we have $\mathbf{A}, \mathbf{B} \in K,\ \mathbf{A} \models \varphi(\vec{a})$, and $\mathbf{A} \leq \mathbf{B}$, then we can choose $\psi(\vec{x}) \in \mathrm{E}$ such that $K \models \varphi \leftrightarrow \psi$. Certainly $\mathbf{B} \models \psi(\vec{a})$, so $\mathbf{B} \models \varphi(\vec{a})$. Thus (ii)–(v) are equivalent. Condition (i) is just a rephrasing of (iii). $\square$

An elementary model complete class need not be complete; consider, for example, the class of all algebraically closed fields.

Definition 1.4.4 A class K has the *joint embedding property* (JEP) if for any $\mathbf{A}, \mathbf{B} \in K$ there is a $\mathbf{C} \in K$ such that both $\mathbf{A}$ and $\mathbf{B}$ can be embedded in $\mathbf{C}$.

Theorem 1.4.5 *Let K be an elementary model complete class. Then K is complete iff K has the* JEP.

Proof Let $\mathbf{A}, \mathbf{B} \in K$.

($\Rightarrow$) As K is complete we know $\mathbf{A} \equiv \mathbf{B}$, and as K is elementary there is a $\mathbf{C} \in K$ with $\mathbf{A}, \mathbf{B} \hookrightarrow \mathbf{C}$ (we can even choose $\mathbf{C}$ to be an ultrapower of $\mathbf{A}$, by 1.1.2). Consequently K has the JEP.

($\Leftarrow$) Use the JEP to choose $\mathbf{C} \in K$ such that $\mathbf{A}, \mathbf{B} \hookrightarrow \mathbf{C}$. As K is model complete, $\mathbf{A} \equiv \mathbf{C} \equiv \mathbf{B}$. Thus $\mathbf{A} \equiv \mathbf{B}$, so K is a complete class. $\square$

1.5 Prime models

If we consider algebraically closed fields of characteristic 0 we know they all have a common subfield which is algebraically closed, the algebraic closure of the rationals.

Definition 1.5.1 A structure $\mathbf{A}$ is a *prime model* of a model complete class K of $\mathcal{L}$-structures if $\mathbf{A} \in K$ and $\mathbf{A}$ can be embedded in every member of K.

This notion is needed for Robinson's well known test for completeness. (His conditions for completeness are sufficient, but certainly not necessary.)

Theorem 1.5.2 *Let K be a class of $\mathcal{L}$-structures. If*

(i) *K is model complete, and*

(ii) *K has a prime model*

then K is complete.

Proof Let $\mathbf{P}$ be a prime model for K. Then for $\mathbf{A}, \mathbf{B} \in K$ we have $\mathbf{P}$ embedded in both $\mathbf{A}$ and $\mathbf{B}$, so by model completeness $\mathbf{A} \equiv \mathbf{P} \equiv \mathbf{B}$. Hence K is complete. $\square$

1.6 Quantifier elimination

The elimination of quantifiers has long been a central method to prove positive decidability results. One can think of proving a class to be model complete as an intermediate step on the road to proving it has elimination of quantifiers. However the reverse direction can be rather illuminating. Schröder [21] struggled in his first two volumes of the *Algebra der Logik* (1890–1895) to demonstrate that power set Boolean algebras admit elimination of quantifiers. However it is easy to see that this class of Boolean algebras (in the usual language) is not model complete, and hence does not have elimination of quantifiers. Skolem [24] added 'numerical' predicates in 1919 and proved that elimination of quantifiers was then possible (indeed his proof works for atomic Boolean algebras).

Definition 1.6.1 A class K of $\mathcal{L}$-structures has *quantifier elimination* if every $\mathcal{L}$-formula is equivalent to an open formula modulo K.

Remark A class K of $\mathcal{L}$-structures has quantifier elimination iff every existential [primitive] $\mathcal{L}$-formula is equivalent to an open formula modulo K.

If a class K has quantifier elimination then it is certainly model complete (since embeddings preserve *open* sentences). The converse is in general false, and the rest of this section is devoted to finding how much information we need to add to model completeness to obtain quantifier elimination.

Definition 1.6.2 For Σ a set of $\mathcal{L}$-sentences we say two $\mathcal{L}$-structures $\mathbf{B}$ and $\mathbf{C}$ *agree* on Σ if for each $\sigma \in \Sigma$ we have

$$\mathbf{B} \models \sigma \Leftrightarrow \mathbf{C} \models \sigma.$$

Lemma 1.6.3 *Let Ω be the set of open sentences of a language $\mathcal{L}$ with at least one constant symbol, and let φ be any $\mathcal{L}$-sentence. Then, for K an elementary class of $\mathcal{L}$-structures, φ is equivalent modulo K to an open sentence iff for all $\mathbf{B}, \mathbf{C} \in K$*

$$\mathbf{B} \textit{ and } \mathbf{C} \textit{ agree on } \Omega \Rightarrow \mathbf{B} \textit{ and } \mathbf{C} \textit{ agree on } \varphi.$$

Proof ($\Rightarrow$) This is obvious since φ is equivalent to an open sentence modulo K.

($\Leftarrow$) Suppose φ is not equivalent to an open sentence modulo K. Then, given $\omega_1, \ldots, \omega_n \in \Omega$ we claim that there are $\mathbf{B}, \mathbf{C} \in K$ with $\mathbf{B}, \mathbf{C}$ agreeing on $\omega_1, \ldots, \omega_n$ but not on φ.

Otherwise $\mathrm{Th}(K) \cup \{\omega_1', \ldots, \omega_n'\} \models \pm\varphi$ for each choice of $\omega_i' = \pm\omega_i$, and this gives $K \models \omega_1' \wedge \cdots \wedge \omega_n' \to \pm\varphi$ for each choice of ω_i'. Propositional logic then shows that

$$K \models \varphi \leftrightarrow \bigvee\{\omega_1' \wedge \cdots \wedge \omega_n' : K \models \omega_1' \wedge \cdots \wedge \omega_n' \to \varphi\}.$$

Thus φ is equivalent to an open sentence modulo K, which is impossible.

Then, by König's Lemma and compactness, there are $\mathbf{B}, \mathbf{C} \in K$ such that $\mathbf{B}, \mathbf{C}$ agree on Ω but not on φ. $\square$

Definition 1.6.4 A class K of $\mathcal{L}$-structures has the *amalgamation property* (AP) if for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{I}(K)$ with $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ there is a $\mathbf{D} \in K$ with embeddings $\beta : \mathbf{B} \hookrightarrow \mathbf{D}$, $\gamma : \mathbf{C} \hookrightarrow \mathbf{D}$ such that β and γ agree on $\mathbf{A}$.

Remark For K a universal class, K has the AP iff for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in K$ with $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ and $\mathrm{A} = \mathrm{B} \cap \mathrm{C}$, the set of sentences $\mathrm{Th}(K) \cup \mathrm{Diag}(\mathbf{B}) \cup \mathrm{Diag}(\mathbf{C})$ is consistent.

Theorem 1.6.5 *Let K be an elementary class. Then K has quantifier elimination iff*

(i) *K is model complete, and*

(ii) $\mathrm{S}(K)$ *has the amalgamation property.*

Proof ($\Rightarrow$) Since open sentences are preserved by embeddings, it follows that K has elimination of quantifiers implies (i) holds. To show that $\mathrm{S}(K)$ has the amalgamation property it suffices to show the following (since every member of $\mathrm{S}(K)$ can be embedded in a member of K):

Claim: For $\mathbf{A} \in \mathrm{S}(K)$ and for $\mathbf{A} \leq \mathbf{B}, \mathbf{C} \in K$ with $\mathrm{A} = \mathrm{B} \cap \mathrm{C}$, we have $\mathrm{Th}(K) \cup \mathrm{Diag}(\mathbf{B}) \cup \mathrm{Diag}(\mathbf{C})$ is consistent.

To see this let $\mathrm{F}_1(\vec{a}, \vec{b})$ be a finite subset of $\mathrm{Diag}(\mathbf{B})$, and let $\mathrm{F}_2(\vec{a}, \vec{c})$ be a finite subset of $\mathrm{Diag}(\mathbf{C})$, where $\vec{a} \in \mathrm{A}$, $\vec{b} \in \mathrm{B} - \mathrm{A}$, and $\vec{c} \in \mathrm{C} - \mathrm{A}$. For $i = 1, 2$ let

$$\varphi_i(\vec{x}) = \exists \vec{y} \bigwedge \mathrm{F}_i(\vec{x}, \vec{y}),$$

and choose open formulas $\omega_i(\vec{x})$ such that

$$K \models \varphi_i(\vec{x}) \leftrightarrow \omega_i(\vec{x}).$$

Now

$$\begin{array}{rl|l} \mathbf{B} \models \varphi_1(\vec{a}) & \Rightarrow \mathbf{B} \models \omega_1(\vec{a}) & \mathbf{B} \in K \\ & \Rightarrow \mathbf{C} \models \omega_1(\vec{a}) & \omega_1 \text{ is open} \\ & \Rightarrow \mathbf{C} \models \varphi_1(\vec{a}) & \mathbf{C} \in K, \end{array}$$

so, by symmetry,

$$\mathbf{B}, \mathbf{C} \models \varphi_1(\vec{a}), \varphi_2(\vec{a}).$$

Thus $\mathrm{Th}(K) \cup \mathrm{F}_1(\vec{a}, \vec{b}) \cup \mathrm{F}_2(\vec{a}, \vec{c})$ is consistent, and the claim follows by compactness.

($\Leftarrow$) Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula (without loss of generality we assume at least one free variable occurs in φ), and let $a_1, \ldots, a_n$ be new constant symbols. Let $K[\vec{a}]$ denote the members of K expanded (in all possible ways) by new constants $a_1, \ldots, a_n$. Let $\mathbf{B}, \mathbf{C} \in K[\vec{a}]$ be such that they agree on the open sentences of the language $\mathcal{L} \cup \{a_1, \ldots, a_n\}$. Let $\mathbf{A}$ be the substructure of $\mathbf{B}$ generated by $\vec{a}$. Then $\mathbf{A}$ is also the substructure of $\mathbf{C}$ generated by $\vec{a}$. By the amalgamation property there is a $\mathbf{D} \in K$ with embeddings $\beta : \mathbf{B} \hookrightarrow \mathbf{D}$ and $\gamma : \mathbf{C} \hookrightarrow \mathbf{D}$ which agree on $\mathbf{A}$. By the model completeness of K we see that $K[\vec{a}]$ is model complete, so

$$\mathbf{B} \models \varphi(\vec{a}) \Leftrightarrow \mathbf{D} \models \varphi(\beta\vec{a}) \Leftrightarrow \mathbf{C} \models \varphi(\vec{a});$$

hence, by 1.6.3, there is an open sentence $\omega(\vec{a})$ such that

$$K[\vec{a}] \models \varphi(\vec{a}) \leftrightarrow \omega(\vec{a}).$$

Thus

$$K \models \varphi(\vec{x}) \leftrightarrow \omega(\vec{x}),$$

proving that K admits quantifier elimination. $\square$

For the special case of a universal class the notions of model complete and quantifier elimination coincide.

Corollary 1.6.6 *Let K be a universal class. Then K has quantifier elimination iff K is model complete.*

Proof Since $\mathrm{S}(K) = K$, by 1.6.5 it suffices to show that if K is a model complete universal class, then it has the amalgamation property. So let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in K$ with $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$. Then, by the model completeness of K we have $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A} \preceq \mathbf{C}$; thus $\mathbf{B}_{\mathrm{A}} \equiv \mathbf{C}_{\mathrm{A}}$, so by 1.1.2 there is a $\mathbf{D} \in K$ and embeddings $\beta : \mathbf{B} \hookrightarrow \mathbf{D}$ and $\gamma : \mathbf{C} \hookrightarrow \mathbf{D}$ which agree on $\mathbf{A}$. $\square$

1.7 Model companions

Definition 1.7.1 Two classes K_1, K_2 of $\mathcal{L}$-structures are *mutually model consistent* if each member of K_1 can be embedded in a member of K_2, and vice-versa, that is, $\mathrm{IS}(K_1) = \mathrm{IS}(K_2)$.

For an example of two mutually model consistent classes, let K_1 be the class of fields and let K_2 be the class of algebraically closed fields.

Definition 1.7.2 An elementary class K_1 is the *model companion* of an elementary class K if

(i) K_1 is model complete, and

(ii) K_1 and K are mutually model consistent.

Proposition 1.7.3 *An elementary class K has at most one model companion.*

Proof Suppose K_1 and K_2 are both model companions of K. Then K_1 and K_2 are mutually model consistent. For any $\mathbf{A} \in K_1$ we can find $\mathbf{A}_n \in K_1$, $\mathbf{B}_n \in K_2$ with $\mathbf{A} = \mathbf{A}_1$ and

$$\mathbf{A}_1 \leq \mathbf{B}_1 \leq \mathbf{A}_2 \leq \mathbf{B}_2 \leq \cdots .$$

Let $\mathbf{C} = \bigcup_{n<\omega} \mathbf{A}_n = \bigcup_{n<\omega} \mathbf{B}_n$. Then $\mathbf{C} \in K_1 \cap K_2$ as both K_1 and K_2 are elementary model complete classes, and hence both are inductive classes. Since K_1 is model complete, we have $\mathbf{A} \preceq \mathbf{C}$, so $\mathbf{A} \in K_2$. Thus $K_1 \subseteq K_2$, and by symmetry $K_2 \subseteq K_1$. $\square$

Definition 1.7.4 Let K be an elementary class. Then K^{mc} denotes the model companion of K, provided it exists.

The following table gives four of the best known examples of model companions.

Examples	
K	K^{mc}
sets	infinite sets
linear orders	dense linear orders without endpoints
Boolean algebras	atomless Boolean algebras
fields	algebraically closed fields

Lemma 1.7.5 *Let K_1 and K_2 be mutually model consistent classes of $\mathcal{L}$-structures. Then K_1 has the* JEP *iff K_2 has the* JEP.

Proof By symmetry it suffices to show that K_1 has the JEP implies K_2 has the JEP. So given that K_1 has the JEP, let $\mathbf{A}, \mathbf{B} \in K_2$. By mutual model consistency we can choose $\mathbf{A}^*, \mathbf{B}^* \in K_1$ such that there are embeddings $\widehat{\alpha} : \mathbf{A} \hookrightarrow \mathbf{A}^*$, $\widehat{\beta} : \mathbf{B} \hookrightarrow \mathbf{B}^*$. As K_1 has the JEP, choose $\mathbf{C}^* \in K_1$ such that there are embeddings $\alpha^* : \mathbf{A}^* \hookrightarrow \mathbf{C}^*$ and $\beta^* : \mathbf{B}^* \hookrightarrow \mathbf{C}^*$. Again by mutual model consistency there is a $\mathbf{D} \in K_2$ such that there is an embedding $\gamma^* : \mathbf{C}^* \hookrightarrow \mathbf{D}$. Composing maps we have embeddings $\alpha : \mathbf{A} \hookrightarrow \mathbf{D}$ and $\beta : \mathbf{B} \hookrightarrow \mathbf{D}$, so K_2 has the JEP. $\square$

Theorem 1.7.6 *Let K be an elementary class with a model companion K^{mc}. Then K^{mc} is complete iff K has the* JEP.

Proof By 1.4.5 we know that K^{mc} is complete iff K^{mc} has the JEP, and then by 1.7.5 K^{mc} has the JEP iff K has the JEP. $\square$

Proposition 1.7.7 *Let K be an elementary class, and let $\overline{K}$ be an elementary model complete class. Then*

$$\overline{K} = K^{mc} \quad \textit{iff} \quad \mathrm{S}(\overline{K}) = \mathrm{S}(K).$$

Proof This is an easy consequence of the definition and uniqueness of the model companion. $\square$

In a sense the study of model companions reduces to the study of model companions of universal classes.

Theorem 1.7.8 *For K an elementary class, the following are equivalent:*

(i) *K has a model companion*

(ii) $\mathrm{S}(K)$ *has a model companion*

(iii) $\mathrm{S}(K)^{ec}$ *is an elementary class*

(iv) $\mathrm{P}_u(\mathrm{S}(K)^{ec}) \subseteq \mathrm{S}(K)^{ec}$.

If any of (i)–(iv) *hold, then*

$$K^{mc} = \mathrm{S}(K)^{mc} = \mathrm{S}(K)^{ec}.$$

Proof (i) $\Leftrightarrow$ (ii): As K and $\mathrm{S}(K)$ are mutually model consistent, a model companion for one is also a model companion for the other.

(ii) $\Rightarrow$ (iii): Let $\mathbf{A} \in \mathrm{S}(K)^{mc}$, $\mathbf{B} \in \mathrm{S}(K)$ with $\mathbf{A} \leq \mathbf{B} \models \varphi(\vec{a})$, where $\varphi \in \mathrm{E}$ and $\vec{a} \in \mathrm{A}$. Choose $\mathbf{C} \in \mathrm{S}(K)^{mc}$ with $\mathbf{B} \leq \mathbf{C}$. Then $\mathbf{C} \models \varphi(\vec{a})$, and as $\mathbf{A} \preceq \mathbf{C}$, $\mathbf{A} \models \varphi(\vec{a})$. Thus $\mathbf{A} \in \mathrm{S}(K)^{ec}$, so $\mathrm{S}(K)^{mc} \subset \mathrm{S}(K)^{ec}$.

For the opposite inclusion, let $\mathbf{A} \in \mathrm{S}(K)^{ec}$. Then choose $\mathbf{B}_1 \in \mathrm{S}(K)^{mc}$ such that $\mathbf{A} \leq \mathbf{B}_1$. Since $\mathbf{B}_1 \in \mathrm{S}(K)$, by 1.2.3 there is $\mathbf{A}_2 \in \mathrm{S}(K)^{ec}$ with $\mathbf{B}_1 \leq \mathbf{A}_2$. Iterating the procedure we can find sequences $\mathbf{A}_n \in K^{ec}$ and $\mathbf{B}_n \in K^{mc}$, where $\mathbf{A} = \mathbf{A}_1$, such that

$$\mathbf{A}_1 \leq \mathbf{B}_1 \leq \mathbf{A}_2 \leq \mathbf{B}_2 \leq \cdots .$$

Let $\mathbf{C} = \bigcup \mathbf{A}_n = \bigcup \mathbf{B}_n$. Then $\mathbf{C} \in \mathrm{S}(K)^{mc}$ as $\mathrm{S}(K)^{mc}$ is inductive. As $\mathbf{A} \leq_{ec} \mathbf{C}$ it follows that $\mathbf{A} \models \mathrm{Th}_{\forall\exists}(\mathbf{C})$, so $\mathbf{A} \in \mathrm{S}(K)^{mc}$; hence $\mathrm{S}(K)^{ec} \subseteq \mathrm{S}(K)^{mc}$.

(iii) $\Rightarrow$ (ii): As $(\mathrm{S}(K)^{ec})^{ec} = \mathrm{S}(K)^{ec}$, by 1.4.3 we see that (iii) implies $\mathrm{S}(K)^{ec}$ is a model complete class. By 1.2.3 $\mathrm{S}(K)$ and $\mathrm{S}(K)^{ec}$ are mutually model consistent, so $\mathrm{S}(K)^{ec}$ is the model companion of $\mathrm{S}(K)$.

(iii) $\Leftrightarrow$ (iv): This follows from 1.2.5.

If (i)–(iv) hold, then the proof of (i) $\Leftrightarrow$ (ii) shows $K^{mc} = \mathrm{S}(K)^{mc}$, and the proof of (iii) $\Rightarrow$ (ii) shows $\mathrm{S}(K)^{ec} = \mathrm{S}(K)^{mc}$. □

The following looks at the important cases where the model companion of K is contained in K.

Theorem 1.7.9 *For K an elementary class, the following are equivalent:*

(i) *K has a model companion K^{mc} and $K^{mc} \subseteq K$*

(ii) *K has a model companion K^{mc} and $K^{mc} = K^{ec}$*

(iii) *K^{ec} is an elementary class which is mutually model consistent with K.*

Proof (i) $\Rightarrow$ (ii): If K has a model companion then from 1.7.8 $K^{mc} = \mathrm{S}(K)^{ec}$, so $K^{mc} \subseteq K$ gives $\mathrm{S}(K)^{ec} \subseteq K$. Now $K^{ec} = K \cap \mathrm{S}(K)^{ec}$ holds by 1.2.6; hence $K^{ec} = \mathrm{S}(K)^{ec}$. Thus $K^{mc} = \mathrm{S}(K)^{ec} = K^{ec}$.

(ii) $\Rightarrow$ (i): (Obvious.)

(ii) $\Rightarrow$ (iii): This follows from the definition of K^{mc}.

(iii) $\Rightarrow$ (ii): If K^{ec} is an elementary class, then by 1.4.3 and the fact that $(K^{ec})^{ec} = K^{ec}$ we see that K^{ec} is model complete. Thus it must be the model companion of K as it is also mutually model consistent with K. □

1.8 Model companions of ∀∃ classes

Theorem 1.8.1 *Let K be an ∀∃ class of $\mathcal{L}$-structures. Then the following are equivalent:*

(i) *K has a model companion*

(ii) *K^{ec} is an elementary class*

(iii) $\mathrm{P}_u(K^{ec}) \subseteq K^{ec}$.

If K does have a model companion, then $K^{mc} = K^{ec}$.

Proof (i) $\Rightarrow$ (ii): If K has a model companion K^{mc}, then for $\mathbf{A} \in K^{mc}$ we can apply mutual model consistency to find sequences $\mathbf{A}_n \in K^{mc}$, $\mathbf{B}_n \in K$ with $\mathbf{A} = \mathbf{A}_1$ such that

$$\mathbf{A}_1 \leq \mathbf{B}_1 \leq \mathbf{A}_2 \leq \mathbf{B}_2 \leq \cdots .$$

Let $\mathbf{C} = \bigcup \mathbf{A}_n = \bigcup \mathbf{B}_n$. As both K and K^{mc} are $\forall\exists$ classes we have $\mathbf{C} \in K \cap K^{mc}$. As K^{mc} is model complete and $\mathbf{A} \leq \mathbf{C}$ we have $\mathbf{A} \preceq \mathbf{C} \in K$. Thus $K^{mc} \subseteq K$. By 1.7.9 we see that (ii) follows.

(ii) $\Rightarrow$ (i): Since K^{ec} is assumed to be elementary, and in general we have $(K^{ec})^{ec} = K^{ec}$, it follows from 1.4.3 that K^{ec} is a model complete class. From 1.2.3, K^{ec} is mutually model consistent with K, so K^{ec} must be the model companion of K.

(ii) $\Leftrightarrow$ (iii): This was proved in 1.2.5.

If K^{mc} exists then from the proof of (ii) $\Rightarrow$ (i) we have $K^{mc} = K^{ec}$. $\square$

Example: The class $\mathcal{G}$ of groups does not have a model companion.

To see this let $\mathbf{G}$ be an existentially closed group, and let $\mathcal{U}$ be a nonprincipal ultrafilter over ω. For $n < \omega$ choose $a_n, b_n \in G$ such that $\text{order}(a_n) = n + 1$ and $\text{order}(b_n) = n + 2$. [Note that every finite group can be embedded into $\mathbf{G}$ since groups have the joint embedding property.] Then in $\mathbf{G}/\mathcal{U}$ both $\vec{a}/\mathcal{U}$ and $\vec{b}/\mathcal{U}$ have infinite order. Consequently in some extension of $\mathbf{G}/\mathcal{U}$ these two elements are conjugate. However they are not conjugate in $\mathbf{G}/\mathcal{U}$, and since being conjugate is expressed by an existential formula, it follows that $\mathbf{G}/\mathcal{U}$ is not an existentially closed group. Thus groups do not have a model companion.

1.9 Model completions

Algebraically closed fields have model-theoretic properties beyond being the model companion of fields. One property isolated by Robinson was the fact that if $\mathbf{F}$ is a field and if $\mathbf{A}_1, \mathbf{A}_2$ are algebraically closed fields with $\mathbf{F} \leq \mathbf{A}_1, \mathbf{A}_2$, then $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy the same sentences with parameters from $\mathbf{F}$.

Definition 1.9.1 An elementary class K_1 is a *model completion* of an elementary class K_2 if

(i) $K_1 \subseteq K_2$,

(ii) K_1 and K_2 are mutually model consistent, and

(iii) for $\mathbf{A} \in K_2$ and $\mathbf{B}, \mathbf{C} \in K_1$ with $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ we have $\mathbf{B}, \mathbf{C}$ satisfying the same sentences with parameters from $\mathbf{A}$.

Theorem 1.9.2 *Let K be an elementary class. Then K has a model completion iff*

(i) *K has a model companion K^{mc},*

(ii) $K^{mc} \subseteq K$, *and*

(iii) K *has the amalgamation property.*

If K *has a model completion, it is unique—indeed it is* K^{mc}.

Proof ($\Rightarrow$) Suppose K^{MC} is a model completion of K. Then (by part (ii), (iii) of the definition of a model completion) the class K^{MC} must be the model companion of K, so (i) holds. As $K^{MC} \subseteq K$, (ii) holds. Now given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in K$ with $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ and $\mathrm{A} = \mathrm{B} \cap \mathrm{C}$, we can find $\widehat{\mathbf{B}}, \widehat{\mathbf{C}} \in K^{MC}$ with $\mathbf{B} \leq \widehat{\mathbf{B}},\ \mathbf{C} \leq \widehat{\mathbf{C}}$ and $\widehat{\mathrm{B}} \cap \widehat{\mathrm{C}} = \mathrm{A}$. As $\widehat{\mathbf{B}}, \widehat{\mathbf{C}}$ satisfy the same sentences with parameters from $\mathbf{A}$, it follows that $\mathrm{Th}(K) \cup \mathrm{Diag}(\widehat{\mathbf{B}}) \cup \mathrm{Diag}(\widehat{\mathbf{C}})$ is consistent, so $\mathrm{Th}(K) \cup \mathrm{Diag}(\mathbf{B}) \cup \mathrm{Diag}(\mathbf{C})$ is consistent. This gives the amalgamation property for K, so (iii) holds.

($\Leftarrow$) Given that (i)–(iii) hold, we want to show that K^{mc} is a model completion of K. So let $\mathbf{A} \in K$ and let $\mathbf{B}, \mathbf{C} \in K^{mc}$ with $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$. By the AP we can find $\mathbf{D} \in K$ such that there are embeddings $\beta : \mathbf{B} \hookrightarrow \mathbf{D}$ and $\gamma : \mathbf{C} \hookrightarrow \mathbf{D}$ which agree on $\mathbf{A}$. We can assume that $\mathbf{D} \in K^{mc}$; hence α, β are elementary embeddings, and thus $\mathbf{B}, \mathbf{C}$ satisfy the same sentences with parameters from $\mathbf{A}$. Consequently K^{mc} is a model completion of K.

The uniqueness of the model completion follows from the uniqueness of the model companion. $\square$

Corollary 1.9.3 *If* K *is a universal class with a model completion* K^{mc}, *then* K^{mc} *has quantifier elimination.*

Proof This follows from 1.9.2 and 1.6.5. $\square$

1.10 Extracting information about K^{ec} from classes which are mutually model consistent with K

We will be interested in looking at the existentially closed members of a universal class, and making good use of choosing a suitable generating subclass. The following result will be important in the next chapter.

Proposition 1.10.1 *If* $\mathrm{IS}(K) = \mathrm{IS}(K_1)$ *and* $\mathbf{A} \in K$, *then*

$$\mathbf{A} \in K^{ec} \ \Leftrightarrow\ \mathbf{A} \underset{ec}{\hookrightarrow} K_1.$$

Proof ($\Rightarrow$) Let $\alpha : \mathbf{A} \to \mathbf{B}$ be an embedding with $\mathbf{B} \in K_1$, and suppose $\mathbf{B} \models \varphi(\vec{a})$ for some $\varphi(\vec{v}) \in \mathrm{E},\ \vec{a} \in \mathrm{A}$. Then choose $\mathbf{C} \in \mathrm{B}$ with $\mathbf{B} \leq \mathbf{C}$. We have

$$\begin{array}{rl|l} \mathbf{B} \leq \mathbf{C} \ \Rightarrow & \mathbf{C} \models \varphi(\vec{a}) & \varphi \in \mathrm{E} \\ \Rightarrow & \alpha(\mathbf{A}) \models \varphi(\vec{a}) & \mathbf{A} \in K^{ec}, \end{array}$$

so $\mathbf{A} \underset{ec}{\hookrightarrow} \mathbf{B}$; hence $\mathbf{A} \underset{ec}{\hookrightarrow} K_1$.

($\Leftarrow$) Let $\mathbf{A} \leq \mathbf{B} \models \varphi(\vec{a})$ where $\mathbf{B} \in K,\ \varphi \in \mathrm{E},\ \vec{a} \in \mathrm{A}$. Then choose $\mathbf{C} \in K_1$ with $\mathbf{B} \leq \mathbf{C}$. We have

$$\begin{array}{rl|l} \mathbf{B} \leq \mathbf{C} & \Rightarrow\ \mathbf{C} \models \varphi(\vec{a}) & \varphi \in E \\ & \Rightarrow\ \mathbf{A} \models \varphi(\vec{a}) & \mathbf{A} \underset{ec}{\hookrightarrow} K_1, \end{array}$$

so $\mathbf{A} \leq_{ec} \mathbf{B}$. Thus $\mathbf{A} \in K^{ec}$. $\square$

In Theorem 1.3.2 we gave an infinitary axiomatization of K^{ec} for K an elementary class. Now we introduce a slight refinement of that axiomatization which will be crucial to our development.

ANOTHER INFINITARY AXIOMATIZATION OF K^{ec}

Proposition 1.10.2 *Let K be an elementary class, and let K_1 be any class such that $\mathrm{S}(K) = \mathrm{IS}(K_1)$, that is, such that K and K_1 are mutually model consistent. For $\varphi(\vec{v}) \in \mathrm{P}_n$ let*

$$\Phi(\vec{v})\ =:\ \bigvee\!\!\!\bigvee \{\eta(\vec{v}) : \eta(\vec{v}) \in \mathrm{P}_n,\ K \models \eta(\vec{v}) \rightarrow \neg\, \varphi(\vec{v})\},$$

and let $\Phi^*(\vec{v})$ be a formula of the form $\bigvee\!\!\!\bigvee \varphi_i(\vec{v}),\ \varphi_i(\vec{v}) \in \mathrm{P}_n$, such that

$$K_1\ \models\ \Phi^*(\vec{v}) \leftrightarrow \Phi(\vec{v}).$$

Then K^{ec} is axiomatized by

$$\mathrm{Th}(K) \cup \Sigma$$

where

$$\Sigma\ =\ \{\varphi(\vec{v}) \vee \Phi^*(\vec{v}) : \varphi(\vec{v}) \in \mathrm{P}_n,\ n < \omega\}.$$

Proof First observe that for $\varphi(\vec{v}) \in \mathrm{P}_n$, we have

$$K\ \models\ \neg\, \varphi(\vec{v}) \vee \neg\, \Phi(\vec{v}),$$

thus

$$\mathrm{S}(K)\ \models\ \neg\, \varphi(\vec{v}) \vee \neg\, \Phi(\vec{v}).$$

Without loss of generality we can assume K_1 is closed under isomorphism.

Let $\mathbf{A} \in K,\ \mathbf{A} \models \Sigma$. If $\mathbf{A} \leq \mathbf{B} \in K$ and $\mathbf{B} \models \varphi(\vec{a})$, where $\varphi(\vec{v}) \in \mathrm{P}_n,\ \vec{a} \in \mathrm{A}$, choose $\mathbf{C} \in K_1,\ \mathbf{D} \in K$ such that

$$\mathbf{A} \leq \mathbf{B} \leq \mathbf{C} \leq \mathbf{D}.$$

Then

$$\begin{array}{rl|l} \mathbf{B} \models \varphi(\vec{a}) & \Rightarrow\ \mathbf{D} \models \varphi(\vec{a}) & \varphi \text{ is } \exists \\ & \Rightarrow\ \mathbf{D} \models \neg\, \Phi(\vec{a}) & \text{as } \mathbf{D} \in K \\ & \Rightarrow\ \mathbf{C} \models \neg\, \Phi(\vec{a}) & \neg\, \Phi \text{ is } \forall \\ & \Rightarrow\ \mathbf{C} \models \neg\, \Phi^*(\vec{a}) & \mathbf{C} \in K_1 \\ & \Rightarrow\ \mathbf{A} \models \neg\, \Phi^*(\vec{a}) & \neg\, \Phi^* \text{ is } \forall \\ & \Rightarrow\ \mathbf{A} \models \varphi(\vec{a}) & \mathbf{A} \models \Sigma. \end{array}$$

Thus $\mathbf{A} \in K^{ec}$.

For the converse, let $\mathbf{A} \in K^{ec}$, and suppose $\varphi(\vec{v}) \in \mathrm{P}_n$ and $\mathbf{A} \models \neg\,\varphi(\vec{a})$. Then $\mathrm{Th}(K) \cup \mathrm{Diag}(\mathbf{A}) \cup \{\varphi(\vec{a})\}$ is inconsistent as $\mathbf{A} \in K^{ec}$, so we have $K \models \eta(\vec{v}) \rightarrow \neg\,\varphi(\vec{v})$ for some $\eta \in \mathrm{P}_n$ with $\mathbf{A} \models \eta(\vec{a})$. Let $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$, $\mathbf{B} \in K_1$, $\mathbf{C} \in K$. Then

$$\begin{array}{rl|l}
\mathbf{A} \models \eta(\vec{a}) & \Rightarrow\ \mathbf{B} \models \eta(\vec{a}) & \text{as } \eta \text{ is } \exists \\
& \Rightarrow\ \mathbf{B} \models \Phi(\vec{a}) & \text{def of } \Phi \\
& \Rightarrow\ \mathbf{B} \models \Phi^*(\vec{a}) & \mathbf{B} \in K_1 \\
& \Rightarrow\ \mathbf{C} \models \Phi^*(\vec{a}) & \Phi^* \text{ is } \exists \\
& \Rightarrow\ \mathbf{A} \models \Phi^*(\vec{a}) & \mathbf{A} \in K^{ec}.
\end{array}$$

Therefore $\mathbf{A} \models \Sigma$. $\square$

Lemma 1.10.3 *For any* K,

$$K \textit{ is elementary } + \mathrm{S}(K)^{mc} \textit{ exists} \Rightarrow K^{ec} \textit{ is elementary model complete.}$$

Proof From 1.2.6 we have $K^{ec} = K \cap \mathrm{S}(K)^{ec}$, so it follows that K^{ec} is elementary; and since K^{ec} is contained in the model complete class $\mathrm{S}(K)^{mc}$ it follows that K^{ec} is model complete. $\square$

Remark For $K = \{\text{ordered sets with endpoints}\}$ we have $K^{ec} = \emptyset$.

In the following we will be primarily interested in showing that if K has a mutually model consistent class K_1 with few existential formulas, then K has a model companion. However, we do obtain numerous results about K^{ec} enroute.

Definition 1.10.4 For K a class of structures define the relation $\sim_K$, *equivalent modulo* K, on formulas by

$$\varphi_1 \sim_K \varphi_2 \Leftrightarrow K \models \varphi_1 \leftrightarrow \varphi_2.$$

Next define $\epsilon_n(K)$, $n < \omega$, to be the number of equivalence classes of $E_n/\neg\,thickspace \sim_K$, and define $\delta_n(K)$, $n < \omega$, to be the number of equivalence classes of formulas $\varphi(v_1, \ldots, v_n)$ modulo K.

K is said to have *few existential formulas* if $\epsilon_n(K) < \infty$ for all $n < \omega$; and

K has *few formulas* if $\delta_n(K) < \infty$ for all $n < \omega$.

K is *existentially complete* if $\epsilon_0(K) = 2$, *complete* if $\delta_0(K) = 2$.

Proposition 1.10.5 *Let* $\mathrm{IS}(K) = \mathrm{IS}(K_1)$. *Then*

$$K^{ec} \models \mathrm{Th}_{\forall\exists}(K_1),$$

and thus

$$\epsilon_n(K^{ec}) \leq \epsilon_n(K_1), \quad n < \omega.$$

Proof We can assume K is closed under isomorphism. Let $\mathbf{A} \in K^{ec}$ and let

$$\varphi = \forall\,\vec{v}\,\exists\,\vec{u}\omega(\vec{u},\vec{v}) \in \mathrm{Th}_{\forall\exists}(K_1),$$

where $\omega(\vec{u},\vec{v})$ is open. Choose $\mathbf{B} \in K_1,\ \mathbf{C} \in K$ such that $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$. Then for $\vec{a} \in \mathrm{A}$ we have $\mathbf{B} \models \exists\,\vec{u}\omega(\vec{u},\vec{a})$. Now

$$\begin{array}{rl|l} \mathbf{B} \models \exists\,\vec{u}\omega(\vec{u},\vec{a}) & \Rightarrow\ \mathbf{C} \models \exists\,\vec{u}\omega(\vec{u},\vec{a}) & \exists\,\vec{u}\omega \in \mathrm{E} \\ & \Rightarrow\ \mathbf{A} \models \exists\,\vec{u}\omega(\vec{u},\vec{a}) & \mathbf{A} \in K^{ec}. \end{array}$$

Thus $\mathbf{A} \models \varphi$. Consequently $K^{ec} \models \mathrm{Th}_{\forall\exists}(K_1)$.

Now if $\varphi_1(\vec{v})$ and $\varphi_2(\vec{v})$ are two existential formulas then

$$\begin{aligned} K_1 \models \varphi_1(\vec{v}) \leftrightarrow \varphi_2(\vec{v}) \ &\Rightarrow\ K_1 \models \forall\vec{v}\,(\varphi_1(\vec{v}) \leftrightarrow \varphi_2(\vec{v})) \\ &\Rightarrow\ K^{ec} \models \forall\vec{v}\,(\varphi_1(\vec{v}) \leftrightarrow \varphi_2(\vec{v})) \end{aligned}$$

since $\forall\vec{v}\,(\varphi_1(\vec{v}) \leftrightarrow \varphi_2(\vec{v}))$ is expressible as an $\forall\exists$-sentence. $\square$

Definition 1.10.6 A class K of $\mathcal{L}$-structures is

(i) $\aleph_0$-*categorical* if it has (up to isomorphism) exactly one member of size $\aleph_0$,

(ii) $\leq\aleph_0$-*categorical* if it has (up to isomorphism) exactly one member of size $\leq\aleph_0$, and

(iii) *almost* $\leq\aleph_0$-*categorical* if it has (up to isomorphism) only finitely many members of size $\leq\aleph_0$.

Theorem 1.10.7 *Let K be a complete elementary class with a countable language. Then the following are equivalent:*

(a) *K is $\aleph_0$-categorical*

(b) *K has no finite models and few formulas.*

Proof (See [4].) $\square$

Corollary 1.10.8 *Let K be an elementary class with a countable language. Then the following are equivalent:*

(a) *K is almost $\leq\aleph_0$-categorical*

(b) *K has few formulas.*

Proof By 1.10.7 it is clear that (a) implies (b). To see the converse note that each member $\mathbf{A}$ of K of size $\leq\aleph_0$ generates an elementary class $\mathrm{E}(\mathbf{A})$ which is $\leq\aleph_0$-categorical. Thus for members $\mathbf{A}$ and $\mathbf{B}$ of K of size $\leq\aleph_0$ we have $\mathbf{A}\cong\mathbf{B}$ iff $\mathrm{Th}(\mathbf{A})=\mathrm{Th}(\mathbf{B})$; and as $\delta_0(K)$ is finite, it follows that there are only finitely many possible members of size $\leq\aleph_0$.

Our basic examples of elementary classes K with few existential formulas are:

(1) K has finitely many finite structures

(2) K is complete, $\aleph_0$-categorical and has a countable language.

Remark If we define the notion of 'few primitive formulas' in the obvious way, then it is easy to see that a class K has few existential formulas iff it has few primitive formulas.

$\boxed{\text{FEW EXISTENTIAL FORMULAS LEAD TO MODEL COMPLETENESS}}$

Proposition 1.10.9 *Let K be an elementary class and let K_1 be any class such that* $\mathrm{S}(K)=\mathrm{IS}(K_1)$. *Then*
K_1 has few existential formulas implies

(1) *K^{ec} is elementary model complete and has few formulas—indeed $\delta_n(K^{ec})\leq\epsilon_n(K_1)$ for $n<\omega$;*

(1′) *if the language is countable, then K^{ec} is almost $\leq\aleph_0$-categorical, and*

(2) *if $K^{ec}\neq\emptyset$ and the language is countable, then the following are equivalent:*

   (a) *K^{ec} is complete*
   (b) *K^{ec} is $\leq\aleph_0$-categorical and complete,*
   (c) *K^{ec} has a prime model.*

Proof

(1) Let $K_0=\mathrm{S}(K)$. Then K_0^{ec} is elementary since we can choose each Φ^* in 1.10.2 to be a finite disjunction of primitive formulas (as K_1 has few existential formulas). By 1.7.8 $\mathrm{S}(K)^{mc}$ exists, so by Lemma 1.10.3, K^{ec} is elementary model complete.

Since K^{ec} is elementary model complete, it follows that every formula is equivalent to an existential formula, modulo K^{ec}. So it suffices to show $\epsilon_n(K^{ec})\leq\epsilon_n(K_1),\ n<\omega$. This holds by 1.10.5.

(1′) Combine (1) above with 1.10.8.

(2) (a) $\Rightarrow$ (b) by 1.10.7 since K^{ec} has few formulas. (b) $\Rightarrow$ (c) as K^{ec} is closed under elementary substructures. (c) $\Rightarrow$ (a) since K^{ec} is model complete; and thus every member of K^{ec} is elementarily equivalent to the prime model. $\Box$

FEW EXISTENTIAL FORMULAS LEAD TO: (1) MODEL COMPANIONS, AND (2) MANY THINGS ARE EQUIVALENT TO $\aleph_0$-CATEGORICAL MODEL COMPANIONS

If an elementary class K has no infinite members, then K has a model companion consisting of the maximal members of K (under embedding). In the next theorem we bypass this rather trivial case.

Theorem 1.10.10 *Let K be an elementary class with an infinite member, and let K_1 be any class such that $\mathrm{S}(K) = \mathrm{S}(K_1)$. Then K_1 has few existential formulas implies*

(1) *K^{mc} exists and has few formulas, indeed $\delta_n(K^{mc}) \leq \epsilon_n(K_1)$, $n < \omega$, and*

(1′) *if the language is countable, then K^{mc} is almost $\leq\aleph_0$-categorical, and*

(2) *if the language is countable, the following are equivalent:*

- (a) *K^{mc} is complete*
- (b) *K^{mc} is $\leq\aleph_0$-categorical and complete*
- (c) *K^{mc} has a prime model*
- (d) *K has the JEP.*

Proof By 1.7.8 K has a model companion K^{mc} iff $\mathrm{S}(K)^{mc}$ exists iff $\mathrm{S}(K)^{ec}$ is elementary; and, if so, $K^{mc} = \mathrm{S}(K)^{ec}$. Thus we can apply Proposition 1.10.9, replacing K by $\mathrm{S}(K)$, to obtain (1) and (1′), and also (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) in (2). Since (a) implies K^{mc} has the JEP, we have (a) $\Rightarrow$ (d). As K^{mc} is model complete, (d) $\Rightarrow$ (a). $\Box$

Proposition 1.10.11 *Let K be an elementary class and let K_1 be any class such that $\mathrm{S}(K) = \mathrm{IS}(K_1)$. Then if K_1 is*

(i) *existentially complete,*

(ii) *nontrivial,*

(iii) *has few existential formulas,*

then either $K^{ec} = \emptyset$ or the following hold:

(a) *K^{ec} is elementary model complete*

(b) *K^{ec} has few formulas*

(c) *K^{ec} is complete*

(d) *K^{ec} has no finite members or, up to isomorphism, contains a single finite structure*

(e) *if the language is countable and K^{ec} has no finite members, then K^{ec} is $\aleph_0$-categorical*

(f) *if the language is countable, then K^{ec} has a prime model.*

Proof We have already proved that (a) and (b) hold. From $\delta_0 K^{ec} \leq \epsilon_0(K_1) = 2$ it follows that (c) holds. (d) is a consequence of (c). (e) and (f) follow from part (2) or Proposition 1.10.9, in view of (c). □

CONDITIONS FOR OBTAINING $\aleph_0$-CATEGORICAL MODEL COMPANIONS

Theorem 1.10.12 *Let K be an elementary class with an infinite model, and let K_1 be any class such that $\mathrm{S}(K) = \mathrm{IS}(K_1)$. If K_1 is*

(i) *existentially complete and*

(ii) *has few existential formulas*

then

(a) *K^{mc} exists*

(b) *K^{mc} has few formulas*

(c) *K^{mc} is complete*

(d) *K^{mc} has no finite members*

(e) *if the language is countable, then K^{mc} is $\aleph_0$-categorical*

(f) *if the language is countable, then K^{mc} has a prime model*

(g) *K has the JEP.*

Proof As in Proposition 1.10.11—the only new observation needed being for (e), namely that if K has an infinite member, then some member of K^{mc} is infinite, so by (c) all members are infinite. (g) follows from (c) as (c) implies K^{mc} has the JEP; hence K has the JEP. □

Corollary 1.10.13 *Let K be a complete $\aleph_0$-categorical elementary class with a countable language. Then K^{mc} exists and is complete $\aleph_0$-categorical.*

Proof Let $K_1 = K$ and use Theorem 1.10.12, parts (c), (e). $\square$

DECIDABLE MODEL COMPANIONS

Definition 1.10.14 Given a class of structures K, let $\pi_n(K)$ be the number of inequivalent primitive formulas modulo K in P_n.

Lemma 1.10.15 *For any class of structures K*

$$\pi_n(K) \leq \epsilon_n(K) \leq 2^{\pi_n(K)}.$$

Furthermore, if $\pi_n(K)$ is recursive [primitive recursive] and $\mathrm{Th}_{\forall\exists}(K)$ is decidable [primitive recursive], then $\epsilon_n(K)$ is recursive [primitive recursive].

Proof We can effectively find a finite $\mathrm{R}_n \subseteq \mathrm{P}_n$ such that every $\varphi \in \mathrm{P}_n$ is equivalent to some $\rho \in \mathrm{R}_n$ modulo K. To do this we simply start listing the members of P_n and use the fact that $\mathrm{Th}_{\forall\exists}(K)$ is decidable to test if any given two are equivalent modulo K. As soon as we find $\pi_n(K)$ inequivalent members we stop. Then we can form all possible disjunctions of the members of R_n and use the fact that $\mathrm{Th}_{\forall\exists}(K)$ is decidable to determine which are equivalent modulo K. The number of equivalence classes thus obtained is $\epsilon_n(K)$. $\square$

Problem 1: If $\epsilon_n(K)$ is recursive and $\mathrm{Th}_{\forall\exists}(K)$ is decidable, does it follow that $\pi_n(K)$ is recursive?

Theorem 1.10.16 *Let K be an elementary class, and K_1 any class such that* $\mathrm{S}(K) = \mathrm{IS}(K_1)$. *If*

(i) *K_1 has few existential formulas,*

(ii) *$\pi_n(K_1)$ is recursive, and*

(iii) *$\mathrm{Th}_{\forall\exists}(K_1)$ is decidable*

then

(a) *for any formula φ there are effective procedures to find existential [universal] formulas $\varphi_{ec}, \varphi_{mc}$ such that*

$$K^{ec} \models \varphi \leftrightarrow \varphi_{ec}$$
$$K^{mc} \models \varphi \leftrightarrow \varphi_{mc},$$

and

(b) *K^{mc} has a decidable theory.*

Proof As in the proof of 1.10.15, we can effectively find a finite $\mathrm{R}_n \subseteq \mathrm{P}_n$ such that every $\varphi \in \mathrm{P}_n$ is equivalent to some $\rho \in \mathrm{R}_n$ modulo K_1. If for $\varphi \in \mathrm{P}_n$ we let

$$\Phi^* = \bigvee \{\eta \in \mathrm{R}_n : \ K_1 \models \eta \to \neg \varphi\},$$

then

$$K_1 \models \Phi^* \leftrightarrow \Phi,$$

so K^{ec} is axiomatized by

$$\{\varphi \vee \Phi^* : \varphi \in \mathrm{P}_n, \ n < \omega\} \cup \mathrm{Th}(K)$$

and K^{mc} by

$$\{\varphi \vee \Phi^* : \varphi \in \mathrm{P}_n, \ n < \omega\} \cup \mathrm{Th}_{\forall}(K).$$

Thus both K^{ec}, K^{mc} satisfy $\neg \varphi \leftrightarrow \Phi^*$ for $\varphi \in \mathrm{P}_n, \ n < \omega$. This gives (a). As $\mathrm{Th}_{\forall}(K_1) = \mathrm{Th}_{\forall}(K^{mc})$, in view of (a) and (iii), we have (b). $\square$

Remark If $\pi_n(K_1)$ and $\mathrm{Th}_{\forall\exists}(K_1)$ are primitive recursive, then we can assume the procedures in (a) as well as $\mathrm{Th}(K^{mc})$ are primitive recursive.

1.11 Finitely generic structures and the finite forcing companion

As we have seen in §1.8, an elementary class K need not have a model companion. In such cases K^{ec} tends to be difficult to work with since, among other things, we lose model-completeness. A number of alternatives to the model companion have been considered, one of the most popular being Robinson's finite forcing companion. As we will only look briefly at the use of Boolean products in the study of the finite forcing companion (at the end of Chapter 2), we simply give an overview of the key definitions and facts needed here.

Theorem 1.11.1 *Let K be a universal class with a countable language. Then K has unique subclasses K^{fg} (the set of* finitely generic *structures in K), and K^{ffc} (the* finite forcing companion *of K) such that*

(i) $K = \mathrm{S}(K^{ffc})$

(ii) $K^{ffc} = \{\mathbf{A} \in K : \ \mathbf{A} \models \mathrm{Th}(K^{fg})\}$

(iii) $K^{fg} = \{\mathbf{A} \in K^{ffc} : \ \mathbf{A} \leq \mathbf{B} \in K^{ff} \ \Rightarrow \ \mathbf{A} \preceq \mathbf{B}\}$.

For such classes K^{fg} and K^{ffc} we have:

(iv) $K^{fg} \subseteq K^{ec}$

(v) *K^{ffc} is complete iff K has the JEP.*

Chapter 2

The Influence of Boolean Products

2.1 Boolean products

Boolean products have proved to be a powerful tool in the study of existentially closed structures. We give a self-contained treatment of this topic. The first five sections are based on papers of Burris and Werner. In §2.6–§2.12 we develop the ideas of Riedel, and in §2.13 a special case of Point's analysis of finitely generic structures is given.

WHAT IS A BOOLEAN PRODUCT?

A Boolean product is a special subdirect product. Before defining Boolean products, let us introduce some convenient notation.

Definition 2.1.1 If $\mathbf{A} \underset{\text{sd}}{\leq} \prod_{x \in X} \mathbf{A}_x$, meaning $\mathbf{A}$ sits in $\prod_{x \in X} \mathbf{A}_x$ as a *subdirect product* of the *stalks* $\mathbf{A}_x$, and if $\varphi(\vec{v})$ is a first-order formula, then for elements $\vec{f}$ from $\mathbf{A}$ we define

$$[\![\varphi(\vec{f})]\!]_{\mathbf{A}} = \{x \in X : \ \mathbf{A} \models \varphi(\vec{f}(x))\}.$$

As a rule we will simply write $[\![\varphi(\vec{f})]\!]$, omitting the subscript $\mathbf{A}$.

Definition 2.1.2 $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x \in X} \mathbf{A}_x$, to be read as "$\mathbf{A}$ *is a Boolean product of the* $\mathbf{A}_x$*'s*", means

(i) $\mathbf{A} \underset{\text{sd}}{\leq} \prod_{x \in X} \mathbf{A}_x$,

and one can endow X with a Boolean space topology such that

(ii) for $\alpha(\vec{v})$ an atomic formula and $\vec{f}$ from A, the set $[\![\alpha(\vec{f})]\!]$ is a clopen subset of X, and

(iii) (*patchwork*) for $f, g \in$ A and N a clopen subset of X, $f|_N \cup g|_{X-N} \in$ A (where $f|_N$ means the restriction of f to N).

Remark If $\mathbf{A}$ is an algebra, then we can replace (ii) by

(ii′) (*equalizers are clopen*) for $f, g \in$ A, the set $[\![f = g]\!]$ is a clopen subset of X.

Definition 2.1.3 For $\mathbf{A} \underset{\mathrm{sd}}{\leq} \prod_{x \in X} \mathbf{A}_x$, for $f, g \in$ A,

$[\![f = g]\!]$ is the *equalizer* of f and g,

$[\![f \neq g]\!]$ is the *difference set* of f and g,

$DE(\mathbf{A}) = \{[\![f \neq g]\!] : f, g \in \mathrm{A}\} \cup \{[\![f = g]\!] : f, g \in \mathrm{A}\}$, and

$Triv(\mathbf{A}) = [\![\forall v_1 \forall v_2\, v_1 = v_2]\!]$.

Lemma 2.1.4 *For* $\mathbf{A} \underset{\mathrm{bp}}{\leq} \prod_{x \in X} \mathbf{A}_x$,

(i) $Triv(\mathbf{A})$ *is a closed subset of* X,

(ii) $DE(\mathbf{A}) = \{N \in X^* : N \cap T = \emptyset$ *or* $T \subseteq N\}$, *where* $T = Triv(\mathbf{A})$, *and*

(iii) $DE(\mathbf{A})$ *is a field of subsets of* X^*.

Proof

(i) If $x \notin Triv(\mathbf{A})$ then there are $f, g \in$ A such that $x \in [\![f \neq g]\!]$; as $[\![f \neq g]\!] \cap Triv(\mathbf{A}) = \emptyset$, by 2.1.2 (ii) $X - Triv(\mathbf{A})$ is an open set.

(ii) For $f, g \in$ A clearly $[\![f \neq g]\!] \cap Triv(\mathbf{A}) = \emptyset$, and $Triv(\mathbf{A}) \subseteq [\![f = g]\!]$. Thus $N \in DE(\mathbf{A})$ implies $N \cap Triv(\mathbf{A}) = \emptyset$ or $Triv(\mathbf{A}) \subseteq N$.

Now suppose $N \in X^*$ and $N \cap Triv(\mathbf{A}) = \emptyset$. Let $f \in$ A. For $x \in N$ choose $g_x \in$ A such that $x \in [\![f \neq g_x]\!]$. Let $h_x = g_x|_N \cup f|_{X-N}$. Then $h_x \in$ A by patchwork, and $x \in [\![f \neq h_x]\!] \subseteq N$. By compactness we can find finitely many h_x's, say $h_{x_1}, \ldots, h_{x_n}$ such that $N = [\![f \neq h_{x_1}]\!] \cup \cdots \cup [\![f \neq h_{x_n}]\!]$. Let $N_i = [\![f \neq h_{x_i}]\!]$, and let

$$h = f|_{X-N} \cup h_{x_1}|_{N_1} \cup h_{x_2}|_{N_2 - N_1} \cup \cdots \cup h_{x_n}|_{N_n - (N_1 \cup \cdots \cup N_{n-1})}.$$

Then $h \in$ A and $N = [\![f \neq h]\!]$, so $N \in DE(\mathbf{A})$.

From this it is easy to see that if $Triv(\mathbf{A}) \subseteq N \in X^*$ then $N \in DE(\mathbf{A})$.

(iii) This follows from (ii). $\square$

Corollary 2.1.5 $\mathbf{A} \underset{\mathrm{bp}}{\leq} \prod\limits_{x \in X} \mathbf{A}_x$ *and* $|Triv(\mathbf{A})| \leq 1$ *implies* $\mathbf{A}$ *determines the topology on* X, *namely*

$$X^* = DE(\mathbf{A}).$$

Definition 2.1.6 For K a class of structures of a given type, we use $\Gamma^a(K)$ for *the class of all Boolean products with stalks from* K.

Definition 2.1.7 For K a class of structures of a given type, we define

$$\Gamma^e(K) = \{\mathbf{A} \in \Gamma^a(K) : [\![\varphi(\vec{f})]\!] \text{ is clopen for all formulas } \varphi \text{ and } \vec{f} \in \mathrm{A}\}.$$

Definition 2.1.8 For X a Boolean space and $\mathbf{A}$ a structure, let $\mathbf{A}[X]^*$ be the algebra of continuous functions from X to A, where A is given the discrete topology. $\mathbf{A}[X]^*$ is called a *bounded Boolean power* of $\mathbf{A}$.

For $\mathbf{B}$ a Boolean algebra, let $\mathbf{A}[\mathbf{B}]^*$ denote $\mathbf{A}[\mathbf{B}^*]^*$, where $\mathbf{B}^*$ is the Stone space of $\mathbf{B}$. It is easy to see that one always has

$$\mathbf{A}[\mathbf{B}]^* \in \Gamma^e(\{\mathbf{A}\}).$$

THE VALUE OF $[\![$*open*$]\!]$ AND $[\![$*existential*$]\!]$

Proposition 2.1.9 *For* $\mathbf{A} \in \Gamma^a(K)$ *and* $\vec{a} \in A$ *we have*

(i) *for* $\omega(\vec{v})$ *an open formula,* $[\![\omega(\vec{a})]\!]$ *is clopen, and*

(ii) *for* $\varphi(\vec{v})$ *an existential formula,* $[\![\varphi(\vec{a})]\!]$ *is open.*

Proof (i) follows from property (ii) of Definition 2.1.2 and

$$\begin{aligned} [\![\neg\varphi(\vec{a})]\!] &= X - [\![\varphi(\vec{a})]\!] \\ [\![\varphi_1(\vec{a}) \vee \varphi_2(\vec{a})]\!] &= [\![\varphi_1(\vec{a})]\!] \cup [\![\varphi_2(\vec{a})]\!]. \end{aligned}$$

For part (ii) let $\varphi(\vec{v}) = \exists\, \vec{u}\omega(\vec{u}, \vec{v})$, where ω is an open formula. Then the result follows from noting that

$$[\![\varphi(\vec{a})]\!] = \bigcup\{[\![\omega(\vec{b}, \vec{a})]\!] : \vec{b} \in \mathrm{A}\}.$$

DIRECT PRODUCTS CONSIDERED AS BOOLEAN PRODUCTS

Proposition 2.1.10 *For* K *a class of structures, every member of* $\mathrm{P}(K)$ *is isomorphic to a member of* $\Gamma^e\mathrm{P}_u(K)$, *that is,*

$$\mathrm{P}(K) \subseteq \mathrm{I}\Gamma^e\mathrm{P}_u(K).$$

Thus

$$\mathrm{ISP}_r(K) = \mathrm{IS}\Gamma^e\mathrm{P}_u(K).$$

Proof Let $(\mathbf{A}_i)_{i\in I}$ be an indexed family of structures, and let I^* be the Boolean space of ultrafilters over I. Then one has the proposition by establishing that the natural mapping

$$\nu : \prod_{i\in I} \mathrm{A}_i \rightarrow \prod_{\mathcal{U}\in I^*} \left(\prod_{i\in I} \mathrm{A}_i/\mathcal{U} \right)$$

defined by

$$\nu(f)(\mathcal{U}) = f/\mathcal{U}$$

is an embedding and $\nu\left(\prod_{i\in I} \mathbf{A}_i\right)$ is in $\Gamma^e \mathrm{P}_u(K)$.

To see this first observe that for $f, g \in \mathrm{A}$, $f \neq g$ implies $[\![f \neq g]\!] \neq \emptyset$, so there is an ultrafilter $\mathcal{U} \in I^*$ such that $[\![f = g]\!] \notin \mathcal{U}$. Thus ν is one-to-one. For $\varphi(\vec{v})$ atomic and $\vec{f} \in \prod_{i\in I} \mathrm{A}_i$ we have:

$$\begin{aligned}
\prod_{i\in I} \mathbf{A}_i \models \varphi(\vec{f}) \text{ iff } [\![\varphi(\vec{f})]\!] &= I \\
&\text{iff } [\![\varphi(\vec{f})]\!] \in \mathcal{U} \quad \text{for } \mathcal{U} \in I^* \\
&\text{iff } \prod_{i\in I} \mathbf{A}_i/\mathcal{U} \models \varphi(\vec{f}/\mathcal{U}) \quad \text{for } \mathcal{U} \in I^* \\
&\text{iff } \prod_{i\in I} \mathbf{A}_i/\mathcal{U} \models \varphi((\nu\vec{f})(\mathcal{U})) \quad \text{for } \mathcal{U} \in I^* \\
&\text{iff } \prod_{\mathcal{U}\in I^*} \left(\prod_{i\in I} \mathbf{A}_i/\mathcal{U} \right) \models \varphi(\nu\vec{f}).
\end{aligned}$$

Thus ν is an embedding, and then we clearly have

$$\nu(\mathbf{A}) \underset{\mathrm{sd}}{\leq} \prod_{\mathcal{U}\in I^*} \left(\prod_{i\in I} \mathbf{A}_i/\mathcal{U} \right).$$

If $\varphi(\vec{v})$ is a first-order formula and $\vec{f} \in \prod_{i\in I} \mathbf{A}_i$ then by Łoś's theorem (see [9])

$$\begin{aligned}
[\![\varphi(\nu\vec{f})]\!] &= \{\mathcal{U} \in I^* : \prod_{i\in I} \mathbf{A}_i/\mathcal{U} \models \varphi((\nu\vec{f})(\mathcal{U}))\} \\
&= \{\mathcal{U} \in I^* : \prod_{i\in I} \mathbf{A}_i/\mathcal{U} \models \varphi(\vec{f}/\mathcal{U})\} \\
&= \{\mathcal{U} \in I^* : [\![\varphi(\vec{f})]\!] \in \mathcal{U}\},
\end{aligned}$$

a clopen subset of I^*.

Finally, if $f, g \in \mathrm{A}$ and N is a clopen subset of I^*, say $N = \{\mathcal{U} \in I^* : J \in \mathcal{U}\}$ for suitable $J \subseteq N$, let $h = f|_J \cup g|_{I-J}$. Then

$$\begin{aligned}
[\![\nu h = \nu f]\!] &= \{\mathcal{U} \in I^* : [\![h = f]\!] \in \mathcal{U}\} \supseteq \{\mathcal{U} \in I^* : J \in \mathcal{U}\} \\
[\![\nu h = \nu g]\!] &= \{\mathcal{U} \in I^* : [\![h = g]\!] \in \mathcal{U}\} \supseteq \{\mathcal{U} \in I^* : I - J \in \mathcal{U}\},
\end{aligned}$$

so $\nu h = \nu f|_N \cup \nu g|_{I^*-N}$. This yields the fact that $\nu\left(\prod_{i\in I}\mathbf{A}_i\right)$ has the patchwork property. $\square$

$\boxed{\text{DIRECT DECOMPOSITION OF BOOLEAN PRODUCTS}}$

Definition 2.1.11 If $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x\in X}\mathbf{A}_x$ and N is a nonempty clopen subset of X, let $\mathrm{A}|_N$, the restriction of A to N, be the subuniverse $\{f|_N : f \in \mathrm{A}\}$ of $\prod_{x\in N}\mathbf{A}_x$, and let $\mathbf{A}|_N$ be the corresponding subalgebra of $\prod_{x\in N}\mathbf{A}_x$.

Proposition 2.1.12 *If* $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x\in X}\mathbf{A}_x$ *and N is a clopen subset of X with* $\emptyset \neq N \neq X$, *then*

(i) $\mathbf{A}|_N \underset{\text{bp}}{\leq} \prod_{x\in N}\mathbf{A}_x$, *and*

(ii) $\mathbf{A} \cong \mathbf{A}|_N \times \mathbf{A}|_{X-N}$.

Proof Part (i) is straightforward to check. For part (ii) use the mapping $\alpha : \mathrm{A} \to \mathrm{A}|_N \times \mathrm{A}|_{X-N}$ given by $f \mapsto \langle f|_N, f|_{X-N}\rangle$. $\square$

$\boxed{\text{THE MOST OBVIOUS CONGRUENCES ON A BOOLEAN PRODUCT}}$

Definition 2.1.13 For $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x\in X}\mathbf{A}_x$ and for Y a subset of X, define θ_Y to be the congruence

$$\{\langle f, g\rangle \in A \times A : Y \subseteq [\![f = g]\!]\},$$

which is the kernel of the projection map $\pi_Y : \prod_{x\in X}\mathbf{A}_x \to \prod_{x\in Y}\mathbf{A}_x$ restricted to $\mathbf{A}$.

$\boxed{\text{STALK CONGRUENCES}}$

Definition 2.1.14 If we have a homomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ and a congruence $\theta \in \mathrm{Con}\,\mathbf{B}$, then define $\alpha^{-1}(\theta)$ to be the congruence

$$\{\langle a_1, a_2\rangle \in \mathbf{A}\times\mathbf{A} : \langle \alpha(a_1), \alpha(a_2)\rangle \in \theta\}$$

on $\mathbf{A}$.

Definition 2.1.15 For $\mathbf{A} \underset{\text{sd}}{\leq} \prod_{x\in X}\mathbf{A}_x$ and for $x \in X$, let $\kappa_x^{\mathbf{A}}$ be the *stalk congruence* defined by

$$\kappa_x^{\mathbf{A}} = \{\langle f, g\rangle \in \mathrm{A}\times\mathrm{A} : fx = gx\},$$

that is, $\kappa_x^{\mathbf{A}}$ is the kernel of the projection map $\pi_x : \prod_{x\in X}\mathbf{A}_x \to \mathbf{A}_x$ restricted to $\mathbf{A}$.

Definition 2.1.16 An algebra $\mathbf{A}$ belonging to $\Gamma^a(K)$ has

(i) the SCRP (*stalk congruence restriction property*) with respect to $\Gamma^a(K)$ if for every $\mathbf{B} \in \Gamma^a(K)$ and embedding $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ we have, for each stalk congruence $\kappa_y^{\mathbf{B}}$ of $\mathbf{B}$, either $\alpha^{-1}(\kappa_y^{\mathbf{B}})$ is a stalk congruence $\kappa_x^{\mathbf{A}}$ of $\mathbf{A}$ or it is ∇_{A}, and

(ii) the SCEP (*stalk congruence extension property*) with respect to $\Gamma^a(K)$ if for every $\mathbf{B} \in \Gamma^a(K)$ and embedding $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ we have, for each stalk congruence $\kappa_x^{\mathbf{A}}$ of $\mathbf{A}$ with $\mathbf{A}_x$ nontrivial, there is a stalk congruence $\kappa_y^{\mathbf{B}}$ of $\mathbf{B}$ such that $\kappa_x^{\mathbf{A}} = \alpha^{-1}(\kappa_y^{\mathbf{B}})$.

In the following we use
$$\Gamma^a(K)_{\mathrm{SCRP}}$$
to denote the class of algebras in $\Gamma^a(K)$ having the SCRP with respect to $\Gamma^a(K)$.

$\boxed{\text{THE STANDARD FORM OF A BOOLEAN PRODUCT}}$

Definition 2.1.17 A Boolean product $\mathbf{A} \underset{\mathrm{bp}}{\leq} \prod_{x\in X} \mathbf{A}_x$ is said to be in *standard form* if

(i) at most one $\mathbf{A}_x$ is trivial, and

(ii) if some $\mathbf{A}_{x_0}$ is trivial, then either $X = \{x_0\}$ or x_0 is a limit point of X.

Proposition 2.1.18 *For algebras it holds that every Boolean product* $\mathbf{A} \underset{\mathrm{bp}}{\leq} \prod_{x\in X} \mathbf{A}_x$ *is isomorphic to a Boolean product in standard form with the same set of nontrivial stalks.*

Proof Let $U = [\![\exists v_1 \exists v_2 (v_1 \neq v_2)]\!]$. If $X = U$ then $\mathbf{A}$ is in standard form. If $U = \emptyset$ then $\mathbf{A}$ is trivial. If $\emptyset \neq U$ then $\mathbf{A} \cong \mathbf{A}|_U$. Thus if U is clopen then $\mathbf{A}|_U$ is in standard form.

If $\emptyset \neq U \neq X$ and U is not clopen, let $x_0 \in X - U,\ Z = U \cup \{x_0\}$. Then $\mathbf{A} \cong \mathbf{A}|_Z$ under the projection map.

Claim: $DE(\mathbf{A}|_Z)$ is a field of subsets of Z which is a basis for a Boolean space topology, and with this topology
$$\mathbf{A}|_Z \underset{\mathrm{bp}}{\leq} \prod_{z\in Z} \mathbf{A}_z.$$

To see that $DE(\mathbf{A}|_Z)$ is a field of subsets observe
$$\begin{aligned} DE(\mathbf{A}|_Z) &= \{[\![f|_Z = g|_Z]\!] : f,g \in \mathrm{A}\} \cup \{[\![f|_Z \neq g|_Z]\!] : f,g \in \mathrm{A}\} \\ &= \{[\![f = g]\!] \cap Z : f,g \in \mathrm{A}\} \cup \{[\![f \neq g]\!] \cap Z : f,g \in \mathrm{A}\} \\ &= \{N \cap Z : N \in DE(\mathbf{A})\}. \end{aligned}$$

Points in Z are separated by $DE(\mathbf{A}|_Z)$, for if $x, y \in Z$ with $x \neq y$, at most one of $x, y \in Triv(\mathbf{A})$. As X is a Boolean space and $Triv(\mathbf{A})$ is closed, we can choose $N \in DE(\mathbf{A})$ such that $x \in N,\ y \notin N$. Then $x \in N \cap Z,\ y \in Z - N$, and $N \cap Z,\ Z - N \in DE(\mathbf{A}|_Z)$.

Certainly Z has a basis of clopen subsets since its topology is generated by the field of sets $DE(\mathbf{A}|_Z)$. Also Z is compact, for if $Z = \bigcup_{i \in Z} M_i,\ M_i$ clopen in Z, choose $N_i \in DE(\mathbf{A})$ such that $M_i = N_i \cap Z$. As $x_0 \in Z,\ x_0 \in N_{i_0}$ for some i_0. As $x \in Triv(\mathbf{A}),\ Triv(\mathbf{A}) \subseteq N_{i_0}$. Thus $X - N_{i_0} \subseteq Z$, so $X - N_{i_0} \subseteq \bigcup_{i \in I} N_i$. As $X - N_{i_0}$ is a compact subset of X, choose I_0 a finite subset of I such that $X - N_{i_0} \subseteq \bigcup_{i \in I_0} N_i$. Then, with $J = I_0 \cup \{i_0\},\ X \subseteq \bigcup_{i \in J} N_i$, so $Z \subseteq \bigcup_{i \in J} M_i$. Thus Z is a Boolean space.

Certainly $\mathbf{A}|_Z$ satisfies *equalizers are clopen* as $[\![f|_Z = g|_Z]\!] \in DE(\mathbf{A}|_Z)$ for $f, g \in \mathrm{A}$.

Finally, to show *patchwork* holds for $\mathbf{A}|_Z$, let $f, g \in \mathrm{A}$ and let M be a clopen subset of Z. Choose N clopen in X such that $M = N \cap Z$. Then

$$\begin{aligned} (f|_Z)|_M \cup (g|_Z)|_{Z-M} &= (f|_N \cup g|_{X-N})|_Z \\ &\in \mathbf{A}|_Z. \qquad \square \end{aligned}$$

WHEN DOES A PRIMITIVE POSITIVE FORMULA HOLD IN A BOOLEAN PRODUCT?

Proposition 2.1.19 *If* $\mathbf{A} \underset{\mathrm{bp}}{\leq} \prod_{x \in X} \mathbf{A}_x$, *then for* $\varphi(\vec{v})$ *a primitive positive formula and for* $\vec{f}$ *from* A *we have*

$$\mathbf{A} \models \varphi(\vec{f}) \quad \textit{iff} \quad [\![\varphi(\vec{f})]\!] = X.$$

Proof Suppose $\mathbf{A} \models \varphi(\vec{f})$. Since φ is a positive formula, $\mathbf{A}_x \models \varphi(\vec{f}x)$ for $x \in X$. Thus $[\![\varphi(\vec{f})]\!] = X$.

For the converse suppose $[\![\varphi(\vec{f})]\!] = X$. Let $\varphi(\vec{v}) = \exists\, \vec{u}\omega(\vec{u}, \vec{v})$, where $\omega(\vec{u}, \vec{v})$ is a conjunction of atomic formulas. Given $x \in X$ choose $\vec{g} \in \mathrm{A}$ such that $\mathbf{A}_x \models \omega(\vec{g}x, \vec{f}x)$. As $[\![\omega(\vec{g}, \vec{f})]\!]$ is a clopen neighbourhood of x, we can invoke the compactness of X to show the existence of a partition $N_1, \ldots, N_k$ of X into clopen subsets, and the existence of elements $\vec{g}^{(1)}, \ldots, \vec{g}^{(k)}$ of A such that $N_i \subseteq [\![\omega(\vec{g}^{(i)}, \vec{f})]\!]$. By patching the $\vec{g}^{(i)}$'s over the N_i's we obtain $\vec{g} \in \mathrm{A}$ such that $[\![\omega(\vec{f}, \vec{g})]\!] = X$. Thus $\mathbf{A} \models \omega(\vec{g}, \vec{f})$, yielding $\mathbf{A} \models \varphi(\vec{f})$. $\square$

WHEN DOES A PRIMITIVE FORMULA HOLD IN A BOOLEAN PRODUCT?

Definition 2.1.20 For K a class of structures of a given type let $\Gamma_0^a(K)$ denote the class of members of $\Gamma^a(K)$ whose base space has no isolated points.

Likewise define $\Gamma_0^e(K)$ to be the class of members of $\Gamma^e(K)$ whose base space has no isolated points.

Proposition 2.1.21 *For $\mathbf{A} \in \Gamma_0^a(K)$, for $\varphi(\vec{v})$ a primitive formula*

$$\exists\, \vec{u} \left(\varphi^+(\vec{u}, \vec{v}) \wedge \bigwedge_{1 \le i \le n} \neg\, \alpha_i(\vec{u}, \vec{v}) \right),$$

where $\varphi^+(\vec{u}, \vec{v})$ is a conjunction of atomic formulas and each α_i is atomic, and for $\vec{f}$ from A,

$$\mathbf{A} \models \varphi(\vec{f}) \quad \textit{iff} \quad \begin{cases} [\![\exists\, \vec{u}\varphi^+(\vec{u}, \vec{f})]\!] \;=\; X \\ \textit{and} \\ [\![\exists\, \vec{u}\left[\varphi^+(\vec{u}, \vec{f}) \,\wedge\, \neg\, \alpha_i(\vec{u}, \vec{f})\right]]\!] \;\neq\; \emptyset \textit{ for } 1 \le i \le n. \end{cases}$$

Proof Let

$$\begin{aligned} \varphi_0(\vec{v}) \;&=: \exists\, \vec{u}\varphi^+(\vec{u}, \vec{v}) \\ \varphi_i(\vec{v}) \;&=: \exists\, \vec{u}[\varphi^+(\vec{u}, \vec{v}) \,\wedge\, \neg\, \alpha_i(\vec{u}, \vec{v})], \quad 1 \le i \le n \\ N_i \;&=: [\![\varphi_i(\vec{u})]\!] \quad 0 \le i \le n. \end{aligned}$$

If $\mathbf{A} \models \varphi(\vec{a})$ then certainly $\mathbf{A} \models \varphi_0(\vec{a})$ and $\mathbf{A} \models \varphi_i(\vec{a}),\ 1 \le i \le n$, so $[\![\varphi_0(\vec{a})]\!] = X$ and $[\![\varphi_i(\vec{a})]\!] \neq \emptyset,\ 1 \le i \le n$.

Conversely, suppose $[\![\varphi_0(\vec{a})]\!] = X$ and $[\![\varphi_i(\vec{a})]\!] \neq \emptyset,\ 1 \le i \le n$. Then each N_i is nonempty open, thus infinite since X has no isolated points. Choose distinct $x_i \in [\![\varphi_i(\vec{a})]\!],\ 1 \le i \le n$, and then choose $\vec{b}_i \in A$ such that $x_i \in [\![\varphi^+(\vec{b}_i, \vec{a}) \,\wedge\, \neg\, \alpha_i(\vec{b}_i, \vec{a})]\!],\ 1 \le i \le n$. Using the Hausdorff property it is possible to choose pairwise disjoint clopen subsets M_i such that

$$x_i \in M_i \subseteq [\![\varphi^+(\vec{b}_i, \vec{a}) \,\wedge\, \neg\, \alpha_i(\vec{b}_i, \vec{a})]\!],\ 1 \le i \le n.$$

Let $M = \bigcup_{1 \le i \le n} M_i$. For each $x \in X - M$ it is possible to choose $\vec{b} \in A$ such that $x \in [\![\varphi^+(\vec{b}, \vec{a}]\!]$. Using the compactness of $X - M$ it is possible to find a partition of $X - M$ into clopen subsets $M_{n+1}, \ldots, M_{n+k}$ and to find $\vec{b}_{n+1}, \ldots, \vec{b}_{n+k} \in A$ such that $M_{n+j} \subseteq [\![\varphi^+(\vec{b}_{n+j}, \vec{a})]\!],\ 1 \le j \le k$. Then patching the $\vec{b}_i$'s over the M_i's, $1 \le i \le n + k$, we have a $\vec{b} \in A$ such that $[\![\varphi^+(\vec{b}, \vec{c})]\!] = X$ and $[\![\varphi^+(\vec{b}, \vec{a}) \,\wedge\, \neg\, \alpha_i(\vec{b}, \vec{a})]\!] \neq \emptyset,\ 1 \le i \le n$. But then

$$\mathbf{A} \;\models\; \varphi^+(\vec{b}, \vec{a}) \,\wedge\, \bigwedge_{1 \le i \le n} \neg\, \alpha_i(\vec{b}, \vec{a}),$$

so $\mathbf{A} \models \varphi(\vec{a})$. $\square$

Lemma 2.1.22 *Let K be a class of structures, and let $\varphi(\vec{v})$ be a primitive formula, say*

$$\exists\, \vec{u} \left(\varphi^+(\vec{u}, \vec{v}) \,\wedge\, \bigwedge_{1 \le i \le s} \neg\, \alpha_i(\vec{u}, \vec{v}) \right),$$

where $\varphi^+(\vec{u},\vec{v})$ is a conjunction of atomic formulas, and each $\alpha_i(\vec{u},\vec{v})$ is atomic. Let

$$\begin{aligned}\varphi_0(\vec{v}) &= \exists\,\vec{u}\varphi^+(\vec{u},\vec{v})\\ \varphi_i(\vec{v}) &= \exists\,\vec{u}\left(\varphi^+(\vec{u},\vec{v}) \wedge \neg\,\alpha_i(\vec{u},\vec{v})\right) \quad 1 \le i \le s.\end{aligned}$$

Then

$$\Gamma_0^a(K) \models \varphi(\vec{v}) \leftrightarrow \bigwedge_{0\le i\le s} \varphi_i(\vec{v}).$$

Proof Certainly

$$\varphi(\vec{v}) \to \bigwedge_{0\le i\le s} \varphi_i(\vec{v}).$$

So suppose $\mathbf{A} \in \Gamma_0^a(K)$, $\vec{a} \in \mathrm{A}$, and

$$\mathbf{A} \models \bigwedge_{1\le i\le s} \varphi_i(\vec{a}).$$

Then one easily concludes that

$$\begin{aligned}[\![\varphi_0(\vec{a})]\!] &= X\\ [\![\varphi_i(\vec{a})]\!] &\ne \emptyset \quad 1 \le i \le s,\end{aligned}$$

so by Macintyre's Theorem (2.1.21) we have

$$\mathbf{A} \models \varphi(\vec{a}).$$

Thus

$$\Gamma_0^a(K) \models \bigwedge_{0\le i\le s} \varphi_i(\vec{v}) \to \varphi(\vec{v}). \qquad \square$$

Definition 2.1.23 A primitive formula $\varphi(\vec{v})$ with at most one conjunct of the matrix being negated atomic is called an M-*formula*.[1] We write

$$\mathbf{A} \le_M \mathbf{B}$$

if $\mathbf{A} \le \mathbf{B}$ and if, for every M-formula $\varphi(\vec{v})$ and elements $\vec{a} \in \mathrm{A}$, we have

$$\mathbf{B} \models \varphi(\vec{a}) \Rightarrow \mathbf{A} \models \varphi(\vec{a}).$$

Parallel to Definition 1.2.1 we write

$$\mathbf{A} \underset{M}{\hookrightarrow} \mathbf{B}$$

if for every embedding $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ we have $\alpha(\mathbf{A}) \le_M \mathbf{B}$. Then for K a class of structures we say

$$\mathbf{A} \underset{M}{\hookrightarrow} K$$

if $\mathbf{A} \underset{M}{\hookrightarrow} \mathbf{B}$ holds for each $\mathbf{B} \in K$.

[1] Macintyre used such formulas in his sheaf-theoretic analysis of model complete elementary classes with an underlying ring structure.

Remark Clearly $\mathbf{A} \underset{ec}{\hookrightarrow} K$ implies $\mathbf{A} \underset{M}{\hookrightarrow} K$.

A GENERALIZATION OF THE ULTRAPRODUCT

Definition 2.1.24 Let $\mathbf{A}_i \in \Gamma^e(K)$, and let $\mathcal{U}$ be an ultrafilter on $\prod_{i\in I} X_i^*$, where $X_i = X_i(\mathbf{A}_i)$. Define the binary relation $\sim_{\mathcal{U}}$ on $\prod_{i\in I} \mathrm{A}_i$ by

$$f \sim_{\mathcal{U}} g \text{ iff } \langle\!\langle [\![f(i) = g(i)]\!] \rangle\!\rangle_{i\in I} \in \mathcal{U}.$$

Lemma 2.1.25 *For $\mathbf{A}_i$, $i \in I$, and $\sim_{\mathcal{U}}$ as in* 2.1.24*, $\sim_{\mathcal{U}}$ is an equivalence relation which is compatible with the fundamental operations of $\prod_{i\in I} \mathbf{A}_i$.*

Proof $\sim_{\mathcal{U}}$ is certainly reflexive and symmetric. If $f \sim_{\mathcal{U}} g$ and $g \sim_{\mathcal{U}} h$ then

$$\langle\!\langle [\![f(i) = g(i)]\!] \rangle\!\rangle_{i\in I} \wedge \langle\!\langle [\![g(i) = h(i)]\!] \rangle\!\rangle_{i\in I} \leq \langle\!\langle [\![f(i) = h(i)]\!] \rangle\!\rangle_{i\in I},$$

so $f \sim_{\mathcal{U}} h$. If F is a fundamental n-ary operation and $f_i \sim_{\mathcal{U}} g_i$, $1 \leq i \leq n$, then

$$\begin{aligned} &\langle\!\langle [\![f_1(i) = g_1(i)]\!] \rangle\!\rangle_{i\in I} \wedge \cdots \wedge \langle\!\langle [\![f_n(i) = g_n(i)]\!] \rangle\!\rangle_{i\in I} \\ &\qquad\qquad \leq \langle\!\langle [\![F(f_1,\ldots,f_n)(i) = F(g_1,\ldots,g_n)(i)]\!] \rangle\!\rangle_{i\in I}, \end{aligned}$$

so $\mathrm{F}(f_1,\ldots,f_n) \sim_{\mathcal{U}} \mathrm{F}(g_1,\ldots,g_n)$. $\square$

Definition 2.1.26 Given $\mathbf{A}_i$, $\sim_{\mathcal{U}}$ as in 2.1.24 then for $f \in \prod_{i\in I} \mathrm{A}_i$ let $f /\!\!/ \mathcal{U}$ denote the equivalence class of f modulo $\sim_{\mathcal{U}}$. We will adopt the notation $\vec{f} /\!\!/ \mathcal{U}$ for the sequence $f_1 /\!\!/ \mathcal{U}, \ldots, f_n /\!\!/ \mathcal{U}$. Then define the *generalized ultraproduct* $\prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U}$ by letting its universe be the quotient set $\prod_{i\in I} \mathrm{A}_i /\!\!/ \sim_{\mathcal{U}}$, and for F an n-ary function symbol and R an n-ary relation symbol, let

$$\prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U} \models F(\vec{f} /\!\!/ \mathcal{U}) = F(\vec{f}) /\!\!/ \mathcal{U}$$
$$\prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U} \models R(\vec{f} /\!\!/ \mathcal{U}) \quad \text{iff} \quad \langle\!\langle [\![R(\vec{f})]\!] \rangle\!\rangle_{i\in I} \in \mathcal{U}.$$

It is easy to check that the above definitions are consistent. If all the X_i's are singletons we have the well-known ultraproduct construction. Next we prove the obvious generalization of Łoś's theorem on ultraproducts.

Proposition 2.1.27 *Given $\mathbf{A}_i$, $\sim_{\mathcal{U}}$ as in* 2.1.24*, then for any first-order formula $\varphi(v_1,\ldots,v_n)$ and elements $f_1,\ldots,f_n \in \prod_{i\in I} \mathrm{A}_i$ we have*

$$\prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U} \models \varphi(\vec{f} /\!\!/ \mathcal{U}) \quad \textit{iff} \quad \langle\!\langle [\![\varphi(\vec{f}(i))]\!] \rangle\!\rangle_{i\in I} \in \mathcal{U}.$$

Proof We proceed by induction on the complexity of the formula $\varphi(\vec{v})$.

(i) If $\varphi(\vec{v})$ is $p(\vec{v}) = q(\vec{v})$ then

$$\begin{aligned}
&\prod_{i\in I} \mathbf{A}_i/\!\!/\mathcal{U} \models p(\vec{f}/\!\!/\mathcal{U}) = q(\vec{f}/\!\!/\mathcal{U}) \\
&\text{iff } \prod_{i\in I} \mathbf{A}_i/\!\!/\mathcal{U} \models p(\vec{f})/\!\!/\mathcal{U} = q(\vec{f})/\!\!/\mathcal{U} \\
&\text{iff } \langle\!\langle p(\vec{f})(i) = q(\vec{f})(i)\rangle\!\rangle_{i\in I} \in \mathcal{U} \\
&\text{iff } \langle\!\langle p(\vec{f}(i)) = q(\vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U}.
\end{aligned}$$

(ii) If $\varphi(\vec{v})$ is $R(t_1(\vec{v}), \ldots, t_n(\vec{v}))$ then

$$\begin{aligned}
&\prod_{i\in I} \mathbf{A}_i/\!\!/\mathcal{U} \models \varphi(\vec{f}/\!\!/\mathcal{U}) \\
&\text{iff } \langle\!\langle R(t_1(\vec{f})(i), \ldots, t_k(\vec{f})(i))\rangle\!\rangle_{i\in I} \in \mathcal{U} \\
&\text{iff } \langle\!\langle R(t_1(\vec{f}(i)), \ldots, t_k(\vec{f}(i)))\rangle\!\rangle_{i\in I} \in \mathcal{U}.
\end{aligned}$$

(iii) If $\varphi(\vec{v})$ is $\neg\varphi_0(\vec{v})$ and the theorem holds for $\varphi_0(\vec{v})$ then

$$\begin{aligned}
&\prod_{i\in I} \mathbf{A}_i/\!\!/\mathcal{U} \models \neg\varphi_0(\vec{f}/\!\!/\mathcal{U}) \\
&\text{iff } \langle\!\langle \varphi_0(\vec{f}(i))\rangle\!\rangle_{i\in I} \notin \mathcal{U} \\
&\text{iff } \langle\!\langle \neg\varphi_0(\vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U} \\
&\text{iff } \langle\!\langle \varphi(\vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U}.
\end{aligned}$$

(iv) If $\varphi(\vec{v})$ is $\varphi_0(\vec{v}) \wedge \varphi_1(\vec{v})$ and the theorem holds for φ_0, φ_1 then

$$\begin{aligned}
&\prod_{i\in I} \mathbf{A}_i/\!\!/\mathcal{U} \models \varphi_0(\vec{f}/\!\!/\mathcal{U}) \wedge \varphi_1(\vec{f}/\!\!/\mathcal{U}) \\
&\text{iff } \langle\!\langle \varphi_0(\vec{f}(i))\rangle\!\rangle_{i\in I}, \langle\!\langle \varphi_1(\vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U} \\
&\text{iff } \langle\!\langle \varphi_0(\vec{f}(i))\rangle\!\rangle_{i\in I} \wedge \langle\!\langle \varphi_1(\vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U} \\
&\text{iff } \langle\!\langle \varphi_0(\vec{f}(i)) \wedge \varphi_1(\vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U} \\
&\text{iff } \langle\!\langle \varphi(\vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U}.
\end{aligned}$$

(v) If the theorem holds for $\varphi(v, \vec{v})$ then

$$\begin{aligned}
&\prod_{i\in I} \mathbf{A}_i/\!\!/\mathcal{U} \models \exists v\varphi(v, \vec{f}/\!\!/\mathcal{U}) \\
&\Rightarrow \prod_{i\in I} \mathbf{A}_i/\!\!/\mathcal{U} \models \varphi(f/\!\!/\mathcal{U}, \vec{f}/\!\!/\mathcal{U}) \quad \text{for some } f \\
&\Rightarrow \langle\!\langle \varphi(f(i), \vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U} \\
&\Rightarrow \langle\!\langle \exists v\varphi(v, \vec{f}(i))\rangle\!\rangle_{i\in I} \in \mathcal{U}.
\end{aligned}$$

Conversely,

$$\begin{aligned}
&[\![\exists\, v\varphi(v,\vec{f}(i))]\!]\rangle_{i\in I} \in \mathcal{U}\\
&\Rightarrow \langle\!\langle [\![\varphi(g_i,\vec{f}(i))]\!]\rangle_{i\in I} \in \mathcal{U} \quad \text{for some } g_i \in \mathrm{A}_i,\ i \in I\\
&\Rightarrow \langle\!\langle [\![\varphi(f(i),\vec{f}(i))]\!]\rangle_{i\in I} \in \mathcal{U} \quad \text{for } f(i) = g_i\\
&\Rightarrow \prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U} \models \varphi(f /\!\!/ \mathcal{U}, \vec{f} /\!\!/ \mathcal{U})\\
&\Rightarrow \prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U} \models \exists\, v\varphi(v, \vec{f} /\!\!/ \mathcal{U}). \qquad \square
\end{aligned}$$

GENERALIZED ULTRAPRODUCTS ARE ELEMENTARY SUBSTRUCTURES OF ULTRAPRODUCTS

Definition 2.1.28 For K a class of structures let $\Gamma_u^e(K)$ be the class of generalized ultraproducts of members of $\Gamma^e(K)$.

Proposition 2.1.29 *For K a class of structures*

$$\Gamma_u^e(K) \subseteq \mathrm{IS}^{(\preceq)}\mathrm{P}_u(K).$$

Proof Let $\mathbf{A}_i \in \Gamma^e(K),\ i \in I$, say $\mathbf{A}_i \underset{\mathrm{sd}}{\leq} \prod_{x\in X_i} \mathbf{A}_x^i$, and let $\mathcal{U}$ be an ultrafilter on $\prod_{i\in I} X_i^*$. Let $\widehat{I} = \{\langle i,x\rangle \mid i \in I,\ x \in X_i\}$. Define

$$\beta : \prod_{i\in I} \mathrm{A}_i \to \prod_{\langle i,x\rangle\in\widehat{I}} A_x^i$$

to be the obvious map $\beta(f)(i) = f(i)(x)$. Then β is an embedding.

Let $\mathcal{W}$ be an ultrafilter over $\widehat{I}$ extending

$$\{\{\langle i,x\rangle \in \widehat{I} \mid x \in N(i)\} \mid N \in \mathcal{U}\},$$

a collection of subsets of $\widehat{I}$ with the finite intersection property. For $\varphi(\vec{v})$ a first-order formula and $\vec{f} \in \prod_{i\in I} \mathrm{A}_i$,

$$\begin{aligned}
&\prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U} \models \varphi(\vec{f} /\!\!/ \mathcal{U})\\
&\text{iff } \langle\!\langle [\![\varphi(\vec{f}(i))]\!]\rangle_{i\in I} \in \mathcal{U}\\
&\text{iff } [\![\varphi(\beta\vec{f})]\!] \in \mathcal{W}\\
&\text{iff } \prod_{\langle i,x\rangle\in\widehat{I}} \mathbf{A}_x^i/\mathcal{W} \models \varphi(\beta\vec{f}/\mathcal{W}).
\end{aligned}$$

Thus the mapping

$$\gamma : \prod_{i\in I} \mathbf{A}_i /\!\!/ \mathcal{U} \;\rightarrow\; \prod_{\langle i,x\rangle \in \widehat{I}} \mathbf{A}_x^i / \mathcal{W}$$

defined by

$$\gamma : f /\!\!/ \mathcal{U} \;\mapsto\; \beta f / \mathcal{W}$$

is a well-defined elementary embedding, so

$$\Gamma_u^e(K) \;\subseteq\; \mathrm{IS}^{(\preceq)}\mathrm{P}_u(K). \qquad \Box$$

EMBEDDING DIRECT PRODUCTS IN NICE BOOLEAN PRODUCTS

Proposition 2.1.30 *For K a class of structures*

$$\mathrm{P}(K) \;\subseteq\; \mathrm{IS}\Gamma_0^e\mathrm{S}^{(\preceq)}\mathrm{P}_u(K).$$

Proof Let $\prod_{i\in I} \mathbf{A}_i$ be given. Then, with C the Cantor discontinuum,

$$\prod_{i\in I} \mathbf{A}_i \;\in\; \mathrm{IS}\left(\prod_{i\in I} \mathbf{A}_i[C]^*\right)$$

as $\mathbf{A}_i$ can be embedded in $\mathbf{A}_i[C]^*$.

Let $X = (C^{*I})^*$, and define

$$\alpha : \prod_{i\in I} \mathbf{A}_i[C]^* \;\rightarrow\; \prod_{\mathcal{U}\in X}\left(\prod_{i\in I} \mathbf{A}_i[C]^* /\!\!/ \mathcal{U}\right)$$

by

$$\alpha(f)(\mathcal{U}) \;=\; f /\!\!/ \mathcal{U}.$$

Then, for φ atomic,

$$\begin{aligned}
&\prod_{i\in I} \mathbf{A}_i[C]^* \;\models\; \varphi(\vec{f}) \\
&\text{iff } [\![\varphi(\vec{f}]\!] = I \\
&\text{iff } \langle\![\![\varphi(\vec{f}(i))]\!]\rangle_{i\in I} \;=\; C^I \\
&\text{iff } \langle\![\![\varphi(\vec{f}(i))]\!]\rangle_{i\in I} \in \mathcal{U} \text{ for all } \mathcal{U} \in X \\
&\text{iff } \prod_{i\in I} \mathbf{A}_i[C]^* /\!\!/ \mathcal{U} \;\models\; \varphi(\vec{f} /\!\!/ \mathcal{U}) \text{ for } \mathcal{U} \in X \\
&\text{iff } \prod_{i\in I} \mathbf{A}_i[C]^* /\!\!/ \mathcal{U} \;\models\; \varphi((\alpha\vec{f})(\mathcal{U})) \text{ for } \mathcal{U} \in X \\
&\text{iff } [\![\varphi(\alpha\vec{f})]\!] = X \\
&\text{iff } \prod_{\mathcal{U}\in X}\left(\prod_{i\in I} \mathbf{A}_i[C]^* /\!\!/ \mathcal{U}\right) \;\models\; \varphi(\alpha\vec{f}).
\end{aligned}$$

Consequently α is an embedding (and it is subdirect).

Now we claim

$$\alpha\left(\prod_{i\in I}\mathbf{A}_i[C]^*\right) \underset{\text{bp}}{\leq} \prod_{\mathcal{U}\in X}\left(\prod_{i\in I}\mathbf{A}_i[C]^*/\!\!/\mathcal{U}\right).$$

First given a first-order formula $\varphi(v_1,\ldots,v_n)$ and $\vec{f}\in\prod_{i\in I}\mathrm{A}_i[C]^*$ we have

$$\begin{aligned}[\![\varphi(\alpha\vec{f})]\!] &= \{\mathcal{U}\in X : \prod_{i\in I}\mathbf{A}_i[C]^*/\!\!/\mathcal{U} \models \varphi(\alpha\vec{f}/\!\!/\mathcal{U})\}\\ &= \{\mathcal{U}\in X : \langle\![\![\varphi(\vec{f}(i))]\!]\rangle_{i\in I}\in\mathcal{U}\},\end{aligned}$$

a clopen subset of X.

For the patchwork property, let $f,g\in\prod_{i\in I}\mathrm{A}_i[C]^*$, and let N be a clopen subset of X. Choose $\langle\langle i,J_i\rangle\rangle_{i\in I}\in C^{*I}$ such that

$$N=\{\mathcal{U}\in X : \langle\langle i,J_i\rangle\rangle_{i\in I}\in\mathcal{U}\}.$$

Define $h: I\to\bigcup_{i\in I}\mathrm{A}_i[C]^*$ by

$$h(i)(x)=\begin{cases} f(i)(x) & \text{if } x\in J_i\\ g(i)(x) & \text{if } x\in C-J_i.\end{cases}$$

Then $h(i)=f(i)|_{J_i}\cup g(i)|_{C-J_i}$, so $h(i)\in\mathrm{A}_i$; hence $h\in\prod_{i\in I}\mathrm{A}_i[C]^*$. Now

$$\begin{aligned}[\![\alpha f=\alpha h]\!] &= \{\mathcal{U}\in X : \langle\langle i,[\![f(i)=h(i)]\!]\rangle\rangle_{i\in I}\in\mathcal{U}\}\\ &\supseteq \{\mathcal{U}\in X : \langle\langle i,J_i\rangle\rangle_{i\in I}\in\mathcal{U}\}\\ &= N\end{aligned}$$

and

$$\begin{aligned}[\![\alpha g=\alpha h]\!] &= \{\mathcal{U}\in X : \langle\langle i,[\![g(i)=h(i)]\!]\rangle\rangle_{i\in I}\in\mathcal{U}\}\\ &\supseteq \{\mathcal{U}\in X : \langle\langle i,C-J_i\rangle\rangle_{i\in I}\in\mathcal{U}\}\\ &= X-N.\end{aligned}$$

Thus $\alpha h=\alpha f|_N\cup\alpha g|_{X-N}$.

Finally, as $(C^{*I})^*$ has no isolated points, we see that

$$\begin{aligned}\prod_{i\in I}\mathbf{A}_i[C]^* &\in \mathrm{I}\Gamma_0^e\Gamma_u^e(\{\mathbf{A}_i : i\in I\})\\ &\subseteq \mathrm{I}\Gamma_0^e\mathrm{S}^{(\preceq)}\mathrm{P}_u(\{\mathbf{A}_i : i\in I\}),\end{aligned}$$

so

$$\prod_{i\in I}\mathbf{A}_i \in \mathrm{IS}\Gamma_0^e\mathrm{S}^{(\preceq)}(\{\mathbf{A}_i : i\in I\}).$$

This leads to the desired result: $\mathrm{P}(K)\subseteq\mathrm{IS}\Gamma_0^e\mathrm{S}^{(\preceq)}\mathrm{P}_u(K)$. $\square$

Corollary 2.1.31 *For any class of $\mathcal{L}$-structures we have*

$$\mathrm{ISP}_r(K) \;=\; \mathrm{IS}\Gamma_0^e\mathrm{S}^{(\preceq)}\mathrm{P}_u(K),$$

and hence for K an elementary class of $\mathcal{L}$-structures

$$\mathrm{ISP}_r(K) \;=\; \mathrm{IS}\Gamma_0^e(K).$$

$\boxed{\text{TESTING FOR BEING EXISTENTIALLY CLOSED}}$

Proposition 2.1.32 *Let K be an elementary class and let $\mathbf{A} \in \mathrm{ISP}(K)$. Then*

$$\mathbf{A} \in \mathrm{ISP}(K)^{ec} \Leftrightarrow \mathbf{A} \underset{ec}{\hookrightarrow} \Gamma_0^e(K).$$

Proof Since $\mathrm{ISP}(K) = \mathrm{IS}\Gamma_0^e(K)$ by 2.1.31, this proposition is an immediate consequence of 1.10.1. $\Box$

Proposition 2.1.33 *Let K be an elementary class and let $\mathbf{A} \in \Gamma_0^a(K)$. Then*

$$\mathbf{A} \in \mathrm{ISP}(K)^{ec} \Leftrightarrow \mathbf{A} \underset{M}{\hookrightarrow} \Gamma_0^e(K).$$

Proof By 2.1.32 it suffices to show

$$\mathbf{A} \underset{M}{\hookrightarrow} \Gamma_0^e(K) \Rightarrow \mathbf{A} \underset{ec}{\hookrightarrow} \Gamma_0^e(K).$$

So let $\alpha : \mathbf{A} \to \mathbf{B} \in \Gamma_0^e(K)$ be an embedding, and let $\varphi(\vec{v})$ be an existential formula and $\vec{a} \in \mathrm{A}$ be such that $\mathbf{B} \models \varphi(\vec{a})$. Let $\varphi_0(\vec{v}), \ldots, \varphi_s(\vec{v})$ be M-formulas as in 2.1.22. Then

$$\begin{array}{rl|l}
\mathbf{B} \models \varphi(\vec{a} \Rightarrow & \mathbf{B} \models \bigwedge\limits_{0\le i\le s} \varphi_i(\vec{a}) & \text{by 2.1.22}\\
\Rightarrow & \mathbf{A} \models \bigwedge\limits_{0\le i\le s} \varphi_i(\vec{a}) & \mathbf{A} \underset{M}{\hookrightarrow} \Gamma_0^e(K)\\
\Rightarrow & \mathbf{A} \models \varphi(\vec{a}) & \text{by 2.1.22,}
\end{array}$$

so $\alpha(\mathbf{A}) \le_{ec} \mathbf{B}$. Consequently $\mathbf{A} \underset{ec}{\hookrightarrow} \Gamma_0^e(K)$. $\Box$

2.2 Universal Horn classes generated by classes with few existential formulas

In this section we will show, with the help of the Boolean product construction, that many well-known universal Horn classes do indeed have a model companion. As the universal Horn class generated by a class K is $\mathrm{ISP}_r(K)$, one might try to apply earlier results on few existential formulas—unfortunately even if K has few existential formulas the class $\mathrm{P}_r(K)$ will usually have lots of such formulas. That is the reason we introduced the class operator Γ_0^e.

$\boxed{\Gamma_0^e \text{ PRESERVES “FEW EXISTENTIAL FORMULAS”}}$

Lemma 2.2.1 *Let K be a class of structures. Then*

(i) $\varepsilon_n(K) < \infty \Rightarrow \varepsilon_n(\Gamma_0^e(K)) < \infty$ *for* $n < \infty$, *and*

(ii) $\varepsilon_0(K) = 2 \Rightarrow \varepsilon_0(\Gamma_0^e(K)) = 2$.

Proof (i) Given $n < \infty$ let $\varepsilon_n(K) = r < \infty$. Let $\rho_1(v_1, \ldots, v_n), \ldots, \rho_r(v_1, \ldots, v_n)$ be r existential formulas which are inequivalent modulo K. Then any existential formula $\varphi(v_1, \ldots, v_n)$ must be equivalent to one of the ρ_i's, $1 \leq i \leq r$, modulo K.

Given a primitive formula $\varphi(v_1, \ldots, v_n)$ we apply 2.1.21 to find primitive formulas $\varphi_0(v_1, \ldots, v_n), \ldots, \varphi_s(v_1, \ldots, v_n)$ such that for $\mathbf{A} \in \Gamma_0^e(K)$, $\vec{f} \in \mathrm{A}$,

$$\mathbf{A} \models \varphi(\vec{f}) \quad \text{iff} \quad \begin{cases} [\![\varphi_0(\vec{f})]\!] = X \\ \text{and} \\ [\![\varphi_i(\vec{f})]\!] \neq \emptyset \qquad 1 \leq i \leq s. \end{cases}$$

Then we can find $\psi_0, \ldots, \psi_s$, each from the set $\{\rho_1, \ldots, \rho_r\}$, such that

$$\mathbf{A} \models \varphi(\vec{f}) \quad \text{iff} \quad \begin{cases} [\![\psi_0(\vec{f})]\!] = X \\ \text{and} \\ [\![\psi_i(\vec{f})]\!] \neq \emptyset \qquad 1 \leq i \leq s. \end{cases}$$

Since there are only r of the ρ_i's, it follows that, modulo $\Gamma_0^e(K)$, there are at most $r2^{r-1}$ inequivalent primitive $\varphi(v_1, \ldots, v_n)$, and hence at most $2^{r2^{r-1}}$ inequivalent existential $\varphi(v_1, \ldots, v_n)$ modulo $\Gamma_0^e(K)$. Thus

$$\varepsilon_n(\Gamma_0^e(K)) \leq 2^{r2^{r-1}} < \infty.$$

For part (ii) suppose $\varepsilon_0(K) = 2$, that is every existential sentence holds for all or no members of K, that is, K is *existentially complete.* Then using 2.1.21 again we see that every primitive sentence φ holds for all or no members of $\Gamma_0^e(K)$ since we have, for suitable primitive sentences $\varphi_0, \ldots, \varphi_s$, for any $\mathbf{A} \in \Gamma_0^e(K)$,

$$\begin{aligned} \mathbf{A} \models \varphi \quad &\text{iff} \quad \begin{cases} [\![\varphi_0]\!] = X \\ \text{and} \\ [\![\varphi_i]\!] \neq \emptyset \qquad 1 \leq i \leq s \end{cases} \\ &\text{iff} \quad \begin{cases} K \models \varphi_0 \\ \text{and} \\ K \models \varphi_i \quad 1 \leq i \leq s. \end{cases} \end{aligned}$$

Thus $\Gamma_0^e(K) \models \varphi$ if $\mathbf{A} \models \varphi$ for any $\mathbf{A} \in \Gamma_0^e(K)$. From this it immediately follows that $\Gamma_0^e(K)$ is existentially complete, so $\varepsilon_0(\Gamma_0^e(K)) = 2$. $\square$

A CLASS WITH FEW EXISTENTIAL FORMULAS GENERATES A UNIVERSAL HORN CLASS WITH MODEL COMPANION ...

Theorem 2.2.2 *Let K be a class of structures with a nontrivial member and with few existential formulas. Then*

(1) $\mathrm{ISP}_r(K)^{mc}$ *exists and has few formulas,*

(1′) *if the language is countable, then* $\mathrm{ISP}_r(K)^{mc}$ *is almost* $\aleph_0$*-categorical, and*

(2) *if the language is countable, then the following are equivalent:*

  (a) $\mathrm{ISP}_r(K)^{mc}$ *is complete*
  (b) $\mathrm{ISP}_r(K)^{mc}$ *is* $\aleph_0$*-categorical and complete*
  (c) $\mathrm{ISP}_r(K)^{mc}$ *has a prime model*
  (d) $\mathrm{ISP}_r(K)$ *has the* JEP
  (e) *there is a single structure* $\mathbf{A}$ [*which is finite if* K *consists of finitely many finite structures*] *with few existential formulas such that* $\mathrm{ISP}_r(K) = \mathrm{ISP}_r(\mathbf{A})$, *and*

(3) *if* $\pi_n(\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(K))$ *and* $\mathrm{Th}_{\forall\exists}(K)$ *are both recursive* [*primitive recursive*] *then*

  (a) *there is an effective* [*primitive recursive*] *procedure to find, for any formula* φ, *an existential* (*or universal*) *formula* φ_{mc} *such that*

$$\mathrm{ISP}_r(K)^{mc} \models \varphi \leftrightarrow \varphi_{mc}$$

  (b) $\mathrm{ISP}_r(K)^{mc}$ *has a decidable* [*primitive recursive*] *theory.*

Proof Let $K_1 = \Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(K)$. By 2.2.1 we see that K_1 has few existential formulas, and by 2.1.31, $\mathrm{ISP}_r(K) = \mathrm{IS}(K_1)$. Thus (1) and (1′) as well as (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) of (2) follow from 1.10.10. To see that (e) is equivalent to the other parts of (2), first suppose (a)–(d) hold. Then we can choose $\mathbf{A}$ to be the countable member of $\mathrm{ISP}_r(K)^{mc}$ [if K consists of finitely many structures, use (d) to find a finite $\mathbf{A}$ in $\mathrm{ISP}_r(K)$ such that each member of K can be embedded in $\mathbf{A}$]. Thus (a)–(d) $\Rightarrow$ (e). Conversely, if (e) holds, then $\varepsilon_0(\mathbf{A}) = 2$ implies $\varepsilon_0(\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(\mathbf{A})) = 2$, and as $\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(\mathbf{A})$ has few existential formulas, it follows from 1.10.12 that (a)–(d) hold.

Since $\pi_n(\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(K))$ is recursive [primitive recursive], for part (3), in view of 1.10.16, we only need to show $\mathrm{Th}_{\forall\exists}(\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(K))$ is decidable [primitive recursive]. We do this as follows. Since atomless Boolean algebras have a primitive recursive decision procedure (using elimination of quantifiers), we can apply Comer's version of the Feferman-Vaught Theorem for Γ^e along with the fact that $\mathrm{Th}_{\forall\exists}(K)$ is decidable [primitive recursive] to show that indeed $\mathrm{Th}_{\forall\exists}(\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(K))$ is decidable [primitive recursive]. $\square$

Problem 2. If $\pi_n(K)$ is recursive and $\mathrm{Th}_{\forall\exists}(K)$ is recursive, does it follow that $\pi_n(\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(K))$ is recursive?

Remark Rings provide a simple example of a class K such that K has few existential formulas but $\mathrm{ISP}_r(K)^{mc}$ is not $\aleph_0$-categorical, namely let $K = \{\mathbb{Z}_2, \mathbb{Z}_3\}$.

Remark Theorem 2.2.2 and the results based on discriminator varieties which we will study later are the *only* general results we know which say that if a class K has certain properties, then $\mathrm{ISP}_r(K)$ will have a model companion. Several obvious directions of inquiry, such as assuming K has a model companion, lead to counterexamples (such as K being the class of linearly ordered groups—see 2.12.6).

FINITELY GENERATED UNIVERSAL HORN CLASSES HAVE MODEL COMPANIONS . . .

Theorem 2.2.3 *Let K be a nonempty finite set of finite structures with a nontrivial member. Then*

(1) $\mathrm{ISP}(K)^{mc}$ *exists and has few formulas,*

(1′) $\mathrm{ISP}(K)^{mc}$ *is almost $\aleph_0$-categorical,*

(2) *the following are equivalent:*

- (a) $\mathrm{ISP}(K)^{mc}$ *is complete*
- (b) $\mathrm{ISP}(K)^{mc}$ *is $\aleph_0$-categorical and complete*
- (c) $\mathrm{ISP}(K)^{mc}$ *has a prime model*
- (d) $\mathrm{ISP}(K)$ *has the* JEP
- (e) *there is a single finite structure* $\mathbf{A}$ *such that* $\mathrm{ISP}(K) = \mathrm{ISP}(\mathbf{A})$, *and*

(3) *if the language is finite, then*

- (a) *there is a primitive recursive procedure to find, for any formula φ, an existential (or universal) formula φ_{mc} such that*

$$\mathrm{ISP}(K)^{mc} \models \varphi \leftrightarrow \varphi_{mc}$$

- (b) $\mathrm{ISP}(K)^{mc}$ *has a decidable (indeed primitive recursive) theory.*

Proof Parts (1), (1′) and (2) are immediate from 2.2.2. For (3) it suffices, in view of 2.2.2, to verify that $\pi_n(\Gamma_0^e(K))$ is primitive recursive. For this it suffices to show that there is a primitive recursive function $f(n)$ such that for every $\varphi \in \mathrm{P}_n$ there is a $\psi \in \mathrm{P}_n$ with *length* $(\psi) \leq f(n)$ such that φ is equivalent to ψ modulo $\Gamma_0^e(K)$—for then we can apply 2.1.21 to determine the number of inequivalent members of P_n modulo $\Gamma_0^e(K)$ with length $\leq f(n)$, and this gives $\pi_n(\Gamma_0^e(K))$.

We proceed as follows. First let P_n^* be the set of primitive formulas in P_n such that each atomic subformula is in one of the following forms:

variable = variable	
variable = constant	
constant = variable	
constant = constant	
r(variables)	r a fundamental relation symbol
f(variables) = variable	f a fundamental operation symbol
variable = f(variables)	f a fundamental operation symbol

We refer to these as *minimal* atomic formulas. One can easily show any member of P_n is equivalent to a formula in P_n^*. Let

$$|K| = N$$
$$\mathrm{Max}\{|\mathrm{A}| : A \in K\} = M.$$

Certainly

$$\pi_n(K) \leq (2^{M^n})^N \tag{2.1}$$

as each member of K has at most 2^{M^n} n-ary relations on it. Let $\varphi(\vec{v}) \in \mathrm{P}_n^*$, say

$$\varphi(\vec{v}) = \exists \vec{u} \left(\varphi^+(\vec{u}, \vec{v}) \wedge \bigwedge_{1 \leq i \leq s} \neg\, \alpha_i(\vec{u}, \vec{v}) \right).$$

Let

$$\varphi_0(\vec{v}) = \exists \vec{u} \varphi^+(\vec{u}, \vec{v})$$
$$\varphi_i(\vec{v}) = \exists \vec{u} \left(\varphi^+(\vec{u}, \vec{v}) \wedge \neg\, \alpha_i(\vec{u}, \vec{v}) \right) \quad 1 \leq i \leq s.$$

If $\varphi_i(\vec{v}) \sim_K \varphi_j(\vec{v})$, for some $1 \leq i,\ j \leq s$, then for $\vec{f}$ from $\mathbf{A} \in \Gamma_0^e(K)$,

$$[\![\varphi_i(\vec{f})]\!] \neq \emptyset \Leftrightarrow [\![\varphi_j(\vec{f})]\!] \neq \emptyset.$$

Thus by Macintyre's Theorem we can assume

$$s \leq \pi_n(K) - 1. \tag{2.2}$$

Now we want to show we can bound the number of bound variables $\vec{u} = u_1, \ldots, u_m$. For $\vec{a}$ an n-tuple from $\mathbf{A} \in K$ and $i \in \{0, \ldots, s\}$ such that $\mathbf{A} \models \varphi_i(\vec{a})$ let

$$\lambda_{i,\vec{a}} : \{u_1, \ldots, u_m\} \rightarrow \mathrm{A}$$

be an assignment such that $\lambda_{i,\vec{a}}\vec{u}$ witnesses $\mathbf{A} \models \varphi_i(\vec{a})$. As there are at most M^n n-tuples $\vec{a}$ from each $\mathbf{A} \in K$ it follows that there are at most

$$T = \pi_n(K) \times N \times M^n$$

maps $\lambda_{i,\vec{a}}$. Let

$$\theta = \bigcap\{\ker\lambda_{i,\vec{a}} :\ 0 \le i \le s,\ \vec{a} \in \mathrm{A}^n,\ \mathbf{A} \in K\}.$$

Then θ has at most M^T cosets. Now let

$$\begin{aligned}\widehat{\varphi}_0(\vec{v}) &= \exists\vec{u}\left(\bigwedge_{\langle u_j,u_k\rangle\in\theta} u_j = u_k \wedge \varphi^+(\vec{u},\vec{v})\right)\\ \widehat{\varphi}_i(\vec{v}) &= \exists\vec{u}\left(\bigwedge_{\langle u_j,u_k\rangle\in\theta} u_j = u_k \wedge \varphi^+(\vec{u},\vec{v}) \wedge \neg\,\alpha_i(\vec{u},\vec{v})\right), \quad 1 \le i \le s.\end{aligned}$$

Then

$$K \models \widehat{\varphi}_i(\vec{v}) \leftrightarrow \varphi_i(\vec{v}), \qquad 0 \le i \le s, \tag{2.3}$$

since clearly

$$K \models \widehat{\varphi}_i(\vec{v}) \rightarrow \varphi_i(\vec{v}),$$

and if $\mathbf{A} \in K,\ \vec{a} \in \mathrm{A}^n$ with $\mathbf{A} \models \varphi_i(\vec{a})$, then using $\lambda_{i,\vec{a}}$ to find witnesses for the u_j's we see that $\mathbf{A} \models \widehat{\varphi}_i(\vec{a})$ as $\theta \subseteq \ker\lambda_{i,\vec{a}}$. Hence

$$K \models \varphi_i(\vec{v}) \rightarrow \widehat{\varphi}_i(\vec{v}).$$

Let $\vec{w}$ be a sequence of M^T variables, and let

$$\lambda : \{u_1, \ldots, u_m\} \rightarrow \{w_1, \ldots, w_{M^T}\}$$

be a mapping with $\ker\lambda = \theta$. Then let

$$\begin{aligned}\varphi_0^*(\vec{v}) &= \exists\vec{w}\varphi^+(\lambda\vec{u},\vec{v})\\ \varphi_i^*(\vec{v}) &= \exists\vec{w}\left(\varphi^+(\lambda\vec{u},\vec{v}) \wedge \neg\,\alpha_i(\lambda\vec{u},\vec{v})\right), \quad 1 \le i \le s.\end{aligned}$$

From (2.3) it follows that

$$K \models \varphi_i^*(\vec{v}) \leftrightarrow \varphi_i(\vec{v}), \quad 0 \le i \le s,$$

so by Macintyre's Theorem

$$\Gamma_0^e(K) \models \varphi(\vec{v}) \leftrightarrow \exists\vec{w}\left(\varphi^+(\lambda\vec{u},\vec{v}) \wedge \bigwedge_{1\le i\le s} \neg\,\alpha_i(\lambda\vec{u},\vec{v})\right).$$

Thus every member of P_n is equivalent modulo $\Gamma_0^{\mathrm{e}}(K)$ to a member of P_n^* with at most M^T bound variables.

Now it is straightforward to show that there is a primitive recursive $f(n)$ such that every member of P_n is equivalent to some member of P_n^* of length $\leq f(n)$. $\square$

Applications

(1) Posets have a complete primitive recursive $\aleph_0$-categorical model companion.

(2) Semilattices have a complete primitive recursive $\aleph_0$-categorical model companion.

(3) [Wheeler] n-colorable graphs have a complete primitive recursive $\aleph_0$-categorical model companion.

(4) A finitely generated congruence distributive variety has a primitive recursive model companion.

Problem 3. Does every finitely generated variety have a model companion?

From 2.2.2 it is clear that we would like to know when $\mathrm{ISP}_r(K)$ has the JEP. We give a sufficient condition in the next lemma.

Lemma 2.2.4 *K has the* JEP $\Rightarrow$ $\mathrm{ISP}(K)$ *has the* JEP.

Proof It suffices to show $\mathrm{P}(K)$ has the JEP. Let $\mathbf{A}_i \in K$ for $i \in I$, $\mathbf{B}_j \in K$ for $j \in J$. Then, for $\langle i, j\rangle \in I \times J$ and $\mathbf{C}_{ij} \in K$, choose embeddings

$$\begin{aligned} \alpha_{ij} : \mathbf{A}_i &\hookrightarrow \mathbf{C}_{ij} \\ \beta_{ij} : \mathbf{B}_j &\hookrightarrow \mathbf{C}_{ij}. \end{aligned}$$

Then

$$\alpha : \prod_{i \in I} \mathbf{A}_i \rightarrow \prod_{\langle i,j\rangle \in I \times J} \mathbf{C}_{ij}$$

defined by

$$\alpha(f)(\langle i, j\rangle) = \alpha_{ij}(f(i))$$

is an embedding; and likewise

$$\beta : \prod_{j \in J} \mathbf{B}_j \rightarrow \prod_{\langle i,j\rangle \in I \times J} \mathbf{C}_{ij}$$

defined by

$$\beta(f)(\langle i, j\rangle) = \beta_{ij}(f(j))$$

is an embedding. $\square$

UNIVERSAL HORN CLASSES GENERATED BY COMPLETE $\aleph_0$-CATEGORICAL CLASSES HAVE MODEL COMPANIONS ...

Theorem 2.2.5 *If* ThK) *is complete* $\aleph_0$*-categorical with a countable language, then* $\mathrm{ISP}_r(K)^{mc}$ *exists, it is complete* $\aleph_0$*-categorical, and if* $\pi_n(\Gamma_0^e \mathrm{S}^{(\preceq)}\mathrm{P}_u(K))$ *and* $\mathrm{Th}_{\forall\exists}(K)$ *are both recursive [primitive recursive], then*

(a) *there is an effective [primitive recursive] procedure to find, for any formula* φ*, an existential (or universal) formula* φ_{mc} *such that*

$$\mathrm{ISP}_r(K)^{mc} \models \varphi \leftrightarrow \varphi_{mc}, \quad \textit{and}$$

(b) $\mathrm{ISP}_r(K)^{mc}$ *has a decidable [primitive recursive] theory.*

Proof Immediate from 2.2.2. $\square$

WORRYING ABOUT THE ROLE OF 1-ELEMENT STRUCTURES

Given a class of structures K, let K_- denote the subclass of nontrivial members.

Lemma 2.2.6 *For* K *an elementary class of structures, we have*

$$(K_-)^{ec} = K^{ec} \cap K_-.$$

Proof Certainly $(K_-)^{ec} \supseteq K^{ec} \cap K_-$. Also it is easy to verify that $(K_-)^{ec} \subseteq K^{ec}$. $\square$

Lemma 2.2.7 *For* K *an elementary class of algebras, either both of* K^{ec} *and* $(K_-)^{ec}$ *are elementary classes, or neither.*

Proof For algebras we have

$$K^{ec} = \begin{cases} (K_-)^{ec} \\ \text{or} \\ (K_-)^{ec} \cup \{\text{trivial algebras}\}. \end{cases}$$

The result follows from this. $\square$

Corollary 2.2.8 *For* V *a variety,* V *has a model companion iff* V_- *has a model companion. If the model companion exists then*

$$(V_-)^{mc} = V^{mc} \cap V_-.$$

Lemma 2.2.9 *Let V be a variety and let $K \subseteq V$. If every nontrivial finitely generated subdirectly irreducible algebra in V can be embedded in a member of K, then*

$$\mathrm{ISP}_r(K) = \begin{cases} V \\ \text{or} \\ V_-. \end{cases}$$

Proof Every algebra in V can be embedded in an ultraproduct of finitely generated members of V (see [9, V §2.18]), and finitely generated members of V can be embedded in products of finitely generated subdirectly irreducible members of V. □

MODEL COMPANIONS OF VARIETIES . . .

Theorem 2.2.10 *Let V be a nontrivial variety and let $K \subseteq V$ be such that every nontrivial finitely generated subdirectly irreducible member of V can be embedded in some member of K. If K has few existential formulas, then*

(1) *V_- has a model companion $(V_-)^{mc}$ with few formulas,*

(2) *the following are equivalent if the language is countable:*

  (a) *$(V_-)^{mc}$ is complete*
  (b) *$(V_-)^{mc}$ is complete and $\aleph_0$-categorical*
  (c) *$(V_-)^{mc}$ has a prime model*
  (d) *V_- has the* JEP
  (e) *there is a single structure $\mathbf{A}$ with few existential formulas such that $\mathrm{ISP}_r(\mathbf{A})_- = V_-$, and*

(3) *if $\pi_n(\Gamma_0^e \mathrm{S}^{(\preceq)} \mathrm{P}_u(K))$ and $\mathrm{Th}_{\forall\exists}(K)$ are both recursive [primitive recursive] then*

  (a) *there is an effective [primitive recursive] procedure to find, for any formula φ, an existential (or universal) formula φ_{mc} such that*

$$(V_-)^{mc} \models \varphi \leftrightarrow \varphi_{mc}$$

  (b) *$(V_-)^{mc}$ has a decidable [primitive recursive] theory.*

Proof This follows directly from 2.2.2, 2.2.7 and 2.2.9. □

Application Nontrivial monadic algebras have a complete $\aleph_0$-categorical model companion with a primitive recursive theory.

Theorem 2.2.11 *Let V be a nontrivial variety and let $K \subseteq V$ be such that every finitely generated subdirectly irreducible member of V can be embedded in some member of K. If K has few existential formulas, then*

(1) *V has a model companion V^{mc} with few formulas,*

(2) *the following are equivalent:*

 (a) *V^{mc} is complete*

 (b) *V^{mc} is complete and $\aleph_0$-categorical*

 (c) *V^{mc} has a prime model*

 (d) *V has the* JEP

 (e) *there is a single structure* $\mathbf{A}$ *with few existential formulas such that* $\mathrm{ISP}_r(\mathbf{A}) = V$, *and*

(3) *if $\pi_n(\Gamma_0^e \mathrm{S}^{(\preceq)} \mathrm{P}_u(K))$ and $\mathrm{Th}_{\forall\exists}(K)$ are both recursive [primitive recursive], then*

 (a) *there is an effective [primitive recursive] procedure to find, for any formula φ, an existential (or universal) formula φ_{mc} such that*

$$V^{mc} \models \varphi \leftrightarrow \varphi_{mc}$$

 (b) *V^{mc} has a decidable [primitive recursive] theory.*

Proof This follows directly from 2.2.2, 2.2.7 and 2.2.9. $\square$

2.3 Discriminator varieties: the basic facts

WHAT IS A DISCRIMINATOR VARIETY?

Definition 2.3.1 A ternary function t on a set A is the *discriminator function* if

$$t(a,b,c) = \begin{cases} a & \text{if } a \neq b \\ c & \text{if } a = b. \end{cases}$$

A ternary term $t(u_1, u_2, u_3)$ is a *discriminator term* for a class of algebras K if, for each $\mathbf{A} \in K$, the term function $t^{\mathbf{A}}$ is the discriminator function on A. [A finite algebra with a discriminator term is called *quasiprimal*.] A variety of the form $\mathrm{V}(K)$, where K has a discriminator term, is said to be a *discriminator variety.*

Definition 2.3.2 The 4-ary function s defined on a set A by

$$s(a,b,c,d) \;=\; \begin{cases} c & \text{if } a=b \\ d & \text{if } a \neq b \end{cases}$$

is called the *switching function* (or *normal transform*) on A. A term $s(u_1, v_1, u_2, v_2)$ is a *switching term* for an algebra $\mathbf{A}$ if the term function $s^{\mathbf{A}} : \mathrm{A}^4 \to \mathrm{A}$ is the switching function. $s(u_1, v_1, u_2, v_2)$ is a switching term for a class of algebras K if it is a switching term for each algebra in K.

Proposition 2.3.3 *A class of algebras K has a discriminator term iff it has a switching term.*

Proof If $t(x, y, z)$ is a discriminator term for K, then

$$s(u_1, v_1, u_2, v_2) \;=:\; t\,(t(u_1, v_1, u_2), t(u_1, v_1, v_2), v_2)$$

is a switching term; and if $s(u_1, v_1, u_2, v_2)$ is a switching term for K, then

$$t(u_1, u_2, u_3) \;=:\; s(u_1, u_2, u_3, u_1)$$

is a discriminator term. $\square$

Proposition 2.3.4 *A term $t(u_1, u_2, u_3)$ is a discriminator term for a class K of algebras iff it is a discriminator term for the universal class* $\mathrm{ISP}_u(K)$ *generated by K.*

Proof This follows from the observation that a term $t(u_1, u_2, u_3)$ is a discriminator term for a class K iff

$$K \models (t(u_1, u_2, u_3) = u_4) \leftrightarrow ((u_1 = u_2 \wedge u_4 = u_3) \text{ or } (u_1 \neq u_2 \wedge u_4 = u_1))\,. \qquad \square$$

Thus when dealing with discriminator varieties $\mathrm{V}(K)$ it frequently suffices to consider those classes K which are universal classes and have a discriminator term.

Definition 2.3.5 For K a class of algebras of a given type, let K_+ denote K union the class of trivial algebras of that type.

Definition 2.3.6 The smallest and largest congruence of an algebra are denoted by Δ, respectively ∇. An algebra $\mathbf{A}$ is *simple* if it has at most two congruences, that is, $\mathrm{Con}\ \mathbf{A} = \{\Delta, \nabla\}$; it is *subdirectly irreducible* if whenever $\theta_i \in \mathrm{Con}\ \mathbf{A},\ i \in I$, then $\bigcap\limits_{i\in I} \theta_i = \Delta$ implies some $\theta_i = \Delta$; it is *finitely subdirectly irreducible* if whenever $\theta_1, \theta_2 \in \mathrm{Con}\ \mathbf{A},\ i \in I$, then $\theta_1 \cap \theta_2 = \Delta$ implies $\theta_1 = \Delta$

or $\theta_2 = \Delta$; and $\mathbf{A}$ is *directly indecomposable* if $\mathbf{A} \cong \mathbf{A}_1 \times \mathbf{A}_2$ implies $\mathbf{A}_1$ or $\mathbf{A}_2$ is a trivial algebra. Let

$$\begin{aligned}
S &= \text{the class of simple algebras}\\
K_S &= \text{the class of simple algebras in } K\\
SI &= \text{the class of subdirectly irreducible algebras}\\
K_{SI} &= \text{the class of subdirectly irreducible algebras in } K\\
FSI &= \text{the class of finitely subdirectly irreducible algebras}\\
K_{FSI} &= \text{the class of finitely subdirectly irreducible algebras in } K\\
DI &= \text{the class of directly indecomposable algebras}\\
K_{DI} &= \text{the class of directly indecomposable algebras in } K.
\end{aligned}$$

Remark We have

$$S \subseteq SI \subseteq FSI \subseteq DI.$$

THE FUNDAMENTAL STRUCTURE THEOREM

Theorem 2.3.7 *Let K be a universal class of algebras with a discriminator term. Then*

(i) $V(K) = \begin{cases} \mathrm{ISP}(K) & \textit{iff some member of } K \textit{ has a trivial subalgebra} \\ \mathrm{ISP}(K)_+ & \textit{if no member of } K \textit{ has a trivial subalgebra} \end{cases}$

(ii) $\mathrm{ISP}(K) = \mathrm{I}\Gamma^a(K)$

(iii) $K = \mathrm{ISP}(K) \cap S = \mathit{ISP}(K) \cap SI = \mathrm{ISP}(K) \cap FSI = \mathit{ISP}(K) \cap DI$

(iv) $\mathrm{V}(K)_S = K_+$.

Proof (See [9, IV §9.4].) $\square$

t-CALCULATIONS FOR V_S

Proposition 2.3.8 *Let V be a discriminator variety, t a discriminator term for V_S, and s a switching term for V_S. Then*

$$V_S \models \begin{cases}
(u_1 = v_1 \vee u_2 = v_2) \leftrightarrow s(u_1, v_1, u_2, v_2) = u_2\\
(u_1 \neq v_1 \wedge u_2 \neq v_2) \leftrightarrow s(u_1, v_1, u_2, v_2) \neq u_2\\
(u_1 = v_1 \wedge u_2 = v_2) \leftrightarrow t(u_1, v_1, u_2) = t(v_1, u_1, v_2)\\
(u_1 \neq v_1 \vee u_2 \neq v_2) \leftrightarrow t(u_1, v_1, u_2) \neq t(v_1, u_1, v_2)\\
(u_1 = v_1 \wedge u_2 \neq v_2) \leftrightarrow s(u_1, v_1, u_2, v_2) \neq v_2\\
(u_1 = v_1 \rightarrow u_2 = v_2) \leftrightarrow s(u_1, v_1, u_2, v_2) = v_2.
\end{cases}$$

Proof These are straightforward consequences of the definitions of discriminator terms and switching functions. $\square$

$\boxed{[\![\cdots]\!]\text{-CALCULATIONS FOR } \Gamma^a(V_S)}$

Proposition 2.3.9 *Let V be a discriminator variety and assume $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x\in X} \mathbf{A}_x,\ \mathbf{A}_x \in V_S$. Then for $f, g, h, k \in \mathrm{A}$ we have*

$$\begin{aligned}
[\![f = g]\!] \cup [\![h = k]\!] \ &= [\![s(f,g,h,k) = h]\!] \\
[\![f \neq g]\!] \cap [\![h \neq k]\!] \ &= [\![s(f,g,h,k) \neq h]\!] \\
[\![f = g]\!] \cap [\![h = k]\!] \ &= [\![t(f,g,h) = t(g,f,k)]\!] \\
[\![f \neq g]\!] \cup [\![h \neq k]\!] \ &= [\![t(f,g,h) \neq t(g,f,k)]\!] \\
[\![f = g]\!] \cap [\![h \neq k]\!] \ &= [\![s(f,g,h,k) \neq k]\!] \\
[\![f \neq g]\!] \cup [\![h = k]\!] \ &= [\![s(f,g,h,k) = k]\!] \\
[\![f \neq g]\!] \subseteq [\![h \neq k]\!] \ &\Leftrightarrow\ s(h,k,f,g) = g \\
[\![f \neq g]\!] \cap [\![h \neq k]\!] = \emptyset \ &\Leftrightarrow\ s(f,g,h,k) = h.
\end{aligned}$$

Proof An easy consequence of 2.3.8. For example, consider the first claim. We have

$$\begin{aligned}
x \in [\![f = g]\!] \cup [\![h = k]\!] \ &\Leftrightarrow\ (f(x) = g(x)) \vee (h(x) = k(x)) \\
&\Leftrightarrow\ s(f(x), g(x), h(x), k(x)) = h(x) \\
&\Leftrightarrow\ x \in [\![s(f,g,h,k) = h]\!]. \qquad \square
\end{aligned}$$

$\boxed{\Theta\text{-CALCULATIONS}}$

Proposition 2.3.10 *Let V be a discriminator variety and assume $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x\in X} \mathbf{A}_x,\ \mathbf{A}_x \in V_S$. Then for $f, g, h, k, h_i, k_i \in \mathrm{A}$ we have*

$$\begin{aligned}
&\text{(i)} & \Theta(h,k) \ &= \ \theta_{[\![h=k]\!]} \\
&& &= \ \{\langle f,g\rangle \in \mathrm{A}\times\mathrm{A} : [\![f \neq g]\!] \subseteq [\![h \neq k]\!]\} \\
&& &= \ \{\langle f,g\rangle \in \mathrm{A}\times\mathrm{A} : s(h,k,f,g) = g\} \\
&\text{(ii)} & \Theta(h_1,k_1) \vee \Theta(h_2,k_2) \ &= \ \theta_{[\![h_1=k_1]\!]\cap[\![h_2=k_2]\!]} \\
&& &= \ \Theta(t(h_1,k_1,h_2), t(k_1,h_1,k_2)) \\
&\text{(iii)} & \Theta(h_1,k_1) \wedge \Theta(h_2,k_2) \ &= \ \theta_{[\![h_1=k_1]\!]\cup[\![h_2=k_2]\!]} \\
&& &= \ \Theta(s(h_1,k_1,h_2,k_2), h_2) \\
&\text{(iv)} & \Theta(h_1,k_1) \subseteq \Theta(h_2,k_2) \ &\Leftrightarrow\ [\![h_1 \neq k_1]\!] \subseteq [\![h_2 \neq k_2]\!] \\
&& &\Leftrightarrow\ s(h_2,k_2,h_1,k_1) = k_1 \\
&\text{(v)} & \Theta(h_1,k_1) \wedge \Theta(h_2,k_2) = \Delta \ &\Leftrightarrow\ [\![h_1 \neq k_1]\!] \cap [\![h_2 \neq k_2]\!] = \emptyset \\
&& &\Leftrightarrow\ s(h_1,k_1,h_2,k_2) = h_2.
\end{aligned}$$

Proof

(i) Suppose $\langle f, g\rangle \in \Theta(h, k)$. Then for some principal congruence formula $\pi(u_1, v_1, u_2, v_2)$ we have

$$\mathbf{A} \models \pi(f, g, h, k);$$

hence

$$\mathbf{A}_x \models \pi(fx, gx, hx, kx) \quad \text{for all } x \in X$$

so

$$hx = kx \Rightarrow fx = gx \quad \text{for all } x \in X.$$

Thus

$$\langle f, g\rangle \in \Theta(h, k) \Rightarrow [\![f \neq g]\!] \subseteq [\![h \neq k]\!].$$

Next, by 2.3.9

$$[\![f \neq g]\!] \subseteq [\![h \neq k]\!] \Rightarrow s(h, k, f, g) = g.$$

If $s(h, k, f, g) = g$ then

$$f = s(h, h, f, g)\Theta(h, k)s(h, k, f, g) = g$$

so $\langle f, g\rangle \in \Theta(h, k)$. Thus

$$s(h, k, f, g) = g \Rightarrow \langle f, g\rangle \in \Theta(h, k).$$

So the first, third and fourth sets in part (i) are equal. Finally, by definition

$$\theta_{[\![h=k]\!]} = \{\langle f, g\rangle \in \mathrm{A} \times \mathrm{A} : [\![h = k]\!] \subseteq [\![f = g]\!]\}.$$

(ii) From

$$s(t(h_1, k_1, h_2), t(k_1, h_1, k_2), h_i, k_i) = k_i \quad \text{for } i = 1, 2$$

follows, by (i) above,

$$\langle h_i, k_i\rangle \in \Theta(t(h_1, k_1, h_2), t(k_1, h_1, k_2)) \quad \text{for } i = 1, 2$$

so

$$\Theta(h_1, k_1) \vee \Theta(h_2, k_2) \subseteq \Theta(t(h_1, k_1, h_2), t(k_1, h_1, k_2)).$$

Conversely,

$$t(h_1, k_1, h_2)\Theta(h_1, k_1)t(k_1, h_1, h_2)\Theta(h_2, k_2)t(k_1, h_1, k_2)$$

so

$$\Theta(t(h_1, k_1, h_2), t(k_1, h_1, k_2)) \subseteq \Theta(h_1, k_1) \vee \Theta(h_2, k_2).$$

Thus we have

$$\Theta(t(h_1, k_1, h_2), t(k_1, h_1, k_2)) = \Theta(h_1, k_1) \vee \Theta(h_2, k_2).$$

Finally note that

$$\begin{aligned}\Theta(t(h_1, k_1, h_2), t(k_1, h_1, k_2)) &= \theta_{[\![t(h_1,k_1,h_2)=t(k_1,h_1,k_2)]\!]}\\ &= \theta_{[\![h_1=k_1]\!]\cap[\![h_2=k_2]\!]}.\end{aligned}$$

(iii) From
$$s(h_i, k_i, s(h_1, k_1, h_2, k_2), h_2) \;=\; h_2 \quad \text{for } i = 1, 2$$
follows
$$\langle s(h_1, k_1, h_2, k_2), h_2\rangle \;\in\; \Theta(h_i, k_i) \quad \text{for } i = 1, 2$$
so
$$\Theta(s(h_1, k_1, h_2, k_2), h_2) \;\subseteq\; \Theta(h_1, k_1) \wedge \Theta(h_2, k_2).$$
Conversely, if
$$\langle f, g\rangle \;\in\; \Theta(h_1, k_1) \wedge \Theta(h_2, k_2)$$
then
$$\begin{aligned}[\![f \neq g]\!] &\subseteq [\![h_1 \neq k_1]\!] \cap [\![h_2 \neq k_2]\!] \\ &= [\![s(h_1, k_1, h_2, k_2) \neq h_2]\!].\end{aligned}$$
Thus by (i)
$$\Theta(h_1, k_1) \wedge \Theta(h_2, k_2) \;\subseteq\; \Theta(s(h_1, k_1, h_2, k_2), h_2),$$
and this gives
$$\Theta(h_1, k_1) \wedge \Theta(h_2, k_2) \;=\; \Theta(s(h_1, k_1, h_2, k_2), h_2).$$

(iv) follows immediately from (i) as $\Theta(h_1, k_1) \subseteq \Theta(h_2, k_2)$ iff $\langle h_1, k_1\rangle \in \Theta(h_2, k_2)$.

(v) We know from (iii) that
$$\Theta(h_1, k_1) \wedge \Theta(h_2, k_2) \;=\; \Theta(s(h_1, k_1, h_2, k_2), h_2),$$
thus
$$\begin{aligned}\Theta(h_1, k_1) \wedge \Theta(h_2, k_2) \;=\; \Delta &\text{ iff } s(h_1, k_1, h_2, k_2) = h_2 \\ &\text{ iff } [\![h_1 \neq k_1]\!] \cap [\![h_2 \neq k_2]\!] = \emptyset. \qquad \square\end{aligned}$$

CONGRUENCES ARE DETERMINED BY CLOSED SETS

Proposition 2.3.11 *Let V be a discriminator variety and assume $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x \in X} \mathbf{A}_x$, $\mathbf{A}_x \in V_S$. Then for $\theta \in \operatorname{Con} \mathbf{A}$ there is a closed subset Y of X such that $\theta = \theta_Y$. Consequently the maximal congruences of $\mathbf{A}$ are precisely the stalk congruences corresponding to nontrivial stalks.*

Proof Let
$$Y = \bigcap_{\langle f, g\rangle \in \theta} [\![f = g]\!].$$
Then certainly $\theta \subseteq \theta_Y$, for if $\langle f, g\rangle \in \theta$ we have $Y \subseteq [\![f = g]\!]$.

Conversely suppose $f, g \in \mathrm{A}$ and $Y \subseteq [\![f = g]\!]$. By compactness there are $\langle f_i, g_i\rangle \in \theta,\ 1 \le i \le n$, such that

$$\bigcap_{1\le i\le n} [\![f_i = g_i]\!] \subseteq [\![f = g]\!].$$

But then

$$\langle f, g\rangle \in \theta_{\bigcap\limits_{1\le i\le n} [\![f_i = g_i]\!]} = \bigvee_{1\le i\le n} \Theta(f_i, g_i) \quad \text{by 2.3.10(ii)}$$

so $\langle f, g\rangle \in \theta$. This shows $\theta = \theta_Y$. $\square$

EXTENDING/RESTRICTING CONGRUENCES

Corollary 2.3.12 *Let K be a class of algebras with a discriminator term. Then*

(i) $\mathrm{V}(K)$ *has the* CEP (*congruence extension property*)

(ii) *every member of* $\Gamma^a(K)$ *has the* SCRP *and* SCEP *with respect to* $\Gamma^a(K)$; *hence*

$$\Gamma^a(K)_{\mathrm{SCRP}} = \Gamma^a(K)_{\mathrm{SCEP}} = \Gamma^a(K).$$

Proof

(i) Let $\mathbf{A} \le \mathbf{B} \in V$, V a discriminator variety. If $\theta \in \mathrm{Con}\ \mathbf{A}$ let $\theta^* \in \mathrm{Con}\ \mathbf{B}$ be the congruence of $\mathbf{B}$ generated by θ, that is, $\theta^* = \Theta_{\mathrm{B}}(\theta)$. Let $\langle a, b\rangle \in \theta^*|_{\mathbf{A}}$. As $\Theta_{\mathbf{B}}$ is an algebraic closure operator on Con $\mathbf{B}$ there are $\langle a_i, b_i\rangle \in \theta,\ 1 \le i \le n$ such that

$$\langle a, b\rangle \in \Theta_{\mathbf{B}}(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle)$$

and thus

$$\langle a, b\rangle \in \Theta_{\mathbf{B}}(\langle a_1, b_1\rangle) \vee \cdots \vee \Theta_{\mathbf{B}}(\langle a_n, b_n\rangle).$$

From 2.3.10(ii) we then see that there are elements $c, d \in \mathrm{A}$ such that

$$\langle a, b\rangle \in \Theta_{\mathbf{B}}(c, d).$$

But then by 2.3.10(i)

$$s(c, d, a, b) = b,$$

so again by 2.3.10(i) we have

$$\langle a, b\rangle \in \Theta_{\mathbf{A}}(c, d).$$

Thus $\theta^*|_{\mathbf{A}} = \theta$.

(ii) Let $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ be an embedding with $\mathbf{A}, \mathbf{B} \in \Gamma^a(K)$. Let θ be a stalk congruence of $\mathbf{B}$. Then $\mathbf{B}/\theta \in V_S$, and as V_S is universal, $\alpha\mathbf{A}/\theta|_{\alpha\mathbf{A}} \in V_S$ as well. Thus by 2.3.11 $\alpha^{-1}(\theta)$ is a stalk congruence of $\mathbf{A}$, or equals ∇_{A}. Hence $\Gamma^a(K)$ has the SCRP.

Now let $\theta \in \mathrm{Con}\ \mathbf{A}$ be a stalk congruence $\kappa_x^{\mathbf{A}}$ of $\mathbf{A}$ with $\mathbf{A}_x$ nontrivial, and let $\theta^* = \Theta_{\mathbf{B}}(\alpha(\theta))$. As $\mathrm{V}(K)$ is semisimple by 2.3.4 and 2.3.7, it follows that $\theta^* \subseteq \mu$ for some μ which is a maximal congruence of $\mathbf{B}$. Then $\alpha(\theta) \subseteq \mu|_{\alpha\mathbf{A}}$. As $\mathbf{A}/\mu|_{\alpha\mathbf{A}}$ is simple, $\alpha(\theta) = \mu|_{\alpha\mathbf{A}}$. Thus we have the SCEP for $\Gamma^a(K)$. $\Box$

$\boxed{\text{TRANSFER OF THE AMALGAMATION PROPERTY/JOINT EMBEDDING PROPERTY}}$

Proposition 2.3.13 *Let K be a universal class with a discriminator term. Then*

(i) K *has the* JEP $\Rightarrow$ $\mathrm{ISP}(K)$ *has the* JEP

(i′) K_+ *has the* JEP $\Rightarrow$ $\mathrm{V}(K)$ *has the* JEP

(ii) $\mathrm{V}(K)$ *has the* AP $\Leftrightarrow$ K_- *has the* AP.

Proof (i) and (i′) are consequences of 2.2.4 [in (i′) note that $V(K) = \mathrm{ISP}(K_+)$].

(ii) ($\Rightarrow$) Given that $V(K)$ has the AP, let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in K_-$ with

$$\begin{aligned} \mathbf{A} &\leq \mathbf{B} \\ \mathbf{A} &\leq \mathbf{C}. \end{aligned}$$

We can assume that $\mathrm{A} = \mathrm{B} \cap \mathrm{C}$. Choose $\mathbf{D} \in V(K)$ such that $\mathbf{B}, \mathbf{C} \leq \mathbf{D}$. As $\mathbf{D}$ is semisimple and $\mathbf{A}$ is simple, there is a maximal congruence θ of $\mathbf{D}$ with $\theta|_{\mathbf{A}} = \Delta_{\mathbf{A}}$. As $\mathbf{A}$ is nontrivial and $\mathbf{B}, \mathbf{C}$ are simple, we have

$$\begin{aligned} \theta|_{\mathbf{B}} &= \Delta_{\mathbf{B}} \\ \theta|_{\mathbf{C}} &= \Delta_{\mathbf{C}}. \end{aligned}$$

Then

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\iota_{\mathbf{B}}"] \arrow[d, "\iota_{\mathbf{C}}"'] & \mathbf{B} \arrow[d] \\ \mathbf{C} \arrow[r] & \mathbf{D}/\theta \end{tikzcd}$$

Figure 1

commutes, where $\iota_{\mathbf{B}}$ and $\iota_{\mathbf{C}}$ are inclusion maps, and the unlabelled maps are the natural maps. Thus K_- has the AP.

($\Leftarrow$) By 2.3.7 $V(K) = I\Gamma^a(K_+)$. So let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \Gamma^a(K_+)$ and let

$$\begin{aligned} \alpha &: \mathbf{A} \hookrightarrow \mathbf{B} \\ \beta &: \mathbf{A} \hookrightarrow \mathbf{C} \end{aligned}$$

be embeddings. If $\mathbf{A}$ is trivial then we can find embeddings α^*, β^* such that

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathbf{B} \arrow[d, "\alpha^*"] \\ \mathbf{C} \arrow[r, "\beta^*"] & \mathbf{B} \times \mathbf{C} \end{tikzcd}$$

Figure 2

commutes. So suppose $\mathbf{A}$ is nontrivial. For θ a maximal congruence on $\mathbf{A}$ choose extensions $\theta_{\mathbf{B}} \in \operatorname{Con} \mathbf{B}$, $\theta_{\mathbf{C}} \in \operatorname{Con} \mathbf{C}$ of $\alpha(\theta)$, respectively $\beta(\theta)$, such that $\theta_{\mathbf{B}}$ and $\theta_{\mathbf{C}}$ are maximal congruences (use the SCEP). Then we have natural embeddings

$$\begin{aligned} \alpha_\theta &: \mathbf{A}/\theta \hookrightarrow \mathbf{B}/\theta_{\mathbf{B}} \\ \beta_\theta &: \mathbf{A}/\theta \hookrightarrow \mathbf{C}/\theta_{\mathbf{C}} \end{aligned}$$

such that the following diagram commutes (the unlabelled maps are the quotient maps):

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\alpha"] \arrow[dr] \arrow[dd, "\beta"'] & \mathbf{B} \arrow[dr] & \\ & \mathbf{A}/\theta \arrow[r, "\alpha_\theta"'] \arrow[dd, "\beta_0"] & \mathbf{B}/\theta_{\mathbf{B}} \\ \mathbf{C} \arrow[dr] & & \\ & \mathbf{C}/\theta_{\mathbf{C}} & \end{tikzcd}$$

Figure 3

Since $\mathbf{A}/\theta,\ \mathbf{B}/\theta_{\mathbf{B}},\ \mathbf{C}/\theta_{\mathbf{C}} \in K_-$, it follows from the AP for K_- that we can choose $D_\theta \in K_-$ and embeddings $\widehat{\alpha}_\theta, \widehat{\beta}_\theta$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, "\alpha"] \arrow[dr] \arrow[dd, "\beta"'] & & \mathbf{B} \arrow[dr] & \\
& \mathbf{A}/\theta \arrow[rr, "\alpha_\theta"'] \arrow[dd, "\beta_\theta"] & & \mathbf{B}/\theta_{\mathbf{B}} \arrow[dd, "\widehat{\alpha}_\theta"] \\
\mathbf{C} \arrow[dr] & & & \\
& \mathbf{C}/\theta_{\mathbf{C}} \arrow[rr, "\widehat{\beta}_\theta"'] & & \mathbf{D}_\theta
\end{tikzcd}$$

Figure 4

So taking products we have the commuting diagram:

$$\begin{tikzcd}
\mathbf{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{B} \arrow[dr] & \\
& \prod_\theta \mathbf{A}/\theta \arrow[rr] \arrow[dd] & & \prod_\theta \mathbf{B}/\theta_{\mathbf{B}} \arrow[dd] \\
\mathbf{C} \arrow[dr] & & & \\
& \prod_\theta \mathbf{C}/\theta_{\mathbf{C}} \arrow[rr] & & \prod_\theta \mathbf{D}_\theta
\end{tikzcd}$$

Figure 5

and this shows $V(K)$ has the AP. $\square$

2.4 Elementary subclasses of a discriminator variety

$\boxed{\text{REDUCING A MATRIX TO A BASIC FORMULA—FOR THE STALKS}}$

Proposition 2.4.1 *Let K be a class of algebras with a discriminator term t and a switching term s. Then every positive open formula is equivalent, modulo K, to an atomic formula; and every nonpositive open formula is equivalent, modulo K, to a basic formula.*

Proof This is immediate from Proposition 2.3.8. $\square$

REDUCING THE MATRIX TO AN ATOMIC FORMULA—FOR NONTRIVIAL STALKS

Proposition 2.4.2 *For* $\mathbf{A}$ *an algebra with a switching term* s, *and for* $a, b \in \mathrm{A}$,

$$\mathbf{A} \models \forall u_2 \forall v_2 (s(a, b, u_2, v_2) = v_2) \quad \textit{iff} \quad \begin{cases} a \neq b \\ \textit{or} \\ \mathbf{A} \textit{ is trivial.} \end{cases}$$

Proof $(\Leftarrow)$ is obvious. If $\mathbf{A}$ is trivial, then $(\Rightarrow)$ is certainly true. So suppose $\mathbf{A}$ is not trivial. Then $s(a, b, c, d) = d$ holds for all $c, d \in \mathrm{A}$ implies $a \neq b$. $\square$

Proposition 2.4.3 *For* $\mathbf{A}$ *an algebra with a switching term* s, *we have, for any terms* $p(\vec{w}, \vec{z})$, $q(\vec{w}, \vec{z})$, *and* $\vec{a} \in \mathrm{A}$

$$\mathbf{A} \models \forall u \forall v \vec{\mathbf{Q}} \vec{w} (s(p(\vec{w}, \vec{a}), q(\vec{w}, \vec{a}), u, v) = v) \quad \textit{iff} \quad \begin{cases} \mathbf{A} \models \vec{\mathbf{Q}} \vec{w} (p(\vec{w}, \vec{a}) \neq q(\vec{w}, \vec{a})) \\ \textit{or} \\ \mathbf{A} \textit{ is trivial.} \end{cases}$$

Proof (Similar to the proof of 2.4.2.) $\square$

Definition 2.4.4 Let t be a 3-ary term, s a 4-ary term. For σ a formula define the formula $\sigma_{s,t}$ by first putting σ in prenex form and then reducing the matrix of σ to an atomic or negated atomic formula as in the first proposition above; and if the matrix is negated atomic, then replace it as in the left side of the previous proposition.

If Σ is a collection of formulas, let $\Sigma_{s,t}$ denote the set of $\sigma_{s,t}$ for $\sigma \in \Sigma$.

Proposition 2.4.5 *Let* $\mathbf{A}$ *be an algebra with a discriminator term* t *and switching term* s, *and let* Σ *be a collection of formulas. Then*

$$\mathbf{A} \models \Sigma_{s,t} \Leftrightarrow \begin{cases} \mathbf{A} \models \Sigma \\ \textit{or} \\ \mathbf{A} \textit{ is trivial.} \end{cases}$$

Proof Immediate from 2.4.3. $\square$

Definition 2.4.6 Given a 4-ary term s, let us define the following formulas:

$$\begin{aligned}
\varphi_{(\subseteq)}(u_1, u_2, v_1, v_2) &=: s(v_1, v_2, u_1, u_2) = u_2 \\
\varphi_{(\emptyset)}(u_1, u_2, v_1, v_2) &=: s(u_1, u_2, v_1, v_2) = v_1 \\
\varphi_{nontriv} &=: \exists u_1 \exists u_2 \forall v_1 \forall v_2 (\varphi_{(\subseteq)}(v_1, v_2, u_1, u_2)) \\
\varphi_{atomless} &=: \forall u_1 \forall u_2 (u_1 \neq u_2 \rightarrow \exists v_1 \exists v_2 (v_1 \neq v_2 \wedge \varphi_{(\subseteq)}(v_1, v_2, u_1, u_2) \\
&\qquad \wedge \neg \varphi_{(\subseteq)}(u_1, u_2, v_1, v_2))).
\end{aligned}$$

Proposition 2.4.7 *For K a class of algebras with a discriminator term t and switching term s, for $\mathbf{A} \in \Gamma^a(K)$, and for $f, g, h, k \in \mathrm{A}$, we have:*

$$\begin{aligned}
&\mathbf{A} \models \varphi_{(\subseteq)}(f,g,h,k) \Leftrightarrow [\![f \neq g]\!] \subseteq [\![h \neq k]\!]\\
&\mathbf{A} \models \varphi_{(\emptyset)}(f,g,h,k) \Leftrightarrow [\![f \neq g]\!] \cap [\![h \neq k]\!] = \emptyset\\
&\mathbf{A} \models \varphi_{nontriv} \Leftrightarrow \textit{all stalks of } \mathbf{A} \textit{ are nontrivial (for } \mathbf{A} \textit{ in standard form)}\\
&\qquad\qquad\qquad \textit{or } \mathbf{A} \textit{ is trivial}\\
&\mathbf{A} \models \varphi_{atomless} \Leftrightarrow X(\mathbf{A})^* \textit{ is atomless (for } \mathbf{A} \textit{ in standard form) or } \mathbf{A} \textit{ is trivial.}
\end{aligned}$$

Proof The first two follow immediately from the last two assertions of 2.3.9. For the third claim above, note that if $\mathbf{A}$ is nontrivial and in standard form, then one can find $f, g \in \mathbf{A}$ such that $[\![f \neq g]\!] = X$ iff there is no trivial stalk of $\mathbf{A}$. For the last claim observe that, for $\mathbf{A}$ nontrivial and in standard form, every nonempty difference set $[\![f \neq g]\!]$ of $\mathbf{A}$ properly contains another nonempty difference set $[\![h \neq k]\!]$ iff $X(\mathbf{A}^*)$ is atomless. □

THE TRANSFER OF ELEMENTARY CLASSES

Proposition 2.4.8 *For K an $\forall\exists$ class with defining sentences Σ, and with a discriminator term t and switching term s, let Σ_V be a set of identities defining $V(K)$. Then we have*

(a) $\mathrm{I}\Gamma^a(K)$ *is a Horn class axiomatized by*

  (i) $\Sigma_V \cup \Sigma_{s,t}$ *if K contains a trivial algebra*

  (ii) $\Sigma_V \cup \Sigma_{s,t} \cup \{\exists u \exists v\ u \neq v\}$ *if no member of K has a trivial subalgebra, and*

  (iii) $\Sigma_V \cup \Sigma_{s,t} \cup \{\exists u \exists v\ u \neq v\} \cup \{\varphi_{nontriv}\}$ *if some member of K has a trivial subalgebra, but no member of K is trivial;*

(b) $\mathrm{I}\Gamma^a_0(K)$ *is a Horn class axiomatized by axioms for* $\mathrm{I}\Gamma^a(K)$ *plus the sentence* $\varphi_{atomless}$.

In particular, if K is an $\forall\exists$-class and no member of K has a trivial subalgebra, or if K has a trivial member, then $\mathrm{I}\Gamma^a(K)$ and $\mathrm{I}\Gamma^a_0(K)$ are both $\forall\exists$-Horn classes.

Proof

(a) Since $K \models \Sigma_{s,t}$ by 2.4.5, and since each member of $\Sigma_{s,t}$ is of the form $\forall\exists$*atomic*, it follows from 2.1.19 that $\Gamma^a(K) \models \Sigma_{s,t}$; hence $\mathrm{I}\Gamma^a(K) \models \Sigma_V \cup \Sigma_{s,t}$.

On the other hand, if $\mathbf{A} \models \Sigma_V \cup \Sigma_{s,t}$, then by 2.3.7 we can assume that $\mathbf{A} \underset{\text{bp}}{\leq} \prod_{x \in X} \mathbf{A}_x$ with each $\mathbf{A}_x \in V(K)_S$. As each member of $\Sigma_{s,t}$ is a positive formula, it follows that $\mathbf{A}_x \models \Sigma_{s,t}$ for $x \in X$. Thus each $\mathbf{A}_x \models \Sigma$ or is trivial, by 2.4.5. Consequently $\mathbf{A} \in \Gamma^a(K_+)$.

For case (i) note that if K has a trivial algebra, then $K_+ = K$.

For case (ii) we see that if no member of K has a trivial subalgebra, then $\mathrm{I}\Gamma^a(K)$ has no trivial member; hence it satisfies $\exists u \exists v\ u \neq v$. On the other hand, a routine compactness argument shows that any nontrivial member of $\Gamma^a(K_+)$ in standard form has no trivial stalk, and thus belongs to $\Gamma^a(K)$.

Case (iii) is similar to case (ii) except that we now add $\varphi_{nontriv}$ to exclude trivial stalks from standard forms.

(b) now follows from 2.4.7. $\square$

Problem 4. If K is an elementary class with a discriminator term, does it follow that $\mathrm{I}\Gamma^a(K)$ is an elementary class?

2.5 The model companion transfer theorem for discriminator varieties

Proposition 2.5.1 *Let K be an $\forall\exists$-class of algebras with a discriminator term. Furthermore assume no member of K has a trivial subalgebra, or K contains a trivial algebra. Then*

$$\mathrm{ISP}(K)^{ec} \subseteq \mathrm{I}\Gamma^a_0(K).$$

Proof As $\mathrm{ISP}(K) = \mathrm{IS}\Gamma^a_0(K)$, by 2.1.31, it follows from 1.10.5 that

$$\mathrm{ISP}(K)^{ec} \models \mathrm{Th}_{\forall\exists}\Gamma^a_0(K).$$

Now by 2.4.8 we see that $\mathrm{I}\Gamma^a_0(K)$ is an $\forall\exists$-class; hence

$$\mathrm{ISP}(K)^{ec} \subseteq \mathrm{I}\Gamma^a_0(K). \quad \square$$

Definition 2.5.2 For K a class of algebras, let

$$K_{(+)} = \begin{cases} K & \text{if no member of } K \text{ properly contains} \\ & \text{a trivial algebra} \\ K \cup \{\text{trivial algebras}\} & \text{otherwise.} \end{cases}$$

Proposition 2.5.3 *Let K be an $\forall\exists$-class of algebras with a discriminator term. If K^{mc} exists, then*

$$\mathrm{ISP}(K)^{ec} \subseteq \mathrm{I}\Gamma^a_0((K^{mc})_{(+)}).$$

Proof As K^{mc} is an $\forall\exists$-class, $(K^{mc})_{(+)}$ is also an $\forall\exists$ class. From

$$\mathrm{ISP}(K) = \mathrm{ISP}((K^{mc})_{(+)})$$

we have, by 2.5.1,

$$\mathrm{ISP}(K)^{ec} \subseteq \mathrm{I}\Gamma^a_0((K^{mc})_{(+)}). \quad \square$$

Definition 2.5.4 An M-formula $\varphi(\vec{v})$ is an M^+-*formula* if it is of the form $\exists$*atomic*; and it is an M^--*formula* if it is of the form $\exists\neg$ *atomic*. Parallel to our previous definitions we introduce the obvious notation

$$\mathbf{A} \underset{M^+}{\hookrightarrow} \mathbf{B} \qquad \mathbf{A} \underset{M^-}{\hookrightarrow} \mathbf{B} \qquad \mathbf{A} \underset{M^+}{\hookrightarrow} K \qquad \mathbf{A} \underset{M^-}{\hookrightarrow} K.$$

Also we will use

$$\mathbf{A} \underset{M^\pm}{\hookrightarrow} \mathbf{B}$$

to mean that $\mathbf{A} \underset{M^+}{\hookrightarrow} \mathbf{B}$ and $\mathbf{A} \underset{M^-}{\hookrightarrow} \mathbf{B}$ hold; and similarly

$$\mathbf{A} \underset{M^\pm}{\hookrightarrow} K$$

means $\mathbf{A} \underset{M^+}{\hookrightarrow} K$ and $\mathbf{A} \underset{M^-}{\hookrightarrow} K$ hold.

WHEN DO $M^\pm$ FORMULAS SUFFICE TO TEST FOR BEING EC?

Proposition 2.5.5 *Let K be an elementary class with a discriminator term. If $\mathbf{A} \in \Gamma_0^a(K)$, then*

$$\mathbf{A} \in \mathrm{ISP}(K)^{ec} \Leftrightarrow \mathbf{A} \underset{M^\pm}{\hookrightarrow} \Gamma_0^e(K).$$

Proof In view of 2.1.32 it suffices to show

$$\mathbf{A} \underset{M^\pm}{\hookrightarrow} \Gamma_0^e(K) \Rightarrow \mathbf{A} \underset{ec}{\hookrightarrow} \Gamma_0^e(K).$$

So suppose $\mathbf{A} \underset{M^\pm}{\hookrightarrow} \Gamma_0^e(K)$.

Let $\varphi(\vec{v})$ be a primitive formula, say (following the notation of 2.1.22)

$$\varphi(\vec{v}) = \exists\vec{u}\left(\varphi^+(\vec{u},\vec{v}) \wedge \bigvee_{1\le i\le k} \neg\,\alpha_i(\vec{u},\vec{v})\right),$$

where $\varphi^+(\vec{u},\vec{v})$ is a conjunction of atomic formulas, and each $\alpha_i(\vec{u},\vec{v})$ is atomic. Let

$$\begin{aligned}\varphi_0(\vec{v}) &= \exists\vec{u}\varphi^+(\vec{u},\vec{v})\\ \varphi_i(\vec{v}) &= \exists\vec{u}(\varphi^+(\vec{u},\vec{v}) \wedge \neg\,\alpha_i(\vec{u},\vec{v})) \qquad 1 \le i \le k.\end{aligned}$$

Now by 2.4.1 we can, modulo K, contract the matrices of the $\varphi_i(\vec{v})$ to basic formulas, namely we can find atomic formulas $\beta_i(\vec{u},\vec{v}),\ 0 \le i \le k$, such that

$$\begin{aligned}K &\models \beta_0(\vec{u},\vec{v}) \leftrightarrow \varphi^+(\vec{u},\vec{v})\\ K &\models \neg\,\beta_i(\vec{u},\vec{v}) \leftrightarrow \varphi^+(\vec{u},\vec{v}) \wedge \neg\,\alpha_i(\vec{u},\vec{v}) \qquad 1 \le i \le k.\end{aligned}$$

Then let

$$\begin{aligned}
\varphi_0^*(\vec{v}) &= \exists\vec{u}\beta_0(\vec{u},\vec{v})\\
\varphi_i^*(\vec{v}) &= \exists\vec{u}(\neg\,\beta_i(\vec{u},\vec{v})) \qquad 1\le i\le k\\
\varphi^*(\vec{v}) &= \exists\vec{u}\left(\beta_0(\vec{u},\vec{v})\wedge\bigwedge_{1\le i\le k}\neg\,\beta_i(\vec{u},\vec{v})\right).
\end{aligned}$$

Certainly

$$K \models \varphi_i(\vec{v})\leftrightarrow\varphi_i^*(\vec{v}) \qquad 0\le i\le k. \tag{2.4}$$

Now for $\mathbf{C}\in\Gamma_0^e(K)$ and $\vec{c}\in \mathrm{C}$,

$$\begin{array}{rl|l}
\mathbf{C}\models\varphi(\vec{c}) \Leftrightarrow & \begin{cases}[\![\varphi_0(\vec{c})]\!]=X\\ [\![\varphi_i(\vec{c})]\!]\neq\emptyset \qquad 1\le i\le k\end{cases} & \text{by 2.1.21}\\
\Leftrightarrow & \begin{cases}[\![\varphi_0^*(\vec{c})]\!]=X\\ [\![\varphi_i^*(\vec{c})]\!]\neq\emptyset \qquad 1\le i\le k\end{cases} & \text{by (2.4)}\\
\Leftrightarrow & \mathbf{C}\models\varphi^*(\vec{c}) & \text{by 2.1.21}\\
\Leftrightarrow & \mathbf{C}\models\bigwedge_{0\le i\le k}\varphi_i^*(\vec{c}) & \text{by 2.1.22.}
\end{array}$$

Thus

$$\Gamma_0^a(K) \models \varphi(\vec{v}) \leftrightarrow \bigwedge_{0\le i\le k}\varphi_i^*(\vec{v}). \tag{2.5}$$

Consequently, if $\alpha:\mathbf{A}\hookrightarrow\mathbf{B}\in\Gamma_0^e(K)$ is an embedding and $\vec{a}\in\mathrm{A}$ with $\mathbf{B}\models\varphi(\vec{a})$ we have

$$\begin{array}{rl|l}
\mathbf{B}\models\varphi(\alpha\vec{a}) \Rightarrow & \mathbf{B}\models\bigwedge_{0\le i\le k}\varphi_i^*(\alpha\vec{a}) & \text{by (2.5)}\\
\Rightarrow & \mathbf{A}\models\bigwedge_{0\le i\le k}\varphi_i^*(\vec{a}) & \text{as } \mathbf{A}\underset{M^\pm}{\hookrightarrow}\Gamma_0^e(K)\\
\Rightarrow & \mathbf{A}\models\varphi(\vec{a}) & \text{by (2.5).}
\end{array}$$

Thus indeed we have proved

$$\mathbf{A}\underset{M^\pm}{\hookrightarrow}\Gamma_0^e(K)\Rightarrow\mathbf{A}\underset{ec}{\hookrightarrow}\Gamma_0^e(K). \qquad\square$$

WHEN DO M^- FORMULAS SUFFICE TO TEST FOR BEING EC?

Proposition 2.5.6 *Let K be an $\forall\exists$ class with a discriminator term. If K^{mc} exists, then we have*

$$\mathbf{A}\in\mathrm{ISP}(K)^{ec}\Leftrightarrow\begin{cases}\mathbf{A}\in\mathrm{I}\Gamma_0^a((K^{mc})_{(+)})\\ \mathbf{A}\underset{M^-}{\hookrightarrow}\Gamma_0^e(K).\end{cases}$$

Proof The direction ($\Rightarrow$) follows from 2.5.3 and 2.5.5.

For ($\Leftarrow$) it suffices, by 2.5.5, to show

$$\mathbf{A} \in \Gamma_0^a((K^{mc})_{(+)}) \;\Rightarrow\; \mathbf{A} \underset{M^+}{\hookrightarrow} \Gamma_0^e(K).$$

So assume $\mathbf{A} \in \Gamma_0^a((K^{mc})_{(+)})$. Let $\alpha : \mathbf{A} \hookrightarrow \Gamma_0^e(K)$ be an embedding, let $\varphi(\vec{v})$ be an M^+-formula, and let $\vec{a} \in$ A. Suppose $\mathbf{B} \models \varphi(\alpha\vec{a})$. Let θ_x be a stalk congruence of $\mathbf{A}$. By the SCEP (see 2.3.12) there is a stalk congruence θ_y of $\mathbf{B}$ which extends $\alpha(\theta_x)$. Let α^* be the map which makes the following diagram commute (unlabelled maps are quotient maps):

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\alpha"] \arrow[d] & \mathbf{B} \arrow[d] \\
\mathbf{A}/\theta_x \arrow[r, "\alpha^*"] & \mathbf{B}/\theta_y
\end{tikzcd}$$

Figure 6

Then

$$\left.\begin{array}{rl}\mathbf{B} \models \varphi(\alpha\vec{a}) \;\Rightarrow\; \mathbf{B}/\theta_y & \models \varphi(\alpha\vec{a}/\theta_y) \\ \Rightarrow\; \mathbf{A}/\theta_x & \models \varphi(\vec{a}/\theta_x),\end{array}\right| \; \varphi \text{ is positive}$$

the last step holding because $\mathbf{A}/\theta_x \in (K^{mc})_{(+)}$, so either $\alpha^*(\mathbf{A}/\theta_x)$ is trivial or $\alpha^*(\mathbf{A}/\theta_x) \leq_{ec} \mathbf{B}/\theta_y$. Thus

$$[\![\varphi(\vec{a})]\!] = X$$

so

$$\mathbf{A} \models \varphi(\vec{a}).$$

Consequently we have proved $\mathbf{A} \underset{M^+}{\hookrightarrow} \Gamma_0^e(K)$, as required. $\square$

$\boxed{\text{DEFINING } \Gamma^*}$

Definition 2.5.7 Given a class K of algebras, let $\Phi^-(K)$ be the set of M^--formulas $\mu(\vec{v})$ such that for some $\mathbf{B} \in K$ and some 1-element subuniverse $\{b\}$ of $\mathbf{B}$, we have $\mathbf{B} \models \mu(\vec{b})$, where $\vec{b}$ denotes a sequence of b's. Now define

$$\Gamma^*(K) \;=\; \{\mathbf{A} \in \Gamma_0^a(K_{(+)}) : \; \mathbf{A} \models \forall\vec{v}\mu(\vec{v}) \text{ for all } \vec{\mu} \in \Phi^-(K)\}.$$

Proposition 2.5.8 *Let K_0 be an $\forall\exists$ class with a discriminator term, and let $K = \mathrm{IS}(K_0)$. Then*

(i) $\mathrm{I}\Gamma^*(K_0)$ *is an $\forall\exists$ Horn class axiomatized by*

$$\mathrm{Th}_{\forall\exists}\left(\Gamma_0^a(K_{0(+)})\right) \cup \{\forall\vec{v}\mu(\vec{v}) : \; \mu(\vec{v}) \in \Phi^-(K)\},$$

(ii) *and we have*

$$\mathrm{ISP}(K) = \mathrm{IS}\Gamma^*(K_0);$$

hence $\mathrm{ISP}(K)$ *is mutually model consistent with* $\mathrm{I}\Gamma^*(K_0)$.

Proof

(i) Since K_0 is an $\forall\exists$-class, it follows that $K_{0(+)}$ is an $\forall\exists$ class. Thus by 2.4.8 $\mathrm{I}\Gamma_0^a(K_{0(+)})$ is an $\forall\exists$ Horn class. Note that $\Phi^-(K) = \Phi^-(K_0)$. From this it follows that $\mathrm{I}\Gamma^*(K_0)$ is axiomatized as indicated; hence it is an $\forall\exists$ class.

(ii) Since $\mathrm{I}\Gamma^*(K_0)$ is a Horn class, it suffices to show

$$K \subseteq \mathrm{IS}\Gamma^*(K_0)$$

as

$$\mathrm{IS}\Gamma^*(K_0) \subseteq \mathrm{IS}\Gamma_0^a(K_{0(+)}) \subseteq \mathrm{ISP}(K).$$

So let $\mathbf{A} \in K$. Let C be the Cantor discontinuum, and let x_0 be a fixed element of C. For $\mu(\vec{v}) \in \Phi^-(K)$ choose $\mathbf{B}_\mu \in K_0$ and a 1-element subuniverse $\{b_\mu\}$ of $\mathbf{B}_\mu$ such that

$$\mathbf{B}_\mu \models \mu(\vec{b}_\mu).$$

(This is possible as $K \subseteq \mathrm{S}(K_0)$.) Let

$$\mathrm{A}_\mu = \{f \in \mathbf{B}_\mu[C]^* : f(x_0) = b_\mu\}.$$

Then A_μ is a subuniverse of $\mathbf{B}_\mu[C]^*$, so let the corresponding subalgebra be $\mathbf{A}_\mu$. Let $b \in \mathbf{A}_\mu$ satisfy $b(x) = b_\mu$ for $x \in C$. Certainly

$$\mathbf{A}_\mu \in \Gamma_0^a(K_{0(+)}).$$

Also, for $\vec{f} \in \mathrm{A}_\mu$,

$$x_0 \in [\![\vec{f} = \vec{b}]\!],$$

so we have

$$\emptyset \neq [\![\vec{f} = \vec{b}]\!] - \{x_0\} \subseteq [\![\mu(\vec{f})]\!].$$

Thus

$$\mathbf{A}_\mu \models \mu(\vec{f}).$$

Now choose $\mathbf{A}_0$ with $\mathbf{A} \leq \mathbf{A}_0 \in K_0$. Then

$$\mathbf{A}_0[C]^* \in \Gamma_0^a(K_0),$$

so letting

$$\mathbf{B} = \mathbf{A}_0[C]^* \times \prod_{\mu \in \Phi^-(K)} \mathbf{A}_\mu$$

we see that

$$\mathbf{B} \in \mathrm{I}\Gamma_0^a(K_{0(+)})$$

since $\mathrm{I}\Gamma_0^a(K_{0(+)})$ is a Horn class. Now we can certainly embed $\mathbf{A}$ into $\mathbf{A}_0[C]^*$; and $\mathbf{A}_0[C]^*$ can be embedded in $\mathbf{B}$ since each of the $\mathbf{A}_\mu$ has a one-element subalgebra. Thus

$$\mathbf{A} \in \mathrm{IS}\Gamma_0^a(K_{0(+)}),$$

as desired. □

THE MODEL COMPANION TRANSFER THEOREM

Theorem 2.5.9 *Let K be an $\forall\exists$ class with a discriminator term. If K^{mc} exists, then* $\mathrm{ISP}(K)^{mc}$ *exists and*

(i) $\mathrm{ISP}(K)^{mc} = \mathrm{I}\Gamma^*(K^{mc})$, *and*

(ii) $\mathrm{ISP}(K)^{mc}$ *is axiomatized by*

$$\Sigma_V \cup (\Sigma^+_{mc})_{s,t} \cup \{\exists u \exists v(u \neq v)\} \cup \{\varphi_{atomless}\} \cup \{\forall \vec{v}\mu(\vec{v}) : \mu(\vec{v}) \in \Phi^-(K)\},$$

unless K has a trivial algebra, but no member of K properly contains a trivial subalgebra, in which case $\mathrm{ISP}(K)^{mc}$ *is axiomatized by*

$$\Sigma_V \cup (\Sigma^+_{mc})_{s,t} \cup \{\varphi_{atomless}\},$$

where

$$V(K) \text{ is axiomatized by } \Sigma_V$$
$$(K^{mc})_{(+)} \text{ is axiomatized by } \Sigma^+_{mc}.$$

Proof

(i) As K^{mc} is an $\forall\exists$-class, we know by 2.5.8(i) that $\mathrm{I}\Gamma^*(K^{mc})$ is also an $\forall\exists$-class; and by 2.5.8(ii) it is mutually model consistent with $\mathrm{ISP}(K)$. Thus if we can show

$$\mathrm{I}\Gamma^*(K^{mc}) \subseteq \mathrm{ISP}(K)^{ec}$$

then, by 1.4.3 and 1.7.2, $\mathrm{I}\Gamma^*(K^{mc})$ is the model companion of $\mathrm{ISP}(K)$. So let $\mathbf{A} \in \Gamma^*(K^{mc})$. By 2.5.6 we only need to show $\mathbf{A} \underset{M^-}{\hookrightarrow} \Gamma^e_0(K)$. Let $\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^e_0(K)$ be an embedding. If $\varphi(\vec{v})$ is an M^--formula and $\vec{a} \in A$ is such that $\mathbf{B} \models \varphi(\alpha\vec{a})$, then for some $y \in X(\mathbf{B})$,

$$\mathbf{B}/\theta_y \models \varphi(\alpha\vec{a}/\theta_y).$$

By the SCRP (see 2.3.12) for $\Gamma^a(K)$ we see that either

(a) $\theta_y \mid_{\alpha\mathbf{A}} = \nabla_A$, or

(b) $\theta_y \mid_{\alpha\mathbf{A}} = \alpha(\theta_x)$ for some stalk congruence θ_x of $\mathbf{A}$ with $\mathbf{A}/\theta_x$ nontrivial.

If (a) occurs, then $\mathbf{B}/\theta_y$ has a one-element subalgebra $\{b/\theta_y\}$ such that $\alpha(\mathrm{A})/\theta_y = \{b/\theta_y\}$, and

$$\mathbf{B}/\theta_y \models \varphi(\vec{b}/\theta_y),$$

as $\alpha\vec{a}/\theta_y = \vec{b}/\theta_y$. Thus $\varphi(\vec{v}) \in \Phi^-(K)$, so

$$\mathbf{A} \models \varphi(\vec{a})$$

as $\mathbf{A} \in \Gamma^*(K^{mc})$.

In case (b) we have an embedding $\alpha^* : \mathbf{A}/\theta_x \hookrightarrow \mathbf{B}/\theta_y$ such that the diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\alpha"] \arrow[d] & \mathbf{B} \arrow[d] \\ \mathbf{A}/\theta_x \arrow[r, "\alpha^*"] & \mathbf{B}/\theta_y \end{tikzcd}$$

Figure 7

commutes. As $\mathbf{A}/\theta_x \in K^{mc}$ ($\mathbf{A}/\theta_x$ is nontrivial), we have

$$\alpha^*(\mathbf{A}/\theta_x) \leq_{ec} \mathbf{B}/\theta_y.$$

As

$$\mathbf{B}/\theta_y \models \varphi(\alpha^*(\vec{a}/\theta_x))$$

it follows that

$$\mathbf{A}/\theta_x \models \varphi(\vec{a}/\theta_x).$$

Hence again we have

$$\mathbf{A} \models \varphi(\vec{a}).$$

(ii) The axioms for $\mathrm{ISP}(K)^{mc}$ now follow from 2.5.8(i) and 2.4.8 after noting that $\Phi^-(K) = \emptyset$ in the second case. □

$\boxed{\text{. . . FOR DISCRIMINATOR VARIETIES}}$

Corollary 2.5.10 *If V is a discriminator variety and V_S, the class of simple algebras in V, has a model companion [model completion], then V has a model companion [model completion which admits elimination of quantifiers].*

Proof From 2.3.7 we see that $V = \mathrm{ISP}(V_S)$, so 2.5.9 applies to show model companions transfer. The result for the model completions follows from 2.3.13(ii), 1.9.2 and 1.9.3. □

Problem 5. If a discriminator variety V has a model companion, does it follow that V_S has a model companion?

$\boxed{\text{TWO SPECIAL CASES}}$

Corollary 2.5.11 *If K is an $\forall\exists$ class with a discriminator term, if no member of K properly contains a trivial subalgebra, and if K has a model companion, then*

$$\mathrm{ISP}(K)^{mc} = \mathrm{I}\Gamma_0^a(K^{mc}).$$

Proof In this case we see that $\Phi^-(K) = \emptyset$. $\square$

Definition 2.5.12 If K is such that some member properly contains a trivial subalgebra, then let

$$\widehat{\Gamma}_0^a(K) \;=\; \{\mathbf{A} \in \Gamma_0^a(K) :\; \mathbf{A} \models \neg\,\varphi_{nontriv}\}.$$

Corollary 2.5.13 *If K is an $\forall\exists$ class with a finite language and with a discriminator term such that every member of K has exactly one trivial subalgebra and if K^{mc} exists and has a complete theory, then*

$$\mathrm{ISP}(K)^{mc} \;=\; \mathrm{I}\widehat{\Gamma}_0^a(K^{mc}).$$

Proof One sees that

$$\mathrm{ISP}(K)^{mc} \;\subseteq\; \mathrm{I}\widehat{\Gamma}_0^a(K^{mc})$$

by noting that every member of $\mathrm{ISP}(K)$ can be embedded in a member satisfying $\neg\,\varphi_{nontriv}$.

Now for $\mu(\vec{u}) \in \Phi^-(K)$ note that some member $\mathbf{B} \in K$ has a one-element subuniverse $\{0_{\mathbf{B}}\}$ with $\mathbf{B} \models \mu(\vec{0}_{\mathbf{B}})$. We can assume $\mathbf{B} \in K^{mc}$. Then, by completeness, every $\mathbf{A} \in K^{mc}$ satisfies $\mu(\vec{0}_{\mathbf{A}})$. From this one can conclude

$$\widehat{\Gamma}_0^a(K^{mc}) \;\models\; \forall\vec{u}\mu(\vec{u}). \qquad \square$$

Remark If K is an $\forall\exists$ class with a discriminator term, but K does *not* have a model companion, then we no longer have a description of $\mathrm{ISP}(K)^{ec}$. By combing back through the above results, it is not difficult to see that we do have

$$\mathrm{I}\Gamma^*(K^{ec}) \;\subseteq\; \mathrm{ISP}(K)^{ec} \;\subseteq\; \mathrm{I}\Gamma^*(Cl_{\forall\exists}K^{ec})$$

where $Cl_{\forall\exists}K^{ec}$ is the $\forall\exists$ closure of K^{ec}.

Problem 6. Find a good description of $\mathrm{ISP}(K)^{ec}$ for the general case just mentioned.

2.6 Discriminator formulas, $(*)$ and $(**)$

Definition 2.6.1 For K a class of algebras of type $\mathcal{L}$ and $K' \subseteq \mathrm{SP}(K)$, we say that K' has the property (D) if there is an $\mathcal{L}$-formula $\tau(u_1, u_2, u_3, v)$ such that for $\mathbf{A} \in K'$ and $f, g, h, k \in \mathrm{A}$,

$$\mathbf{A} \;\models\; \tau(f,g,h,k) \;\Leftrightarrow\; \begin{cases} [\![f = g]\!] \subseteq [\![h = k]\!] \\ \text{and} \\ [\![f \neq g]\!] \subseteq [\![f = k]\!]. \end{cases} \tag{2.6}$$

τ is called a *discriminator formula*. K' has the property (**O**D) if one can choose τ open, the property ($\exists$D) if one can choose τ existential, etc.

Proposition 2.6.2 *Let $\tau(u_1, u_2, u_3, v)$ be given. Then τ is a discriminator formula for $K' \subseteq \mathrm{SP}(K)$ iff for $f, g, h, k \in \mathrm{A}$, $\mathbf{A} \in K'$, we have*

$$\mathbf{A} \models \tau(f, g, h, k) \Leftrightarrow k = f|_{[[f \neq g]]} \cup h|_{[[f=g]]}. \tag{2.7}$$

Proof This is just a rephrasing of (2.6) above. $\square$

Proposition 2.6.3 *If K is a class of algebras with a discriminator term t, then a formula τ is a discriminator formula for $K' \subseteq \mathrm{SP}(K)$ iff we have*

$$K' \models \tau(u_1, u_2, u_3, v) \leftrightarrow t(u_1, u_2, u_3) = v. \tag{2.8}$$

Proof This follows from (2.7) since one has, for $\mathbf{A} \in \mathrm{SP}(K)$ and $f, g, h, k \in \mathrm{A}$,

$$t(f, g, h) = k \Leftrightarrow k = f|_{[[f \neq g]]} \cup h|_{[[f=g]]}. \qquad \square$$

If we look at the conditions involved in the right-hand side of (2.6), then we are led to consider the following definitions.

Definition 2.6.4 A class $K' \subseteq \mathrm{SP}(K)$ has the property $(*)$ if there is a formula $\varphi_{(\emptyset)}(u_1, v_1, u_2, v_2)$ such that for $\mathbf{A} \in K'$ and $f, g, h, k \in \mathrm{A}$,

$$\mathbf{A} \models \varphi_{(\emptyset)}(f, g, h, k) \Leftrightarrow [\![f \neq g]\!] \cap [\![h \neq k]\!] = \emptyset.$$

K' has $(\mathbf{O}*)$ if one can choose the formula $\varphi_{(\emptyset)}$ to be an open formula, $(\exists *)$ if one can choose $\varphi_{(\emptyset)}$ to be an existential formula, etc.

Definition 2.6.5 A class $K' \subseteq \mathrm{SP}(K)$ has the property $(**)$ if there is a formula $\varphi_{(\subseteq)}$ such that for $\mathbf{A} \in K'$ and $f, g, h, k \in \mathrm{A}$,

$$\mathbf{A} \models \varphi_{(\subseteq)}(f, g, h, k) \Leftrightarrow [\![f \neq g]\!] \subseteq [\![h \neq k]\!].$$

As before, K' has $(\mathbf{O}**)$, $(\exists **)$, etc., if $\varphi_{(\subseteq)}$ can be chosen in the appropriate form.

In the following, $\vec{Q}$ will denote a string of quantifiers, for example, $\forall\exists\forall$. The notation $(\vec{Q}_1 \wedge \vec{Q}_2 : \mathrm{D})$ means that we have a discriminator formula in the form $\vec{Q}_1\omega_1 \wedge \vec{Q}_2\omega_2$, where ω_1 and ω_2 are open.

GOING FROM ONE OF (D), $(*)$ and $(**)$ TO THE OTHERS

Proposition 2.6.6 *For $K' \subseteq \Gamma^a(K) \cup \mathrm{P}(K)$ we have the following:*

(a) $(\mathrm{D}) \Leftrightarrow (*) \Leftrightarrow (**)$

(b) $(\vec{Q}\mathrm{D}) \Rightarrow (\exists\vec{Q}*)$, $(\exists\vec{Q}**)$

(c) $(\vec{Q}**) \Rightarrow (\forall\neg\vec{Q}*)$, $(\vec{Q}\wedge\forall\neg\vec{Q} : \mathrm{D})$

(d) $(\vec{Q}*) \Rightarrow (\forall\vec{Q}\vee\forall\neg\vec{Q} : **)$, $(\forall\vec{Q}\wedge\forall\neg\vec{Q} : \mathrm{D})$

(e) $(\vec{Q}*)$ *and* $(\vec{Q}**) \Rightarrow (\vec{Q}\mathrm{D})$

(f) $(\exists*)$ *and* $(\exists**) \Leftrightarrow (\exists\mathrm{D})$.

Proof

(a) Let $\tau(u_1, u_2, u_3, v)$ be a discriminator formula for K'. Then we can use (see 2.3.3 and 2.3.9):

$$\varphi_{(\emptyset)}(u_1, v_1, u_2, v_2) \;=\; \exists w_1\exists w_2(\tau(u_1, v_1, u_2, w_1)\wedge\tau(u_1, v_1, v_2, w_2)\wedge\tau(w_1, w_2, v_2, u_2))$$

and

$$\varphi_{(\subseteq)}(u_1, v_1, u_2, v_2) \;=\; \exists w_1\exists w_2(\tau(u_2, v_2, u_1, w_1)\wedge\tau(u_2, v_2, v_1, w_2)\wedge\tau(w_1, w_2, v_1, v_1)).$$

Now let $\varphi_{(\emptyset)}(u_1, v_1, u_2, v_2)$ be a $(*)$ formula. Then the following formulas suffice:

$$\varphi_{(\subseteq)}(u_1, v_1, u_2, v_2) \;=\; \forall w_1\forall w_2(\varphi_{(\emptyset)}(w_1, w_2, u_2, v_2)\rightarrow\varphi_{(\emptyset)}(w_1, w_2, u_1, v_1))$$

and then

$$\tau(u_1, u_2, u_3, v) \;=\; \varphi_{(\subseteq)}(u_3, v, u_1, u_2)\;\wedge\;\varphi_{(\emptyset)}(u_1, u_2, u_1, v).$$

Next let $\varphi_{(\subseteq)}(u_1, v_1, u_2, v_2)$ be a $(**)$ formula. Then we can choose

$$\varphi_{(\emptyset)}(u_1, v_1, u_2, v_2) \;=\; \forall w_1\forall w_2(w_1 = w_2\vee\neg\,\varphi_{(\subseteq)}(w_1, w_2, u_1, v_1)\vee\neg\,\varphi_{(\subseteq)}(w_1, w_2, u_2, v_2))$$

and then τ as above.

Now (b)–(f) follow from the formulas constructed above. $\Box$

2.7 t-Compatible algebras

EXPANDING BY t

Definition 2.7.1 For $\mathbf{A}$ an algebra of type $\mathcal{L}$, $\mathbf{A}^t$ denotes the *expansion* of $\mathbf{A}$ to the type $\mathcal{L}\cup\{t\}$ with $t^{\mathbf{A}}$ being the ternary discriminator function on A. For K a class of algebras of type $\mathcal{L}$, let

$$K^t \;=\; \{\mathbf{A}^t :\; \mathbf{A}\in K\}.$$

For $\mathbf{A}^*\in \mathrm{SP}(K^t)$ let $\mathrm{Red}_t\mathbf{A}^*$ be the *reduct* of $\mathbf{A}^*$ to the language $\mathcal{L}$, and for $K^*\subseteq \mathrm{SP}(K^t)$,

$$\mathrm{Red}_t(K^*) \;=\; \{\mathrm{Red}_t\mathbf{A}^* :\; \mathbf{A}^*\in K^*\}.$$

Definition 2.7.2 For $\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$ we say that $\mathbf{A}$ is *t-compatible* if A is a subuniverse of $\prod_{i\in I} \mathbf{A}_i^t$. If $\mathbf{A}$ is t-compatible, let $\mathbf{A}(t)$ be the subalgebra of $\prod_{i\in I} \mathbf{A}_i^t$ with universe A.

Proposition 2.7.3 *For* $\mathbf{A} \in \mathrm{SP}(K)$, $\mathbf{A}$ *is* t*-compatible iff for all* $f, g, h \in \mathrm{A}$ *there is a* $k \in \mathrm{A}$ *such that*

$$[\![f = g]\!] \subseteq [\![h = k]\!] \quad \textit{and} \quad [\![f \neq g]\!] \subseteq [\![f = k]\!].$$

(*The element* k *must be* $f|_{[\![f\neq g]\!]} \cup h|_{[\![f=g]\!]}$.)

Proof ($\Rightarrow$) Let $\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$, $\mathbf{A}_i \in K$. If $\mathbf{A}$ is t-compatible, then, for $f, g, h \in \mathrm{A}$ the element $k = t(f, g, h)$ in the algebra $\prod_{i\in I} \mathbf{A}_i^t$ satisfies

$$[\![f = g]\!] \subseteq [\![h = k]\!] \quad \text{and} \quad [\![f \neq g]\!] \subseteq [\![f = k]\!].$$

As $\mathbf{A}$ is t-compatible, $k \in \mathrm{A}$.

($\Leftarrow$) Given $f, g, h \in \mathrm{A}$ let $k \in \mathrm{A}$ be such that

$$[\![f = g]\!] \subseteq [\![h = k]\!] \quad \text{and} \quad [\![f \neq g]\!] \subseteq [\![f = k]\!].$$

Then

$$\prod_{i\in I} \mathbf{A}_i^t \models t(f, g, h) = k,$$

so A is a subuniverse of $\prod_{i\in I} \mathbf{A}_i^t$. Thus $\mathbf{A}$ is t-compatible. $\square$

Corollary 2.7.4 *An algebra* $\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$ *is* t*-compatible iff* $\mathbf{A}$ *has the following patchwork property over* $DE(\mathbf{A})$:

$$f, g \in \mathrm{A} \quad \textit{and} \quad J \in DE(\mathbf{A}) \Rightarrow f|_J \cup g|_{I-J} \in \mathrm{A}.$$

Proof ($\Rightarrow$) Let $J \in DE(\mathbf{A})$, and let $f, g \in \mathrm{A}$. If $J = [\![\widehat{f} \neq \widehat{g}]\!]$, $\widehat{f}, \widehat{g} \in \mathrm{A}$, then

$$\prod_{i\in I} \mathbf{A}_i^t \models t(t(\widehat{f}, \widehat{g}, g), t(\widehat{f}, \widehat{g}, f), f) \;=\; f|_J \cup g|_{I-J};$$

and if $J = [\![\widehat{f} = \widehat{g}]\!]$, $\widehat{f}, \widehat{g} \in \mathrm{A}$, then

$$\prod_{i\in I} \mathbf{A}_i^t \models t(t(\widehat{f}, \widehat{g}, f), t(\widehat{f}, \widehat{g}, g), g) \;=\; f|_J \cup g|_{I-J}.$$

As $\mathbf{A}$ is t-compatible, $f|_J \cup g|_{I-J} \in \mathrm{A}$.

($\Leftarrow$) For $f, g, h \in \mathrm{A}$, by hypothesis

$$k \;=\; f|_{[\![f\neq g]\!]} \cup h|_{[\![f=g]\!]} \in \mathrm{A}.$$

But then

$$[\![f = g]\!] \subseteq [\![h = k]\!] \quad \text{and} \quad [\![f \neq g]\!] \subseteq [\![f = k]\!].$$

so 2.7.3 applies. $\square$

BOOLEAN PRODUCTS ARE t-COMPATIBLE

Corollary 2.7.5 *If* $\mathbf{A} \in \Gamma^a(K)$, *then* $\mathbf{A}$ *is* t*-compatible.*

Proof This follows immediately from 2.7.4 since $\Gamma^a(K)$ has the patchwork property. $\square$

Proposition 2.7.6 *For any class* K *of algebras,*

$$\mathrm{Red}_t \Gamma^a(K^t) \ = \ \Gamma^a(K).$$

Proof Certainly $\mathrm{Red}_t \Gamma^a(K^t) \subseteq \Gamma^a(K)$ as a reduct of a Boolean product is a Boolean product. On the other hand, as each $\mathbf{A} \in \Gamma^a(K)$ is t-compatible, $\mathbf{A}(t)$ is defined, $\mathbf{A}(t) \in \Gamma^a(K^t)$ and $\mathbf{A} = \mathrm{Red}_t \mathbf{A}(t)$. Thus $\Gamma^a(K) \subseteq \mathrm{Red}_t \Gamma^a(K^t)$. $\square$

DISCRIMINATOR FORMULAS AND t-COMPATIBILITY

Proposition 2.7.7 *Let* $K' \subseteq \mathrm{SP}(K)$ *have a discriminator formula* $\tau(u_1, u_2, u_3, v)$. *Then*

$$\mathbf{A} \in K' \ \textit{is } t\textit{-compatible} \quad \textit{iff} \quad \mathbf{A} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v).$$

Proof ($\Rightarrow$) Let $\mathbf{A} \in K'$ be t-compatible. Then $\mathbf{A}(t)$ is defined. So for $f, g, h \in$ A, let $k = t(f, g, h) \in \mathrm{A}(t) = \mathrm{A}$. By §2.6, formula (2.8),

$$\mathbf{A} \models \tau(f, g, h, k).$$

Thus

$$\mathbf{A} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v).$$

($\Leftarrow$) Let $\mathbf{A} \in K'$ be such that

$$\mathbf{A} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v).$$

Given $f, g, h \in \mathrm{A}$ choose $k \in \mathrm{A}$ such that

$$\mathbf{A} \models \tau(f, g, h, k).$$

Then

$$[\![f = g]\!] \subseteq [\![h = k]\!] \quad \text{and} \quad [\![f \neq g]\!] \subseteq [\![f = k]\!]$$

by §2.6, formulas (2.6). Thus by 2.7.3 $\mathbf{A}$ is t-compatible. $\square$

($\forall$D) AND EC STRUCTURES

The last result in this section says that if K is a $\forall$ class and $\Gamma^a(K)$ has a $\forall$ discriminator formula, then the existentially closed Boolean products are definable in $\mathrm{ISP}(K)^{ec}$ by an $\forall\exists\forall$ sentence.

Proposition 2.7.8 *Let K be a universal class such that $\Gamma^a(K)$ has* (∀D). *Then*

$$\mathrm{I}\Gamma^a(K) \cap \mathrm{ISP}(K)^{ec} = \{\mathbf{A} \in \mathrm{ISP}(K)^{ec} : \ \mathbf{A} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v)\},$$

where τ is a $\forall$ discriminator formula witnessing (∀D) *for $\Gamma^a(K)$.*

Proof (⊆): Let $\mathbf{A} \in \Gamma^a(K) \cap \mathrm{ISP}(K)^{ec}$. By 2.7.5 $\mathbf{A}$ is t-compatible, and then by 2.7.7 $\mathbf{A} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v)$.

(⊇): Let $\mathbf{A} \in \mathrm{ISP}(K)^{ec}$ be such that $\mathbf{A} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v)$. Then choose $\mathbf{B} \in \Gamma^a(K)$ such that $\mathbf{A}$ embeds in $\mathbf{B}$. Without loss of generality we can assume $\mathbf{A} \leq \mathbf{B}$. Given $a_1, a_2, a_3 \in$ A choose $a \in$ A such that $\mathbf{A} \models \tau(a_1, a_2, a_3, a)$. As $\mathbf{A} \in \mathrm{ISP}(K)^{ec}$ and τ is a universal formula, it follows that $\mathbf{B} \models \tau(a_1, a_2, a_3, a)$. Now τ is a discriminator formula for $\mathbf{B}$; hence for $\mathbf{B}(t)$ (which exists by 2.7.5). Thus by 2.6.3

$$\mathbf{B} \models t(a_1, a_2, a_3) = a.$$

Thus $\mathbf{A}$ is t-compatible. As $\mathbf{A} \in \mathrm{ISP}(K)$ it follows that $\mathbf{A}(t) \in \mathrm{ISP}(K^t)$, so by 2.3.7 $\mathbf{A}(t) \in \mathrm{I}\Gamma^a(K^t)$. Consequently $\mathbf{A} \in \mathrm{I}\Gamma^a(K)$. □

2.8 t-Faithful Boolean products

Now we look at the general question: for $\mathbf{A} \in \Gamma^a(K)$, when is $\mathbf{A}$ existentially closed in $\mathrm{ISP}(K)$? That is, find a more transparent description of the members of

$$\mathrm{ISP}(K)^{ec} \cap \Gamma^a(K).$$

As we have already seen in 2.1.32, if $\mathbf{A} \in \Gamma^a(K)$, then $\mathbf{A} \in \mathrm{ISP}(K)^{ec}$ iff $\mathbf{A} \underset{ec}{\hookrightarrow} \Gamma^a(K)$. If we knew that

$$\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^a(K) \ \Rightarrow \ \alpha : \ \mathbf{A}(t) \hookrightarrow \mathbf{B}(t),$$

that is, that embeddings of $\mathbf{A}$ into members of $\Gamma^a(K)$ were compatible with t, then we could move over into discriminator varieties and try to apply our earlier results.

$\boxed{t\text{-FAITHFUL BOOLEAN PRODUCTS}}$

Definition 2.8.1 An algebra $\mathbf{A} \in \Gamma^a(K)$ is *t-faithful with respect to* $\Gamma^a(K)$ if for every embedding

$$\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^a(K)$$

we have

$$\alpha : \mathbf{A}(t) \hookrightarrow \mathbf{B}(t).$$

Let $\Gamma^a(K)_{t-faithful}$ denote the class of all algebras in $\Gamma^a(K)$ which are t-faithful with respect to $\Gamma^a(K)$.

$\boxed{t\text{-FAITHFUL ALGEBRAS HAVE } K\text{-SIMPLE STALKS}}$

Definition 2.8.2 Let K be a class of algebras. Then $\mathbf{A} \in K$ is K-*simple* if, for every homomorphism

$$\alpha : \ \mathbf{A} \ \rightarrow \ \mathbf{B} \in K,$$

either α is an embedding or $\alpha\mathbf{A}$ is trivial. Let S_K denote the class of K-simple algebras.

Proposition 2.8.3 *For K a class of algebras, we have*

$$\Gamma^a(K)_{t-faithful} \subseteq \Gamma^a(S_K).$$

Proof Suppose $\mathbf{A} \in \Gamma^a(K) - \Gamma^a(S_K)$. Then for some stalk $\mathbf{A}_x$ of $\mathbf{A}$ there is a homomorphism $\alpha : \mathbf{A}_x \rightarrow \mathbf{B} \in K$ where α is not injective and $\alpha\mathbf{A}$ is not trivial. Then let

$$\alpha^* : \mathbf{A} \hookrightarrow \mathbf{A} \times \mathbf{B}$$

be the embedding defined by

$$\alpha^*(f) \ = \ \langle f, \alpha(fx) \rangle.$$

Choose $a, b, c \in \mathrm{A}_x$ with $a \neq b, \ \alpha a = \alpha b$, and $\alpha a \neq \alpha c$. Then use patchwork to find $f, g, h \in \mathrm{A}$ such that $fx = a, \ gx = b, \ hx = c$, and $[\![f \neq g]\!] = [\![f \neq h]\!]$. We have

$$\mathbf{A}(t) \ \models \ t(f, g, h) \ = \ f$$

and

$$\mathbf{B}^t \ \models \ t(\alpha(fx), \alpha(gx), \alpha(hx)) \ = \ \alpha(hx).$$

Thus

$$\mathbf{A}(t) \times \mathbf{B}^t \ \not\models \ t(\alpha^* f, \alpha^* g, \alpha^* h) \ = \ \alpha^* f,$$

so $\mathbf{A}$ is not in $\Gamma^a(K)_{t-faithful}$. $\square$

Lemma 2.8.4 *Let $\mathbf{A}, \mathbf{B} \in \Gamma^a(K)$ and let $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ be an embedding. Then $\alpha : \mathbf{A}(t) \hookrightarrow \mathbf{B}(t)$ iff for $a, b, c, d \in \mathrm{A}$,*

$$\mathbf{A}(t) \ \models \ t(a, b, c) \ = \ d \ \Rightarrow \ \mathbf{B}(t) \ \models \ t(\alpha a, \alpha b, \alpha c) \ = \ \alpha d.$$

Proof As $\mathbf{A}, \mathbf{B} \in \Gamma^a(K)$ we know by 2.7.5 that $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are defined. Then $\alpha : \mathbf{A}(t) \hookrightarrow \mathbf{B}(t)$ holds iff for $a, b, c \in \mathrm{A}$,

$$\alpha t^{\mathbf{A}}(a, b, c) \ = \ t^{\mathbf{B}}(\alpha a, \alpha b, \alpha c).$$

This is just a rephrasing of the condition stated above. $\square$

Proposition 2.8.5 *If $\Gamma^a(K)$ has the property* ($\exists$D), *then*

$$\Gamma^a(K)_{t-faithful} \ = \ \Gamma^a(K).$$

Proof Let $\mathbf{A} \in \Gamma^a(K)$ and let $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ be an embedding. Choose an existential discriminator formula $\tau(u_1, u_2, u_3, v)$ for $\Gamma^a(K)$. Then τ is also a discriminator formula for $\Gamma^a(K^t)$ by 2.7.6. For $a, b, c, d \in \mathrm{A}$ we have

$$\begin{array}{rl|l} \mathbf{A} \models t(a,b,c) = d & \Rightarrow \mathbf{A}(t) \models \tau(a,b,c,d) & \text{by 2.6.3} \\ & \Rightarrow \mathbf{A} \models \tau(a,b,c,d) & \\ & \Rightarrow \mathbf{B} \models \tau(\alpha a, \alpha b, \alpha c, \alpha d) & \tau \text{ is existential} \\ & \Rightarrow \mathbf{B}(t) \models \tau(\alpha a, \alpha b, \alpha c, \alpha d) & \\ & \Rightarrow \mathbf{B}(t) \models t(\alpha a, \alpha b, \alpha c) = \alpha d & \text{by 2.6.3.} \end{array}$$

Thus from 2.8.4 we have $\alpha : \mathbf{A}(t) \hookrightarrow \mathbf{B}(t)$, so $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$. $\square$

t-FAITHFUL IS EQUIVALENT TO THE SCRP

Lemma 2.8.6 *For $\mathbf{A} \in \Gamma^a(K)$, $\mathbf{A}$ and $\mathbf{A}(t)$ have the same stalk congruences.*

Proof This is obvious since $\mathbf{A}(t)$ is just an expansion of $\mathbf{A}$. $\square$

Proposition 2.8.7 *For any class of algebras K,*

$$\Gamma^a(K)_{t-faithful} = \Gamma^a(K)_{\mathrm{SCRP}}.$$

Proof Let $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$. For $\mathbf{B} \in \Gamma^a(K)$ and an embedding $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ suppose θ is a stalk congruence of $\mathbf{B}$. Then by 2.8.6 θ is a stalk congruence of $\mathbf{B}(t)$. As $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$ we have $\alpha : \mathbf{A}(t) \to \mathbf{B}(t)$. Since $\Gamma^a(K^t)$ has the SCRP by 2.3.12(ii), $\alpha^{-1}(\theta)$ is a stalk congruence of $\mathbf{A}(t)$, or $\alpha^{-1}(\theta) = \nabla_{\mathbf{A}}$. Thus by 2.8.6 $\alpha^{-1}(\theta)$ is a stalk congruence of $\mathbf{A}$, or $\alpha^{-1}(\theta) = \nabla_{\mathbf{A}}$. This shows $\mathbf{A} \in \Gamma^a(K)_{\mathrm{SCRP}}$; hence

$$\Gamma^a(K)_{t-faithful} \subseteq \Gamma^a(K)_{\mathrm{SCRP}}.$$

For the other inclusion, let $\mathbf{A} \in \Gamma^a(K)_{\mathrm{SCRP}}$, and suppose $\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^a(K)$ is an embedding. Let $a, b, c, d \in \mathrm{A}$ with

$$\mathbf{A}(t) \models t(a,b,c) = d.$$

For θ a stalk congruence of $\mathbf{B}$, $\alpha^{-1}(\theta)$ is a stalk congruence of $\mathbf{A}$, or $\alpha^{-1}(\theta) = \nabla_{\mathbf{A}}$. Let α^* be the embedding which makes the following diagram commute (the unlabelled maps being natural):

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\alpha"] \arrow[d] & \mathbf{B} \arrow[d] \\ \mathbf{A}/\alpha^{-1}(\theta) \arrow[r, "\alpha^*"] & \mathbf{B}/\theta \end{tikzcd}$$

Figure 8

Noting that t is the ternary discriminator function on both $\mathbf{A}(t)/\alpha^{-1}(\theta)$ and $\mathbf{B}(t)/\theta$ (these quotients are defined in view of 2.8.6) we have, using $\overline{a}$ for $a/\alpha^{-1}(\theta)$, etc.,

$$\begin{aligned} t(a,b,c) = d \Rightarrow\ & t(\overline{a},\overline{b},\overline{c}) = \overline{d} \\ \Rightarrow\ & ((\overline{a}=\overline{b}) \wedge (\overline{c}=\overline{d})) \vee ((\overline{a}\neq\overline{b}) \wedge (\overline{a}=\overline{d})) \\ \Rightarrow\ & ((\alpha a/\theta = \alpha b/\theta) \wedge (\alpha c/\theta = \alpha d/\theta)) \vee ((\alpha a/\theta \neq \alpha b/\theta) \\ & \qquad \wedge (\alpha a/\theta = \alpha d/\theta)) \\ \Rightarrow\ & \mathbf{B}(t)/\theta \models t(\alpha a/\theta, \alpha b/\theta, \alpha c/\theta) = \alpha d/\theta. \end{aligned}$$

As θ is an arbitrary stalk congruence on $\mathbf{B}(t)$,

$$\mathbf{B}(t) \models t(\alpha a, \alpha b, \alpha c) = \alpha d.$$

By 2.8.4 it follows that $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$, as desired. □

SCRP ⇒ SCEP

Proposition 2.8.8 *For any class of algebras K*

$$\Gamma^a(K)_{\mathrm{SCRP}} \subseteq \Gamma^a(K)_{\mathrm{SCEP}}.$$

Proof Let $\mathbf{A} \in \Gamma^a(K)_{\mathrm{SCRP}}$, and let $\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^a(K)$ be an embedding. By 2.8.7 $\alpha : \mathbf{A}(t) \to \mathbf{B}(t)$, and by 2.3.12(ii) $\Gamma^a(K^t)$ has the SCEP, so by 2.8.6 $\mathbf{A} \in \Gamma^a(K)_{\mathrm{SCEP}}$. □

t-FAITHFUL REDUCTS OF EC'S ARE EC

Theorem 2.8.9 *Let K be any class of algebras. Then for $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$ we have*

$$\mathbf{A}(t) \in \mathrm{ISP}(K^t)^{ec} \Rightarrow \mathbf{A} \in \mathrm{ISP}(K)^{ec},$$

that is,

$$\mathrm{Red}_t \left[\mathrm{ISP}(K^t)^{ec}\right] \cap \Gamma^a(K)_{t-faithful} \subseteq \mathrm{ISP}(K)^{ec} \cap \Gamma^a(K).$$

Proof Let $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$ with $\mathbf{A}(t) \in \mathrm{ISP}(K^t)^{ec}$. To show $\mathbf{A} \in \mathrm{ISP}(K)^{ec}$ it suffices by 2.1.32 to show $\mathbf{A} \underset{ec}{\hookrightarrow} \Gamma^a(K)$. So let $\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^a(K)$ be an embedding. Then $\alpha : \mathbf{A}(t) \hookrightarrow \mathbf{B}(t)$. By hypothesis $\alpha : \mathbf{A}(t) \underset{ec}{\hookrightarrow} \mathbf{B}(t)$. Thus $\alpha : \mathbf{A} \underset{ec}{\hookrightarrow} \mathbf{B}$. □

(∀D) AND EC STRUCTURES

Proposition 2.8.10 *Let K be a class of structures such that $\Gamma^a(K)$ satisfies* (∀D). *Then*

$$\mathrm{ISP}(K)^{ec} \cap \Gamma^a(K) \subseteq \Gamma^a(K)_{t-faithful}.$$

Proof Let $\mathbf{A} \in \mathrm{ISP}(K)^{ec} \cap \Gamma^a(K)$, and suppose we have an embedding $\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^a(K)$. Choose τ to be a universal formula witnessing ($\forall$D) for $\Gamma^a(K)$. Then for $a_1, a_2, a_3, a \in \mathrm{A}$, since $\mathbf{A} \in \mathrm{ISP}(K)^{ec}$ and τ is a $\forall$ formula, we have

$$\mathbf{A} \models \tau(a_1, a_2, a_3, a) \Rightarrow \mathbf{B} \models \tau(\alpha a_1, \alpha a_2, \alpha a_3, \alpha a).$$

As $\mathbf{A}$ and $\mathbf{B}$ are both t-compatible by 2.7.5, and since τ is a discriminator formula for both, we thus have from 2.6.3

$$\mathbf{A}(t) \models t(a_1, a_2, a_3) = a \Rightarrow \mathbf{B}(t) \models t(\alpha a_1, \alpha a_2, \alpha a_3) = \alpha a.$$

Thus $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$. $\square$

Definition 2.8.11 For K a class of $\mathcal{L}$-structures, define the *completing part* K^{cp} of K by

$$K^{cp} = \{\mathbf{A} \in K : \mathbf{A} \leq \mathbf{B} \Rightarrow \mathbf{A} \preceq \mathbf{B} \text{ for } \mathbf{B} \in \mathrm{I}(K)\}.$$

Remark A class K is model complete iff $K^{cp} = K$.

Proposition 2.8.12 *Let K be a $\forall$ class with a model companion and such that $\Gamma^a(K)$ satisfies ($\forall$D). Then*

$$\left[\mathrm{Red}_t \mathrm{ISP}(K^t)^{mc}\right]^{cp} = \mathrm{I}\left[\mathrm{Red}_t \mathrm{ISP}(K^t)^{mc} \cap \Gamma^a(K)_{t-faithful}\right].$$

Proof First note that $\mathrm{ISP}(K^t)^{mc}$ exists by 2.5.9.

($\subseteq$): We know that $\mathrm{ISP}(K^t)^{mc} \subseteq \mathrm{I}\Gamma^a(K)$ by 2.3.7, so let

$$\mathbf{A} \in \left[\mathrm{Red}_t \mathrm{ISP}(K^t)^{mc}\right]^{cp} \cap \Gamma^a(K).$$

If $\alpha : \mathbf{A} \hookrightarrow \mathbf{B} \in \Gamma^a(K)$ choose

$$\mathbf{C} \in \mathrm{Red}_t \mathrm{ISP}(K^t)^{mc} \cap \Gamma^a(K)$$

such that we have some $\beta : \mathbf{B} \hookrightarrow \mathbf{C}$. Then

$$\gamma = \beta \circ \alpha : \mathbf{A} \hookrightarrow \mathbf{C}$$

is an elementary embedding as

$$\mathbf{A} \in \left[\mathrm{Red}_t \mathrm{ISP}(K^t)^{mc}\right]^{cp}.$$

Let τ be a $\forall$ formula witnessing ($\forall$D). Then for $a_1, a_2, a_3 \in \mathrm{A}$ we have (since $\mathbf{A}$ and $\mathbf{B}$ are t-compatible by 2.7.5)

$$\begin{array}{rl|l}
\mathbf{A}(t) \models t(a_1, a_2, a_3) = a & \Rightarrow \mathbf{A} \models \tau(a_1, a_2, a_3, a) & \text{by 2.6.3} \\
& \Rightarrow \mathbf{C} \models \tau(\gamma a_1, \gamma a_2, \gamma a_3, \gamma a) & \gamma \text{ is elementary} \\
& \Rightarrow \mathbf{B} \models \tau(\alpha a_1, \alpha a_2, \alpha a_3, \alpha a) & \tau \text{ is } \forall \\
& \Rightarrow \mathbf{B}(t) \models t(\alpha a_1, \alpha a_2, \alpha a_3) = \alpha a & \text{by 2.6.3.}
\end{array}$$

Thus $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$.

($\supseteq$): Let

$$\mathbf{A} \in \mathrm{Red}_t \mathrm{ISP}(K^t)^{mc} \cap \Gamma^a(K)_{t-faithful}.$$

Then for

$$\mathbf{B} \in \mathrm{Red}_t \mathrm{ISP}(K^t)^{mc} \cap \Gamma^a(K)$$

and $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ we have $\alpha : \mathbf{A}(t) \hookrightarrow \mathbf{B}(t)$ as $\mathbf{A} \in \Gamma^a(K)_{t-faithful}$. As $\mathbf{A}(t), \mathbf{B}(t) \in \mathrm{ISP}(K^t)^{mc}$, a model-complete class, the embedding $\alpha : \mathbf{A}(t) \hookrightarrow \mathbf{B}(t)$ is an elementary embedding. Thus $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ is elementary, so

$$\mathbf{A} \in \left[\mathrm{Red}_t \mathrm{ISP}(K^t)^{mc}\right]^{cp}. \qquad \square$$

2.9 Discriminator formulas: A model companion transfer theorem

In this section we look at the transfer of the property "is elementary" from a class K to $\mathrm{I}\Gamma^a(K)$, and apply it to the study of model companions.

Proposition 2.9.1 *Let K be an elementary class of algebras such that $\Gamma^a(K)$ satisfies* ($\exists$D). *Then*

$$\mathrm{Red}_t(\mathrm{ISP}(K^t)^{ec}) \cap \Gamma^a(K) \subseteq \mathrm{ISP}(K)^{ec} \cap \Gamma^a(K) \subseteq \mathrm{I}\Gamma^a_0(K).$$

Proof From 2.8.5 we see that

$$\Gamma^a(K)_{t-faithful} = \Gamma^a(K),$$

and then from 2.8.9

$$\mathrm{Red}_t(\mathrm{ISP}(K^t)^{ec}) \cap \Gamma^a(K) \subseteq \mathrm{ISP}(K)^{ec}.$$

For the second containment we use 2.1.31 to obtain

$$\mathrm{ISP}(K) \subseteq \mathrm{IS}\Gamma^e_0(K).$$

Then by 2.6.6(f) we know $\Gamma^a(K)$ satisfies ($\exists *$) and ($\exists **$), so let $\varphi_{(\emptyset)}$ be an existential formula witnessing ($\exists *$), and let $\varphi_{(\subseteq)}$ be an existential formula witnessing ($\exists **$). Let ψ_0 be the $\forall\exists$ sentence

$$\forall u_1 u_2 (u_1 \neq u_2 \rightarrow \exists v_1 v_2 v_3 v_4 \chi(u_1, u_2, v_1, v_2, v_3, v_4))$$

where $\chi(u_1, u_2, v_1, v_2, v_3, v_4)$ is the formula

$$v_1 \neq v_2 \wedge v_3 \neq v_4 \wedge \varphi_{(\subseteq)}(v_1, v_2, u_1, u_2) \wedge \varphi_{(\subseteq)}(v_3, v_4, u_1, u_2) \wedge \varphi_{(\emptyset)}(v_1, v_2, v_3, v_4).$$

For $\mathbf{A} \in \Gamma^a(K)$,

$$\mathbf{A} \models \psi_0 \Leftrightarrow \mathbf{A} \in \mathrm{I}\Gamma^a_0(K).$$

In particular, from $\Gamma_0^a(K) \models \psi_0$ and $\mathrm{ISP}(K) = \mathrm{IS}\Gamma_0^e(K)$ follows (by 1.10.5)

$$\mathrm{ISP}(K)^{ec} \models \psi_0.$$

Thus

$$\mathrm{ISP}(K)^{ec} \cap \Gamma^a(K) \subseteq \mathrm{I}\Gamma_0^a(K). \qquad \square$$

Definition 2.9.2 Let $\tau(u_1, u_2, u_3, v)$ be a formula in the language $\mathcal{L}$, and let σ be a formula in the language $\mathcal{L} \cup \{t\}$. Define σ^τ as follows: first put σ into an equivalent form (via some fixed procedure) such that all occurrences of t are in the form

$$t(\mathit{variable}\ 1, \mathit{variable}\ 2, \mathit{variable}\ 3) = \mathit{variable}\ 4,$$

and then replace each such atomic subformula by

$$\tau(\mathit{variable}\ 1, \mathit{variable}\ 2, \mathit{variable}\ 3, \mathit{variable}\ 4).$$

For Σ a set of $\mathcal{L} \cup \{t\}$-formulas let

$$\Sigma^\tau = \{\sigma^\tau : \sigma \in \Sigma\}.$$

Lemma 2.9.3 *If $\mathbf{A} \le \prod_{i \in I} \mathbf{A}_i$ is t-compatible and $\mathbf{A}$ has a discriminator formula τ, then for any $\mathcal{L} \cup \{t\}$-formula σ we have*

$$\mathbf{A}(t) \models \sigma \leftrightarrow \sigma^\tau.$$

Proof This is evident from 2.6.3, namely

$$\mathbf{A}(t) \models t(u_1, u_2, u_3) = v \leftrightarrow \tau(u_1, u_2, u_3, v). \qquad \square$$

DISCRIMINATOR FORMULAS AND REDUCTS OF ELEMENTARY CLASSES

Proposition 2.9.4 *Let K be a class of algebras such that $\Gamma^a(K)$ has a discriminator formula. If $K'' \subseteq \Gamma^a(K^t)$ and $K' = \mathrm{I}(K'')$ is an elementary class, then:*

(a) $\mathrm{Red}_t(K')$ *is also an elementary class,*

(b) *if K is an $\forall\exists$-class, then $\mathrm{I}\Gamma^a(K)$ is an elementary class,*

(c) *if K' is an $\forall\exists$-class, and $\Gamma^a(K)$ satisfies* $(\exists\mathrm{D})$, *then* $\mathrm{Red}_t(K')$ *is also an $\forall\exists$-class, and*

(d) *if K is an $\forall\exists$-class and $\Gamma^a(K)$ satisfies* $(\exists\mathrm{D})$, *then the elementary class* $\mathrm{Red}_t(K')$ *is axiomatized by*

$$\Sigma_{\mathrm{ISP}(K)} \cup (\Sigma_{K'})^\tau \cup \{\forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v)\} \tag{2.9}$$

where

τ is an existential discriminator formula for $\Gamma^a(K)$,

$\Sigma_{\mathrm{ISP}(K)}$ *is a set of axioms for* $\mathrm{ISP}(K)$, *and*

$\Sigma_{K'}$ *is a set of axioms for* K'.

Proof

(a) As K' is closed under ultraproducts it follows that $\mathrm{Red}_t K'$ is closed under ultraproducts. So let $\mathbf{A} \preceq \mathbf{B} \in K''$ and let τ be a discriminator formula for $\Gamma^a(K)$. Then $\mathbf{B}$ is t-compatible by 2.7.5, so $\mathbf{B} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v)$ by 2.7.7. Thus $\mathbf{A} \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v)$. Let $f, g, h \in \mathrm{A}$. Then there is a $k \in \mathrm{A}$ such that $\mathbf{A} \models \tau(f, g, h, k)$. Consequently $\mathbf{B} \models \tau(f, g, h, k)$, so we see by 2.6.3 that $\mathbf{B} \models t(f, g, h) = k$. Thus $\mathbf{A}$ is t-compatible, so $\mathbf{A}(t)$ is defined and $\mathbf{A}(t) \leq \mathbf{B}(t)$. Now τ is also a discriminator formula for $\mathbf{A}$ as

$$\begin{aligned} \mathbf{A} \models \tau(f,g,h,k) &\Leftrightarrow \mathbf{B} \models \tau(f,g,h,k) \\ &\Leftrightarrow [\![f = g]\!] \subseteq [\![h = k]\!] \quad \text{and} \quad [\![f \neq g]\!] \subseteq [\![f = k]\!]. \end{aligned}$$

Thus by 2.6.3

$$\mathbf{A}(t) \models \tau(u_1, u_2, u_3, v) \leftrightarrow t(u_1, u_2, u_3) = v,$$

and this leads to $\mathbf{A}(t) \preceq \mathbf{B}(t)$ as any $\mathcal{L} \cup \{t\}$-formula is equivalent modulo $\{\mathbf{A}(t), \mathbf{B}(t)\}$ to an $\mathcal{L}$-formula. Hence $\mathbf{A}(t) \in K'$ as K' is an elementary class. Therefore $\mathbf{A} \in \mathrm{Red}_t K'$.

(b) Since K^t is also an $\forall\exists$-class (just take the axioms for K and an open formula defining the ternary discriminator), it follows from 2.4.8 that $\mathrm{I}\Gamma^a(K^t)$ is an elementary class. Now apply (a).

(c) We know $\mathrm{Red}_t(K')$ is elementary from (a). Let

$$\mathbf{A}_1 \hookrightarrow \mathbf{A}_2 \hookrightarrow \cdots$$

be a sequence of embeddings with $\mathbf{A}_i \in \mathrm{Red}_t(K'')$. Then by 2.8.5

$$\mathbf{A}_1(t) \hookrightarrow \mathbf{A}_2(t) \hookrightarrow \cdots$$

is a sequence of embeddings with $\mathbf{A}_i(t) \in K''$. As K' is an $\forall\exists$-class, we have

$$\mathrm{Lim}_{i \geq 1} \mathbf{A}_i(t) \in K',$$

and thus

$$\mathrm{Lim}_{i \geq 1} \mathbf{A}_i \in \mathrm{Red}_t(K').$$

Consequently $\mathrm{Red}_t(K')$ is an $\forall\exists$-class.

(d) As

$$\left.\begin{aligned}\mathrm{Red}_t(K') &\subseteq \mathrm{Red}_t \mathrm{I}\Gamma^a(K^t) \\ &= \mathrm{I}\Gamma^a(K) \\ &\subseteq \mathrm{ISP}(K)\end{aligned}\right| \text{by } 2.7.6$$

we have

$$\mathrm{Red}_t(K') \models \Sigma_{\mathrm{ISP}(K)}.$$

From 2.7.5 and 2.7.7 we see that

$$\Gamma^a(K) \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v),$$

so

$$\mathrm{Red}_t(K') \models \forall u_1 u_2 u_3 \exists v \tau(u_1, u_2, u_3, v).$$

And by 2.9.3, as $K' \models \Sigma_{K'}$,

$$K' \models (\Sigma_{K'})^\tau ;$$

hence

$$\mathrm{Red}_t(K') \models (\Sigma_{K'})^\tau .$$

Thus $\mathrm{Red}_t(K')$ satisfies the proposed set of axioms in (2.9).

Conversely, suppose $\mathbf{A}$ is an $\mathcal{L}$-algebra and $\mathbf{A}$ satisfies the axioms in (2.9). As $\mathbf{A} \models \Sigma_{\mathrm{ISP}(K)}$, without loss of generality we can assume

$$\mathbf{A} \leq \mathbf{B} \underset{\mathrm{bp}}{\leq} \prod_{x \in X} \mathbf{B}_x, \qquad \mathbf{B}_x \in K,$$

as $\mathrm{ISP}(K) = \mathrm{IS}\Gamma_0^e(K)$ by 2.1.31. For $f, g, h \in \mathrm{A}$ choose $k \in \mathrm{A}$ such that

$$\mathbf{A} \models \tau(f, g, h, k).$$

(Such a k must exist by the last axiom in (2.9).) Then

$$\mathbf{B} \models \tau(f, g, h, k)$$

as τ is existential, so by 2.6.1

$$[\![f = g]\!] \subseteq [\![h = k]\!] \quad \text{and} \quad [\![f \neq g]\!] \subseteq [\![f = k]\!].$$

Thus τ is a discriminator formula for $\mathbf{A}$ (this follows from the uniqueness of k in the last expression above), so $\mathbf{A}$ is t-compatible by 2.7.3. Certainly

$$\mathbf{A}(t) \models (\Sigma_{K'})^\tau$$

as $\mathbf{A} \models (\Sigma_{K'})^\tau$, so from 2.9.3

$$\mathbf{A}(t) \models \Sigma_{K'};$$

hence

$$\mathbf{A}(t) \in K'.$$

Thus

$$\mathbf{A} \in \mathrm{Red}_t(K'). \qquad \square$$

THE TRANSFER OF MODEL COMPANIONS

Theorem 2.9.5 *Let K be an $\forall\exists$ class such that*

(i) $\Gamma^a(K_{(+)})$ *has a discriminator formula*

(ii) K^{mc} *exists*

(iii) $\mathrm{Red}_t(\Gamma^*(K^{mc})^t)) \subseteq \Gamma^a(K_{(+)})_{t-faithful}$.

Then

$$\mathrm{ISP}(K)^{mc} = \mathrm{Red}_t(\mathrm{I}\Gamma^*((K^{mc})^t)).$$

Proof Let $H = \mathrm{Red}_t(\mathrm{I}\Gamma^*((K^{mc})^t))$. Certainly K^t is $\forall\exists$ and $(K^{mc})^t$ is $\forall\exists$. If $\mathbf{A} \in K^{mc}$, $\mathbf{B} \in K$, and $\mathbf{A} \leq \mathbf{B}$, then $\mathbf{A}^t \underset{\mathrm{ec}}{\leq} \mathbf{B}^t$ since any existential formula φ in the language $\mathcal{L} \cup \{t\}$ is equivalent modulo K^t to an existential formula ψ in the language of K as

$$K^t \models t(u_1, u_2, u_3) = v \leftrightarrow (u_1 = u_2 \wedge u_3 = v) \vee (u_1 \neq u_2 \wedge u_1 = v).$$

Thus $(K^{mc})^t$ is the model companion of K^t. By 2.5.9 $\mathrm{ISP}(K^t)^{mc}$ exists and

$$\mathrm{ISP}(K^t)^{mc} = \mathrm{I}\Gamma^*((K^{mc})^t).$$

From 2.9.4(a) we see that H is an elementary class. From 2.8.9 and (iii) above we know

$$H \subseteq \mathrm{ISP}(K)^{ec};$$

hence H is model-complete. To finish the proof that $H = \mathrm{ISP}(K)^{mc}$ we only need to show that H and $\mathrm{ISP}(K)$ are mutually model consistent. Certainly $H \subseteq \mathrm{ISP}(K)$, so it remains to show $\mathrm{ISP}(K) \subseteq \mathrm{IS}(H)$. From

$$\mathrm{ISP}(K^t)^{mc} = \mathrm{I}\Gamma^*((K^{mc})^t)$$

follows

$$\mathrm{P}(K^t) \subseteq \mathrm{IS}\Gamma^*((K^{mc})^t).$$

Thus

$$\mathrm{P}(K) \subseteq \mathrm{Red}_t(\mathrm{IS}\Gamma^*((K^{mc})^t)) \subseteq \mathrm{IS}(H),$$

so

$$\mathrm{ISP}(K) \subseteq \mathrm{IS}(H). \qquad \Box$$

Remark One can easily find axioms for $\mathrm{ISP}(K)^{mc}$ above by using 2.5.9 and 2.9.4.

Corollary 2.9.6 *Let K be an $\forall\exists$ class such that $\Gamma^a(K_{(+)})$ has an existential discriminator formula. If the model companion K^{mc} exists, then $\mathrm{ISP}(K)^{mc}$ exists and*

$$\mathrm{ISP}(K)^{mc} = \mathrm{Red}_t(\mathrm{I}\Gamma^*((K^{mc})^t)).$$

Proof As $\Gamma^a(K_{(+)})$ has an existential discriminator formula, we know by 2.8.5 that

$$\Gamma^a(K_{(+)}) = \Gamma^a(K_{(+)})_{t-faithful}.$$

Also, by definition of Γ^*,

$$\mathrm{Red}_t(\Gamma^*((K^{mc})^t)) \subseteq \Gamma^a(K_{(+)}),$$

so

$$\mathrm{Red}_t(\Gamma^*((K^{mc})^t)) \subseteq \Gamma^a(K_{(+)})_{t-faithful}.$$

Now 2.9.5 applies. $\square$

TWO SPECIAL CASES

Corollary 2.9.7 *If K is an $\forall\exists$ class such that $\Gamma^a(K)$ has an existential discriminator formula, if no member of K properly contains a trivial subalgebra, and if K has a model companion, then*

$$\mathrm{ISP}(K)^{mc} = \mathrm{I}(\Gamma^a_0(K^{mc})).$$

Proof By 2.5.11 and 2.9.6. $\square$

Corollary 2.9.8 *If K is a nontrivial $\forall\exists$ class with a finite language such that $\Gamma^a(K_{(+)})$ has an existential discriminator formula and such that every member of K has exactly one trivial subalgebra and if K^{mc} exists and has a complete theory, then*

$$\mathrm{ISP}(K)^{mc} = \mathrm{I}\widehat{\Gamma}^a_0(K^{mc}).$$

Proof By 2.5.13 and 2.9.6. $\square$

Applications:

(1) discriminator varieties

(2) K a model-complete class of fields

(3) filtral varieties

(4) discrete o-groups

2.10 The Stalk Congruence Adequacy Property

Definition 2.10.1 A class K of algebras satisfies the SCAP (*stalk congruence adequacy property*) if for $\mathbf{A} \in \Gamma^a(S_K)$ and $\theta \in \mathrm{Con}(\mathbf{A})$ we have

$$\mathbf{A}/\theta \in \mathrm{IS}(K) \Rightarrow \theta \text{ is a stalk congruence of } \mathbf{A} \text{ or } \theta = \nabla_{\mathbf{A}}.$$

$\boxed{\text{SCAP AND } t\text{-FAITHFUL}}$

Proposition 2.10.2 *If K has the SCAP, then*

$$\Gamma^a(K)_{t-faithful} = \Gamma^a(S_K).$$

Proof From 2.8.3 we know $\Gamma^a(K)_{t-faithful} \subseteq \Gamma^a(S_K)$. For the opposite inclusion let $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ be an embedding with $\mathbf{A} \in \Gamma^a(S_K)$, $\mathbf{B} \in \Gamma^a(K)$. For θ a stalk congruence of $\mathbf{B}$ let α^* be the homomorphism which makes the following diagram commute (unlabelled maps are natural):

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\alpha"] \arrow[d] & \mathbf{B} \arrow[d] \\
\mathbf{A}/\alpha^{-1}(\theta) \arrow[r, "\alpha^*"] & \mathbf{B}/\theta
\end{tikzcd}$$

Figure 9

Thus $\alpha^*(\mathbf{A}/\alpha^{-1}(\theta)) \in \mathrm{IS}(K)$, and since K has the SCAP, it follows that $\alpha^{-1}(\theta)$ is a stalk congruence on $\mathbf{A}$, or $\alpha^{-1}(\theta) = \nabla_{\mathbf{A}}$. Consequently $\mathbf{A} \in \Gamma^a(K)_{\mathrm{SCRP}}$, so by 2.8.7 $\Gamma^a(S_K) \subseteq \Gamma^a(K)_{t-faithful}$. $\square$

Proposition 2.10.3 *If K has the SCAP, then*

$$\mathrm{Red}_t(\mathrm{ISP}(K^t)^{ec}) \cap \Gamma^a(S_K) \subseteq \mathrm{ISP}(K)^{ec}.$$

Proof This follows immediately from 2.8.9 and 2.10.2. $\square$

$\boxed{\text{SCAP AND A MODEL COMPANION TRANSFER RESULT}}$

Proposition 2.10.4 *Suppose K is an $\forall\exists$ class such that*

(i) $K_{(+)}$ *has the SCAP,*

(ii) K^{mc} *exists,*

(iii) $K^{mc} \subseteq S_K$, *and*

(iv) $\Gamma^a(K_{(+)})$ *has a discriminator formula.*

Then $\mathrm{ISP}(K)$ *has a model companion and*

$$\mathrm{ISP}(K)^{mc} \;=\; \mathrm{Red}_t(\mathrm{I}\Gamma^*((K^{mc})^t)).$$

Proof This follows from 2.9.5 and 2.10.2 since (iii) guarantees

$$\mathrm{Red}_t(\Gamma^*((K^{mc})^t)) \;\subseteq\; \Gamma^a(S_{K_{(+)}}). \qquad \Box$$

2.11 The Fraser-Horn-Hu Property

Definition 2.11.1 For r_i a binary relation on a set $S_i,\ 1 \leq i \leq n$, define the binary relation $r_1 \otimes \cdots \otimes r_n$ on $S_1 \times \cdots \times S_n$ by

$$r_1 \otimes \cdots \otimes r_n \;=\; \{\langle\langle a_1, \ldots, a_n\rangle, \langle b_1, \ldots, b_n\rangle\rangle : \langle a_i, b_i\rangle \in r_i,\ 1 \leq i \leq n\}.$$

Lemma 2.11.2 *For algebras* $\mathbf{A}_i,\ 1 \leq i \leq n$, *and congruences* $\theta_i \in \mathrm{Con}\ \mathbf{A}_i$ *we have*

$$\theta_1 \otimes \cdots \otimes \theta_n \;\in\; \mathrm{Con}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n).$$

Proof (Straightforward.) $\Box$

Definition 2.11.3 A class K of algebras has the FHHP (*Fraser-Horn-Hu property*) if for $\mathbf{A}_1, \mathbf{A}_2 \in K$ every congruence on $\mathbf{A}_1 \times \mathbf{A}_2$ is of the form $\theta_1 \otimes \theta_2$ where θ_i is a congruence on $\mathbf{A}_i$ for $i = 1, 2$.

Remark Every *congruence distributive variety* has the FHHP; and the class of rings (with 1) has the FHHP.

Lemma 2.11.4 *If K has the FHHP and* $\mathrm{I}(K)$ *is closed under finite products, then, for* $\mathbf{A}_1, \ldots, \mathbf{A}_n \in K$, *each congruence on* $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ *is of the form*

$$\theta_1 \otimes \cdots \otimes \theta_n,$$

where θ_i *is a congruence on* $\mathbf{A}_i$.

Proof (Straightforward.) $\Box$

Definition 2.11.5 For $\theta \in \mathrm{Con} \prod_{i \in I} \mathbf{A}_i$ and $i \in I$, let

$$\pi_i(\theta) \;=\; \{\langle f(i), g(i)\rangle : \langle f, g\rangle \in \theta\}.$$

Lemma 2.11.6 *For* $\theta_i \in \mathrm{Con}\ \mathbf{A}_i,\ i = 1, 2$, *we have*

(i) $\mathbf{A}_1 \times \mathbf{A}_2/\theta_1 \otimes \theta_2 \cong \mathbf{A}_1/\theta \times \mathbf{A}_2/\theta_2$

(ii) $\pi_i(\theta_1 \otimes \theta_2) = \theta_i,\ i = 1, 2.$

Proof For (i) use the obvious map

$$\langle a_1, a_2\rangle \;\mapsto\; \langle a_1/\theta_1, a_2/\theta_2\rangle.$$

Part (ii) is evident from the definition of $\otimes$. $\square$

Lemma 2.11.7 *Suppose K has the FHHP. Then for $\mathbf{A}_1, \mathbf{A}_2 \in K$ and $\theta \in \mathrm{Con}(\mathbf{A}_1 \times \mathbf{A}_2)$ we have $\pi_i(\theta) \in \mathrm{Con}\ \mathbf{A}_i,\ i = 1, 2,$ and*

$$\theta \;=\; \pi_1(\theta) \otimes \pi_2(\theta).$$

Proof We know $\theta = \theta_1 \otimes \theta_2$ for some $\theta_i \in \mathrm{Con}\ \mathbf{A}_i,\ i = 1, 2$, by the FHHP. By 2.11.6 it follows that $\pi_i(\theta) = \theta_i$; hence $\pi_i(\theta) \in \mathrm{Con}\ \mathbf{A}_i,\ i = 1, 2$, and $\theta = \pi_1(\theta) \otimes \pi_2(\theta)$. $\square$

Definition 2.11.8 A class K of $\mathcal{L}$-structures is *hereditary* if $\mathrm{S}(K) \subseteq K$.

WHEN DOES THE SCAP HOLD?

Proposition 2.11.9 *Let K be a hereditary class of directly indecomposable algebras such that $\Gamma^a(K)$ has the FHHP. Then K has the SCAP; hence*

$$\Gamma^a(K)_{t-faithful} \;=\; \Gamma^a(S_K).$$

Proof From 2.8.3 we know $\Gamma^a(K)_{t-faithful} \subseteq \Gamma^a(S_K)$. For the other inclusion let $\mathbf{A} \in \Gamma^a(S_K)$ and let N be a clopen subset of $X = X(\mathbf{A})$. For $\theta \in \mathrm{Con}\ \mathbf{A}$ define

$$\theta|_N \;=\; \{\langle f|_N, g|_N\rangle : \langle f, g\rangle \in \theta\}.$$

Now let $\theta \in \mathrm{Con}\ \mathbf{A} - \{\nabla\}$ be such that

$$\mathbf{A}/\theta \;\in\; \mathrm{IS}(K).$$

We want to show that θ is indeed a stalk congruence on $\mathbf{A}$. Let

$$\mathcal{S} \;=\; \{N \in X^* : \theta|_N \neq \nabla|_N\}.$$

Then we have, for $\mathbf{A}$ nontrivial:

(a) $X \in \mathcal{S}$ and $\emptyset \notin \mathcal{S}$

(b) $N \in \mathcal{S},\ N \subseteq M \in X^* \Rightarrow M \in \mathcal{S}$

(c) for $N \in X^*$, exactly one of $N,\ X - N$ is in $\mathcal{S}$.

For $N \in \{X, \emptyset\}$ this is covered in (a). Otherwise let φ be the congruence on $\mathbf{A}|_N \times \mathbf{A}|_{X-N}$ corresponding to θ (under the canonical isomorphism). Then by 2.11.7

$$\varphi \;=\; \theta|_N \otimes \theta|_{X-N},$$

so

$$\mathbf{A}/\theta \;\cong\; (\mathbf{A}|_N/\theta|_N) \times (\mathbf{A}|_{X-N}/\theta|_{X-N})\,.$$

As members of $\mathrm{IS}(K)$ are directly indecomposable it follows that exactly one of $\theta|_N,\ \theta|_{X-N}$ is not the ∇ of its algebra.

(d) If $N_1, N_2 \in \mathcal{S}$, then $N_1 \cap N_2 \neq \emptyset$.
Otherwise $N_1 \subseteq X - N_2$ so $X - N_2 \in \mathcal{S}$ by (b). This contradicts (c).

(e) If $N_1, N_2 \in \mathcal{S}$, then $N_1 \cap N_2 \in \mathcal{S}$.
Otherwise we have $N_1 - N_2 \neq \emptyset$ and $N_2 - N_1 \neq \emptyset$, so letting φ be the congruence on

$$\mathbf{A}|_{N_1-N_2} \times \mathbf{A}|_{N_1\cap N_2} \times \mathbf{A}|_{N_2-N_1} \times \mathbf{A}|_{X-(N_1\cup N_2)}$$

corresponding to θ we would have

$$\varphi = \theta|_{N_1-N_2} \otimes \theta|_{N_1\cap N_2} \otimes \theta|_{N_2-N_1} \otimes \theta|_{X-(N_1\cup N_2)}.$$

(We drop the last factor in each of the above if $X = N_1 \cup N_2$.) The first and third congruences in this decomposition must be proper on the appropriate factors as $\theta|_{N_1\cap N_2} = \nabla|_{N_1\cap N_2}$ by assumption, but $\theta|_{N_1} \neq \nabla|_{N_1}$ and $\theta|_{N_2} \neq \nabla|_{N_2}$. This leads to the decomposability of $\mathbf{A}/\theta$, which is not the case.

Consequently $\mathcal{S}$ is an ultrafilter on X^*, so there is an $x \in X$ such that

$$\bigcap \mathcal{S} = \{x\}.$$

Now we claim that θ is the stalk congruence κ_x (recall 2.1.15). Certainly $\theta \supseteq \kappa_x$, for

$$\begin{aligned} \langle f, g\rangle \in \kappa_x &\Rightarrow x \in [\![f = g]\!] \\ &\Rightarrow [\![f = g]\!] \in \mathcal{S} \\ &\Rightarrow \langle f, g\rangle \in \theta, \end{aligned}$$

the last step following from observing that either $f = g$ or, letting φ be the congruence on $\mathbf{A}|_{[[f=g]]} \times \mathbf{A}_{[[f\neq g]]}$ corresponding to θ, we have

$$\varphi = \theta|_{[[f=g]]} \otimes \theta|_{[[f\neq g]]}$$

holds, and consequently

$$\theta|_{[[f\neq g]]} = \nabla|_{[[f\neq g]]}$$

as $[\![f = g]\!] \in \mathcal{S}$.

Now by assumption $\mathbf{A}/\kappa_x$ is in $\mathrm{I}(S_K)$, and from the second isomorphism theorem

$$\mathbf{A}/\theta \cong (\mathbf{A}/\kappa_x)/(\theta/\kappa_x),$$

so it follows that $\theta = \kappa_x$. □

Corollary 2.11.10 *Let K be a universal class of directly indecomposable algebras such that $\Gamma^a(K)$ has the FHHP. Then*

$$\mathrm{Red}_t(\mathrm{ISP}(K^t)^{ec}) \cap \Gamma^a(S_K) \subseteq \mathrm{ISP}(K)^{ec}.$$

Proof By 2.10.3 and 2.11.9. □

Corollary 2.11.11 *Let K be a universal class of directly indecomposable algebras such that*

(i) *K has a model companion*

(ii) *$K^{mc} \subseteq S_K$*

(iii) *$\Gamma^a(K)$ has the FHHP*

(iv) *$\Gamma^a(K)$ has a discriminator formula.*

Then ISP(K) *has a model companion and*

$$\mathrm{ISP}(K)^{mc} \;=\; \mathrm{Red}_t(\mathrm{I}\Gamma^*((K^{mc})^t)).$$

Proof By 2.10.4 and 2.11.9. □

Definition 2.11.12 Let $\mathcal{L}$ be a language of algebras. An $\mathcal{L}$-formula $\pi \in \mathrm{P}_4^+$ is a *principal congruence formula* if given any $\mathcal{L}$-algebra $\mathbf{A}$ and $a, b, c, d \in \mathrm{A}$ we have

$$\mathbf{A} \models \pi(a, b, c, d) \;\Rightarrow\; \langle a, b\rangle \in \Theta_{\mathbf{A}}(c, d).$$

Let $\prod_{\mathcal{L}}$ be the set of principal congruence formulas for the language $\mathcal{L}$.

Theorem 2.11.13 *For $\mathbf{A}$ an $\mathcal{L}$-algebra and $a, b, c, d \in \mathrm{A}$ we have*

$$\langle a, b\rangle \in \Theta_{\mathbf{A}}(c, d) \;\Leftrightarrow\; \mathbf{A} \models \bigvee\!\!\!\bigvee \left\{\pi(a, b, c, d) : \pi \in \textstyle\prod_{\mathcal{L}}\right\}.$$

Proof (See [9].) □

Definition 2.11.14 A class K of $\mathcal{L}$-algebras has *directed principal congruence formulas* if for every pair $\pi_1, \pi_2 \in \prod_{\mathcal{L}}$ there is a $\pi \in \prod_{\mathcal{L}}$ such that

$$K \models \pi \to \pi_1 \vee \pi_2.$$

Theorem 2.11.15 *Let V be a variety with the FHHP. Then V has directed principal congruence formulas.*

Proof (See [7].) □

Lemma 2.11.16 *If V is a variety with the FHHP, then $V \models \widehat{\pi}(x, x, u, v)$ for some $\widehat{\pi} \in \prod_{\mathcal{L}}$.*

Proof From 2.11.13 we have

$$V \models \bigvee\!\!\!\bigvee \left\{\pi(x,x,u,v) : \pi \in \prod\nolimits_{\mathcal{L}}\right\},$$

so the result follows using compactness and the fact that V has directed principal congruences. $\square$

Lemma 2.11.17 *Let K be a universal class of finitely subdirectly irreducible algebras in a variety V with the FHHP. Then*

$$u_1 = v_1 \ \vee\ u_2 = v_2$$

is definable in K by a formula $\Phi(u_1, v_1, u_2, v_2)$ of the form

$$\forall w_1 \forall w_2 (\pi(w_1, w_2, u_1, v_1) \ \wedge\ \pi(w_1, w_2, u_2, v_2) \ \rightarrow\ w_1 \ =\ w_2),$$

where π is a suitable principal congruence formula.

Proof Since the members of K are finitely subdirectly irreducible, it suffices to show that

$$\Theta(a,b) \cap \Theta(a',b') \ =\ \Delta$$

is definable in K by a formula of the above form.

By 2.11.15, V has directed principal congruence formulas. Let $\mathbf{A} \in K,\ a, b, a', b' \in$ A. Then

$$\begin{aligned}
&\Theta(a,b) \cap \Theta(a',b') \neq \Delta \\
&\qquad \Leftrightarrow\ \langle c,d\rangle \in \Theta(a,b) \cap \Theta(a',b') && \text{for some } c \neq d \\
&\qquad \Leftrightarrow\ \mathbf{A} \models \pi_1(c,d,a,b) \wedge \pi_2(c,d,a',b') && \text{for some } c \neq d \text{ and } \pi_1, \pi_2 \in \prod \\
&\qquad \Leftrightarrow\ \mathbf{A} \models \pi(c,d,a,b) \wedge \pi(c,d,a',b') && \text{for some } c \neq d \text{ and } \pi \in \prod
\end{aligned}$$

where $\prod$ is the set of principal congruence formulas for the language of K.

Therefore

$$\Theta(a,b) \cap \Theta(a',b') = \Delta \ \Leftrightarrow\ \mathbf{A} \models \forall w_1 \forall w_2 (\pi(w_1,w_2,a,b) \wedge \pi(w_1,w_2,a',b') \rightarrow w_1 = w_2)$$

for all $\pi \in \prod$. Hence

$$K \ \models\ (u_1 = v_1 \vee u_2 = v_2) \ \leftrightarrow\ \lambda(u_1, v_1, u_2, v_2)$$

where $\lambda(u_1, v_1, u_2, v_2)$ is the infinitary formula

$$\bigwedge\!\!\!\bigwedge \left\{\forall w_1 \forall w_2 [\pi(w_1, w_2, u_1, v_1) \ \wedge\ \pi(w_1, w_2, u_2, v_2) \ \rightarrow w_1 = w_2] : \pi \in \prod\right\}.$$

By the compactness theorem, and the fact that K has directed principal congruence formulas, we see that

$$u_1 = v_1 \ \vee\ u_2 = v_2$$

is equivalent modulo K to a universal formula $\Phi(u_1, v_1, u_2, v_2)$ of the form

$$\forall w_1 \forall w_2 (\pi(w_1, w_2, u_1, v_1) \wedge \pi(w_1, w_2, u_2, v_2) \rightarrow w_1 = w_2). \qquad \square$$

$\boxed{\text{THE FHHP AND } (\forall^*)}$

Proposition 2.11.18 *Let K be a universal class of finitely subdirectly irreducible algebras in a variety V with the FHHP. Then $\Gamma^a(K) \cup \mathrm{P}(K)$ has the property $(\forall^*)$.*

Proof Let $\mathbf{A} \underset{\mathrm{bp}}{\leq} \prod_{x \in X} \mathbf{A}_x$ or $\mathbf{A} = \prod_{x \in X} \mathbf{A}_x$, where $\mathbf{A}_x \in K$ for $x \in X$. Let $f_1, g_1, f_2, g_2 \in \mathrm{A}$, and suppose

$$[\![f_1 \neq g_1]\!] \cap [\![f_2 \neq g_2]\!] = \emptyset.$$

Then for all $x \in X$,

$$\mathbf{A}_x \models f_1(x) = g_1(x) \vee f_2(x) = g_2(x),$$

and hence

$$\mathbf{A}_x \models \Phi(f_1(x), g_1(x), f_2(x), g_2(x)),$$

where Φ is the universal formula of Lemma 2.11.17. Since Φ is equivalent to a universal Horn formula, it follows that

$$\mathbf{A} \models \Phi(f_1, g_1, f_2, g_2).$$

For the converse, we must show that

$$[\![f_1 \neq g_1]\!] \cap [\![f_2 \neq g_2]\!] \neq \emptyset \Rightarrow \mathbf{A} \not\models \Phi(f_1, g_1, f_2, g_2).$$

Assume there exists an $x \in [\![f_1 \neq g_1]\!] \cap [\![f_2 \neq g_2]\!]$. Since $\mathbf{A}_x \in K$, it follows that

$$\mathbf{A}_x \models \neg \Phi(f_1(x), g_1(x), f_2(x), g_2(x)),$$

that is,

$$\mathbf{A}_x \models \exists w_1 \exists w_2 (\pi(w_1, w_2, f_1(x), g_1(x)) \wedge \pi(w_1, w_2, f_2(x), g_2(x)) \wedge w_1 \neq w_2).$$

Pick $c_0, d_0 \in \mathbf{A}_x$ such that $c_0 \neq d_0$ and

$$\mathbf{A}_x \models \pi(c_0, d_0, f_1(x), g_1(x)) \wedge \pi(c_0, d_0, f_2(x), g_2(x)).$$

Let h and k be any elements of $\mathbf{A}$ such that $h(x) = c_0$ and $k(x) = d_0$. Since π is a primitive positive formula, there exists a clopen subset N of X such that

$$N \subseteq [\![\pi(h, k, f_1, g_1) \wedge \pi(h, k, f_2, g_2)]\!],$$

and $x \in N$. Define $h' = h$ and $k' = k|_N \cup h|_{X-N}$.

We can assume that $V \models \widehat{\pi} \to \pi$, where $\widehat{\pi}$ is as in 2.11.16. Then, since $V \models w_1 = w_2 \to \pi(w_1, w_2, u, v)$, we have

$$[\![\pi(h', k', f_1, g_1) \wedge \pi(h', k', f_2, g_2)]\!] = X,$$

so (see 2.1.19) it follows that

$$\mathbf{A} \models \pi(h', k', f_1, g_1) \wedge \pi(h', k', f_2, g_2),$$

and $h' \neq k'$. Hence $\mathbf{A} \not\models \Phi(f_1, g_1, f_2, g_2)$. □

Problem 7. Let K be a universal class of finitely subdirectly irreducible algebras in a variety V with the FHHP. Does $\mathrm{SP}(K)$ have the property (O^*)?

A MODEL COMPANION TRANSFER THEOREM

Theorem 2.11.19 *Let K be a universal class of finitely subdirectly irreducible algebras in a variety with the FHHP and such that*

(i) *K has a model companion, and*

(ii) *$K^{mc} \subseteq S_K$.*

Then $\mathrm{ISP}(K)$ *has a model companion and*

$$\mathrm{ISP}(K)^{mc} = \mathrm{Red}_t(\mathrm{I}\Gamma^*((K^{mc})^t)).$$

Proof By 2.11.18 and 2.6.6 we know that $\Gamma^a(K)$ has a discriminator formula, so the result follows from 2.11.11. □

WHEN DO EC BOOLEAN PRODUCTS HAVE K-SIMPLE STALKS?

Theorem 2.11.20 *Let K be a class of algebras such that*

(i) *$\Gamma^a(K)$ satisfies the property $(\exists^*)$, and*

(ii) *$\Gamma^a(K)$ has the FHHP.*

Then

$$\mathrm{ISP}(K)^{ec} \cap \Gamma^a(K) \subseteq \Gamma^a(S_K).$$

Proof Let $\mathbf{A} \underset{\mathrm{bp}}{\leq} \prod_{x \in X} \mathbf{A}_x$ be a Boolean product, where $\mathbf{A}_x \in K$ for $x \in X$, and suppose some $\mathbf{A}_x$ is not K-simple.

Choose $\theta = \Theta(a, b) \in \mathrm{Con}\, \mathbf{A}_x$ be such that $\Delta < \theta < \nabla$ and $\mathbf{A}_x/\theta \in K$. Let $c \in \mathrm{A}_x$ be such that $\langle a, c \rangle \notin \theta$. Choose $f, g, h \in \mathrm{A}$ such that $f(x) = a,\ g(x) =$

b, $h(x) = c$, and such that $[\![f = g]\!] = [\![f = h]\!] = [\![g = h]\!]$. Let $\mathbf{B} = (\mathbf{A}_x/\theta) \times \mathbf{A}$. Then we have an embedding $\alpha : \mathbf{A} \hookrightarrow \mathbf{B}$ defined by

$$\alpha(e) \;=\; \langle e(x)/\theta, e\rangle.$$

Let $\overline{f} = \alpha(f)$, $\overline{g} = \alpha(g)$, $\overline{h} = \alpha(h)$. Then let $\overline{k} = \langle a/\theta, h\rangle$. By the FHHP for $\Gamma^a(K)$ we see that $\langle \overline{h}, \overline{k}\rangle \in \Theta(\overline{f}, \overline{h})$ since this holds for the two projections (onto $\mathbf{A}_x/\theta$ and $\mathbf{A}$). Hence there is a principal congruence formula $\pi(w_1, w_2, u, v)$ such that

$$\mathbf{B} \models \pi(\overline{h}, \overline{k}, \overline{f}, \overline{h}).$$

Now let $\varphi_{(\emptyset)}$ express $(\exists^*)$ in $\Gamma^a(K)$. Then $\mathbf{B} \models \varphi_{(\emptyset)}(\overline{f}, \overline{g}, \overline{h}, \overline{k})$. Thus

$$\mathbf{B} \models \exists v[\overline{h} \neq v \wedge \pi(\overline{h}, v, \overline{f}, \overline{h}) \wedge \varphi_{(\emptyset)}(\overline{f}, \overline{g}, \overline{h}, v)].$$

However $\mathbf{A}$ does not satisfy this existential formula (replacing $\overline{h}$ by h, etc.) since the last two parts of the formula imply "$[\![f = h]\!] \subseteq [\![h = v]\!]$" and "$[\![f \neq g]\!] \subseteq [\![h = v]\!]$". $\square$

$\boxed{\text{A TEST FOR NO MODEL COMPANION}}$

Theorem 2.11.21 *Let K be a universal class of directly indecomposable algebras such that*

(i) $\Gamma^a(K)$ *satisfies* $(\exists^*)$

(ii) $\Gamma^a(K)$ *has the FHHP*

and either

(iii) *no member of K properly contains a trivial subalgebra*

or

(iii′) *we have*

- (a) *K has a model companion*
- (b) *K^{mc} is complete*
- (c) *every member of K has exactly one trivial subalgebra, and*
- (d) *the language of K is finite.*

Then if some member of $K^{ec} \cap S_K$ has an ultrapower which is not K-simple, we can conclude that $\mathrm{ISP}(K)$ *does not have a model companion.*

Proof Let $\mathbf{S} \in S_K$, but $\widehat{\mathbf{S}} = \mathbf{S}^I/\mathcal{U} \notin S_K$, where $\mathcal{U}$ is an ultrafilter on I. Let C be the Cantor discontinuum. If case (iii) holds, we have, by 2.11.10 and 2.5.11,

$$\mathbf{S}[C]^* \in \mathrm{ISP}(K)^{ec}.$$

Now

$$\mathbf{S} \equiv \widehat{\mathbf{S}} \Rightarrow \mathbf{S}[C]^* \equiv \widehat{\mathbf{S}}[C]^*,$$

but by 2.11.20

$$\widehat{\mathbf{S}}[C]^* \notin \mathrm{ISP}(K)^{ec};$$

hence $\mathrm{ISP}(K)^{ec}$ is not an elementary class.

If (iii$'$) holds, choose $x \in C$ and let $\mathbf{A}$ be the subalgebra of $\mathbf{S}[C]^*$ with universe

$$\mathrm{A} = \{f \in S[C]^* : f(x) = \mathrm{O}_S\},$$

$\{\mathrm{O}_S\}$ being the unique one-element subuniverse of $\mathbf{S}$. Likewise let $\widehat{\mathbf{A}}$ be the subalgebra of $\mathbf{S}[C]^*$ with universe

$$\widehat{\mathbf{A}} = \{f \in \widehat{\mathrm{S}}[C]^* : f(x) = \mathrm{O}_{\widehat{S}}\}.$$

Then, by 2.11.10 and 2.5.13,

$$\mathbf{A} \in \mathrm{ISP}(K)^{ec}.$$

By [11]

$$\mathbf{A} \equiv \widehat{\mathbf{A}},$$

but again by 2.11.20

$$\widehat{\mathbf{A}} \notin \mathrm{ISP}(K)^{ec},$$

so $\mathrm{ISP}(K)$ does not have a model companion. $\square$

2.12 In congruence distributive varieties

Lemma 2.12.1 *If V is a congruence distributive variety, then there are ternary terms $p_0(u,v,w), \ldots, p_n(u,v,w)$ such that*

$$V \models p_i(u,v,u) = p_i(u,w,u), \qquad 1 \le i \le n-1$$
$$V \models u \ne v \to (p_1(u,u,v) \ne p_1(u,v,v) \vee \cdots \vee p_{n-1}(u,u,v) \ne p_{n-1}(u,v,v)).$$

Proof (See [9].) $\square$

Lemma 2.12.2 *Let K be a universal class of finitely subdirectly irreducible algebras in a congruence distributive variety V, and let $p_1, \ldots, p_n$ be ternary terms as in* 2.12.1. *Then for $\mathbf{A} \in K$ and $a, b, a', b' \in \mathrm{A}$ we have*

$$\Theta(a,b) \cap \Theta(a',b') \ne \Delta$$

iff

$$\mathbf{A} \models p_i(p(a), q(a'), p(b)) \neq p_i(p(a), q(b'), p(b))$$

for some $p, q \in Pol$ *and some* $i,\ 1 \le i \le n-1$, *where* Pol *is the set of unary polynomials with constants from* $\{a, b, a', b'\}$.

Proof $(\Leftarrow)$ It is clear that since $\mathbf{A}$ is finitely subdirectly irreducible, $\Theta(a,b) \cap \Theta(a',b') \neq \Delta$ iff $a \neq b$ and $a' \neq b'$. We may therefore assume without loss of generality that $\mathbf{A}$ is generated by the elements a, b, a', b'.

Suppose $c \neq d$ and

$$\langle c, d\rangle \in \Theta(a,b) \cap \Theta(a',b').$$

Then we claim that for some $\widehat{p}(x) \in Pol$ and for some j, we have

$$p_j(c, \widehat{p}(a), d) \neq p_j(c, \widehat{p}(b), d).$$

To see this, first note that the equivalence relation on A generated by $\{\langle \widehat{p}(a), \widehat{p}(b)\rangle : \widehat{p} \in Pol\}$ is $\Theta(a,b)$.

As $\langle c, d\rangle \in \Theta(a,b)$, we see that for each i, $\langle p_i(c,c,d), p_i(c,d,d)\rangle$ is in the equivalence relation generated by $\{\langle p_i(c, \widehat{p}(a), d), p_i(c, \widehat{p}(b), d)\rangle : \widehat{p} \in Pol\}$.

As $c \neq d$, for some j we know $p_j(c,c,d) \neq p_j(c,d,d)$ by Lemma 2.12.1; hence for some $\widehat{p}$, and the same j,

$$p_j(c, \widehat{p}(a), d) \neq p_j(c, \widehat{p}(b), d),$$

proving the claim.

Since we are assuming that $\mathbf{A} = \mathrm{Sg}(a, b, a', b')$, c and d can be represented as polynomials in a, b, a', b', and hence we have a $p \in Pol$ such that $p(a) \neq p(b)$, and furthermore, $\langle p(a), p(b)\rangle \in \Theta(a',b')$, as $\langle p(a), p(b)\rangle \in \Theta(c,d)$, because of Lemma 2.12.1.

Now starting with $\langle p(a), p(b)\rangle$ instead of $\langle c, d\rangle$, we can repeat the above argument to find $q \in Pol$, and $i,\ 1 \le i \le n-1$, such that

$$p_i(p(a), q(a'), p(b)) \neq p_i(p(a), q(b'), p(b)),$$

as desired.

$(\Rightarrow)$ If for some i,

$$p_i(p(a), q(a'), p(b)) \neq p_i(p(a), q(b'), p(b))$$

then, as the ordered pair consisting of these two distinct elements is in both $\Theta(a,b)$ and $\Theta(a',b')$ by Lemma 2.12.1, we have $\Theta(a,b) \cap \Theta(a',b') \neq \Delta$. $\square$

Lemma 2.12.3 *Let* K *be a universal class of finitely subdirectly irreducible algebras in a congruence distributive variety* V. *Then*

$$u_1 = v_1 \vee u_2 = v_2$$

is definable in K *by a formula* $\Phi(u_1, v_1, u_2, v_2)$ *of the form* $\bigwedge$*atomic.*

Proof By Lemma 2.12.2 and the fact that for a finitely subdirectly irreducible algebra $\Theta(a,b) \cap \Theta(a',b') = \Delta$ holds iff $a = b$ or $a' = b'$, we have for any $\mathbf{A} \in K$ and $a, b, a', b' \in A$,

$$\mathbf{A} \models a = b \vee a' = b' \Leftrightarrow \mathbf{A} \models \Sigma,$$

where

$$\Sigma = \{p_i(p(a), q(a'), p(b)) = p_i(p(a), q(b'), p(b)) : p, q \in Pol,\ 1 \le i \le n-1\}.$$

Σ is a set of atomic sentences $\{\alpha_i(a,b,a',b') : i \in I\}$, and we have

$$\mathbf{A} \models (u_1 = v_1 \vee u_2 = v_2) \Leftrightarrow \bigwedge\!\!\!\bigwedge \{\alpha_i(u_1, v_1, u_2, v_2) : i \in I\}.$$

Hence

$$K \models (u_1 = v_1 \vee u_2 = v_2) \Leftrightarrow \bigwedge\!\!\!\bigwedge \{\alpha_i(u_1, v_1, u_2, v_2) : i \in I\}.$$

By the compactness theorem, there exist $\alpha_1(u_1, v_1, u_2, v_2), \ldots, \alpha_n(u_1, v_1, u_2, v_2)$ such that

$$K \models (u_1 = v_1 \vee u_2 = v_2) \Leftrightarrow \bigwedge_{i=1}^{n} \alpha_i(u_1, v_1, u_2, v_2),$$

and from this the conclusion follows, with $\Phi(u_1, v_1, u_2, v_2) = \bigwedge_{i=1}^{n} \alpha_i(u_1, v_1, u_2, v_2)$. $\square$

CONGRUENCE DISTRIBUTIVE VARIETIES AND (**O**$*$)

Proposition 2.12.4 *Let K be a universal class of finitely subdirectly irreducible algebras in a congruence distributive variety. Then* $\mathrm{SP}(K)$ *has the property* (**O**$*$).

Proof Let $\mathbf{A} \le_{\mathrm{sd}} \prod_{x \in X} \mathbf{A}_x$, where $\mathbf{A}_x \in K$ for $x \in X$. For $f_1, g_1, f_2, g_2 \in A$,

$$\begin{aligned}
&[\![f_1 \neq g_1]\!] \cap [\![f_2 \neq g_2]\!] = \emptyset \\
&\Leftrightarrow f_1(x) = g_1(x) \text{ or } f_2(x) = g_2(x) && \text{for all } x \in X \\
&\Leftrightarrow \mathbf{A}_x \models \Phi(f_1(x), g_1(x), f_2(x), g_2(x)) && \text{for all } x \in X \\
&\Leftrightarrow [\![\Phi(f_1, g_1, f_2, g_2)]\!] = X \\
&\Leftrightarrow \mathbf{A} \models \Phi(f_1, g_1, f_2, g_2),
\end{aligned}$$

where Φ is the open formula of Lemma 2.12.3. $\square$

Example The variety of ℓ-groups is congruence distributive. If we let K be the class of totally ordered ℓ-groups, $\Phi(u_1, v_1, u_2, v_2)$ can be taken to be $|u_1 - v_1| \wedge |u_2 - v_2| = 0$, where $|u| = (u \vee 0) - (u \wedge 0)$.

A TEST FOR NO MODEL COMPANION IN CONGRUENCE DISTRIBUTIVE VARIETIES

Corollary 2.12.5 *Let K be a universal class of finitely subdirectly irreducible algebras in a congruence distributive variety such that either*

(a) *no member of K properly contains a trivial subalgebra, or*

(b) *every member of K has exactly one trivial subalgebra, K has a complete model companion, and the language of K is finite.*

If $\widehat{\mathrm{P}}_u(K^{ec} \cap S_K) \nsubseteq S_K$, then $\mathrm{ISP}(K)$ has no model companion.

Proof By Proposition 2.12.4 we have $(\exists *)$ and the FHHP holds, so 2.11.21 applies. $\square$

Corollary 2.12.6 *The variety of ℓ-groups has no model companion.*

Proof Let K be the class of totally ordered ℓ-groups. Then K is a $\forall$ class of finitely subdirectly irreducible algebras, and (b) holds since K^{mc} is the class of divisible ℓ-groups in K.

The ℓ-group of rational numbers $\mathbb{Q}$ is in $K^{mc} \cap S_K$. But $\mathbb{Q}$ has an ultrapower $\mathbb{Q}^*$, the non-standard rational numbers, which is not Archimedean, and hence not simple. Since K is closed under homomorphic images, the concepts simple and K-simple coincide. By Corollary 2.12.5, the variety of ℓ-groups, which equals $\mathrm{ISP}(K)$, has no model companion since $\widehat{P}_u(K^{mc} \cap S_K) \nsubseteq S_K$. $\square$

2.13 Finitely generic Boolean products

After Saracino and Wood [20] had described the finitely generic lattice ordered abelian group through the construction of a prime existentially closed member of this class, Point [17] showed how the characterization of such finitely generic structures could be derived from a knowledge of discriminator varieties.

Theorem 2.13.1 *Let K be a $\forall$ class of algebras such that*

(i) *K has a model companion,*

(ii) *$\Gamma^a(K)$ has the property $(\forall \mathrm{D})$, and*

(iii) $\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \Gamma^a(K)_{t-faithful} \neq \emptyset$.

Then

$$\begin{aligned} \mathrm{ISP}(K)^{ffc} &= \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \\ \mathrm{ISP}(K)^{fg} &= \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{ISP}(K)^{ec} \end{aligned}$$

holds iff

$$\mathrm{Th}(\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc}) = \mathrm{Th}(\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{ISP}(K)^{ec}). \qquad (*)$$

Proof We want to apply 1.11.1. The direction ($\Rightarrow$) is obvious by 1.11.1(ii). For ($\Leftarrow$) first note that

$$\mathrm{P}(K) \subseteq \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \subseteq \mathrm{ISP}(K)$$

so

$$\mathrm{ISP}(K) = \mathrm{S}\ \mathrm{Red}_t\mathrm{ISP}(K^t).$$

This gives 1.11.1(i). By 2.9.4(a) we see that $\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc}$ is an elementary class, so 1.11.1(ii) follows from the assumption ($*$). From 2.8.12 we see that to finish the proof we only need to show that

$$\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{ISP}(K)^{ec} = \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \Gamma^a(K)_{t-faithful}$$

holds. This follows from:

$$\begin{array}{l|l}
\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{ISP}(K)^{ec} & \\
\subseteq \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{I}\Gamma^a(K) \cap \mathrm{ISP}(K)^{ec} & \text{by 2.3.7 \& 2.7.6} \\
\subseteq \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{I}\Gamma^a(K)_{t\text{-}faithful} & \text{by 2.8.10} \\
\subseteq \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{I}\Gamma^a(K) \cap \mathrm{ISP}(K)^{ec} & \text{by 2.8.9} \\
\subseteq \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{ISP}(K)^{ec} & \text{by 2.3.7 \& 2.7.6} \quad \square
\end{array}$$

Corollary 2.13.2 *Let K be a $\forall$ class of algebras such that*

(i) *K has a model companion,*

(ii) *$\Gamma^a(K)$ has the property* ($\forall$D),

(iii) $\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \Gamma^a(K)_{t-faithful} \neq \emptyset$, *and*

(iv) *K has the JEP.*

Then

$$\begin{aligned}
\mathrm{ISP}(K)^{ffc} &= \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \\
\mathrm{ISP}(K)^{fg} &= \mathrm{Red}_t\mathrm{ISP}(K^t)^{mc} \cap \mathrm{ISP}(K)^{ec}.
\end{aligned}$$

Proof From 2.3.13 we see that $\mathrm{ISP}(K^t)$ has the JEP, so by 1.7.6 the model companion $\mathrm{ISP}(K^t)^{mc}$ has a complete theory. Consequently $\mathrm{Red}_t\mathrm{ISP}(K^t)^{mc}$ has a complete theory, and we have the desired condition ($*$) of 2.13.1. $\square$

Corollary 2.13.3 *Let K be a universal class of finitely subdirectly irreducible algebras such that*

(i) *K has a model companion,*

(ii) $\mathrm{V}(K)$ *is congruence distributive,*

(iii) $K^{mc} \cap S_K \neq \emptyset$, *and*

(iv) *either no member of* K *properly contains a trivial subalgebra, or*

(iv′) K^{mc} *is complete and every member* K^{mc} *has exactly one trivial subalgebra.*

Then

$$\mathrm{ISP}(K)^{ffc} = \begin{cases} \mathrm{I}\Gamma_0^a(K^{mc}) & \textit{if } \text{(iv)} \\ \mathrm{I}\widehat{\Gamma}_0^a(K^{mc}) & \textit{if } \text{(iv}'\text{)} \end{cases}$$

$$\mathrm{ISP}(K)^{fg} = \begin{cases} \mathrm{I}\Gamma_0^a(K^{mc} \cap S_K) & \textit{if } \text{(iv)} \\ \mathrm{I}\widehat{\Gamma}_0^a(K^{mc} \cap S_K) & \textit{if } \text{(iv}'\text{)} \end{cases}$$

holds iff

$$\begin{cases} \mathrm{Th}(\Gamma_0^a(K^{mc})) = \mathrm{Th}(\Gamma_0^a(K^{mc} \cap S_K)) & \textit{if } \text{(iv)} \\ \mathrm{Th}(\widehat{\Gamma}_0^a(K^{mc})) = \mathrm{Th}(\widehat{\Gamma}_0^a(K^{mc} \cap S_K)) & \textit{if } \text{(iv}'\text{)}. \end{cases}$$

Proof A straightforward application of 2.13.1 in view of 2.11.9, 2.11.18, 2.15.11 and 2.15.13. □

Corollary 2.13.4 *Let* K *be a universal class of finitely subdirectly irreducible algebras such that*

(i) K *has a model companion,*

(ii) $\mathrm{V}(K)$ *is congruence distributive,*

(iii) $K^{mc} \cap S_K \neq \emptyset$,

(iv) *either no member of* K *properly contains a trivial subalgebra, or*

(iv′) K^{mc} *is complete and every member* K^{mc} *has exactly one trivial subalgebra, and*

(v) K *has the JEP.*

Then

$$\mathrm{ISP}(K)^{ffc} = \begin{cases} \mathrm{I}\Gamma_0^a(K^{mc}) & \textit{if } \text{(iv)} \\ \mathrm{I}\widehat{\Gamma}_0^a(K^{mc}) & \textit{if } \text{(iv}'\text{)} \end{cases}$$

$$\mathrm{ISP}(K)^{fg} = \begin{cases} \mathrm{I}\Gamma_0^a(K^{mc} \cap S_K) & \textit{if } \text{(iv)} \\ \mathrm{I}\widehat{\Gamma}_0^a(K^{mc} \cap S_K) & \textit{if } \text{(iv}'\text{)}. \end{cases}$$

Proof As in 2.13.3, except apply 2.13.2. □

Example Let K be the class of linearly ordered abelian groups, so $\mathrm{ISP}(K)$ is the class of lattice ordered abelian groups. We noted in the last section that this class does not have a model companion; however we do have a nice description of the finitely generic members and the finite forcing companion using 2.13.4.

Credits and Notes

Chapter 1

Section 1.1

1.1.1 Tarski & Vaught 1957 (see [10, Th. 3.1.13 p. 115, and p. 518]).

1.1.2 Shelah 1971 (see [10, Th. 6.1.15 p. 319, and p. 524]).

1.1.3 See [9, Th. V.2.16 p. 213].

1.1.4 Łoś 1955/Tarski 1954 (see [10, Th. 3.2.2 p. 124, and p. 519]) (see [9, Th. V.2.20 p. 215]).

1.1.5 Łoś & Suzko 1957/Chang 1959 (see [10, Th. 3.2.3 p. 125, and p. 519]).

1.1.6 See [10, Exercise 6.2.5 p. 339].

1.1.7 See [9, Th. V.2.23 p. 218].

1.1.8 See [9, Th. V.2.25 p. 219].

1.1.9 Birkhoff 1935 (see [9, Th. II.11.9 p. 75]), and Tarski 1946 (see [9, Th. II.9.5, p. 61]).

Section 1.2

1.2.1 Primitive formulas are introduced in [19, p. 15]. The notion, but not the terminology, "**A** is existentially closed in **B**" is in [19, p. 16]. The definition of "existentially closed in K" usually means our "existentially closed in $\mathrm{S}(K)$."

1.2.2 Eklof & Sabbagh 1971 [12, Prop. 7.6 p. 284].

Existentially closed groups were introduced by Scott 1951 and arbitrary existentially closed structures were introduced by Eklof & Sabbagh 1971.

1.2.3 Scott 1951/Eklof & Sabbagh 1971 (see [12, Th. 7.12 p. 288]).

1.2.4 Eklof & Sabbagh 1971 [12, Cor. 7.8 p. 286].

1.2.5 Eklof & Sabbagh 1971 [12, Prop. 7.1 p. 287].

Section 1.3

1.3.1 Simmons 1972 [22, Th. 2.1 p. 297].

1.3.2 Simmons 1972 [22, Th. 3.1 p. 300].

1.3.3 Based on Robinson 1956 [19, Th. 2.4.2 p. 21].

Section 1.4

1.4.1 Model completeness was introduced by Robinson 1956 [19, p. 13].

1.4.2 (i) $\Leftrightarrow$ (ii) Robinson 1956 [19, Th. 2.3.1 p. 16].

1.4.3 (ii) $\Rightarrow$ (v) Robinson 1956 [19, Th. 2.4.2 p. 21] — (ii) $\Leftrightarrow$ (v) is stated in [22, Th. 4.1 p. 302].

Section 1.5

1.5.1 Robinson 1956 [19, p. 72].

1.5.2 Robinson 1956 [19, Th. 4.1.6 p. 74].

Section 1.6

1.6.1 Quantifier elimination originated with Schröder 1890 [21, §21 p. 446], and is the subject of Skolem 1920 [24].

1.6.3 See [10, Th. 3.1.16 p. 118].

1.6.4 See [12, p. 254].

1.6.5 See [15, Lemma 12, p. 155].

Section 1.7

1.7.1 Appears in [3, 1970 p. 132].

1.7.2 Eli Bers (see [3, 1970 p. 135]).

1.7.3 Barwise & Robinson 1970 [3, Th. 5.3 p. 135].

1.7.6 See [19, Th. 4.2.1 p. 74 and Th. 4.2.2 p. 75].

Section 1.8

1.8.1 Eklof & Sabbagh 1971 [12, Cor. 7.13 p. 288]. In Th. 7.17 p. 291 they proved that groups have no model companion.

Section 1.9

1.9.2 Eli Bers (see [12, Lemma 2.1 p. 254]).

1.9.3 See [15, Lemma 12 p. 155].

Section 1.10

1.10.2 Burris & Werner 1979 [8, Th. 8.4 p. 292].

1.10.4 The $\epsilon_n(K)$ defined in [8, p. 291] is our π_n in Def. 1.10.12.

1.10.6 Ryll-Nardzewski 1959 (see [4, Th. 5.6.9 p. 208]).

1.10.7 – 1.10.10 These are based on [8, Th. 8.4 p. 292].

1.10.11 Saracino 1973 (see [15, Th. 24 p. 163]).

1.10.16 Based on ideas in [6].

Section 1.11

1.11.1 Simmons 1972 [23, Th. 1].

Chapter 2

Section 2.1

2.1.1 Birkhoff 1944 introduced the concept of subdirect products (see [9, Th. 8.6 p. 58]).

2.1.2 Burris & Werner 1979 [8, §1 p. 271] introduced Boolean products as an alternative to Boolean sheaves—for a detailed history see [5].

2.1.3 Burris & Werner 1979, the $[\![\quad]\!]$ notation is introduced in [8, §1 p. 271].

2.1.5 Werner 1978 [27, p. 42].

2.1.6 – 2.1.7 Burris & Werner 1979 [8, §1].

2.1.8 Arens & Kaplansky 1948 introduced bounded Boolean powers in [2].

2.1.10 Burris & Werner 1979 [8, Th. 2.1(b) p. 273].

2.1.11 Burris & Werner 1979 [8, p. 295].

2.1.13 Burris & Werner 1979 [8, p. 296].

2.1.15 Riedel 1984 [18, p. 41].

2.1.16 Riedel 1984 [18, p. 42].

2.1.17 – **2.1.18** See [27, p. 55–56].

2.1.19 Burris & Werner 1979 [8, Lemma 9.3 p. 294].

2.1.20 Burris & Werner 1979 [8, §7 p. 288].

2.1.21 Macintyre 1973/1974 [1, §3.3 Part 1 p. 79].

2.1.24 Burris & Werner 1979 [8, §7 p. 288].

2.1.25 Burris & Werner 1979 [8, §7 p. 288].

2.1.26 Burris & Werner 1979 [8, §7 p. 288].

2.1.27 Burris & Werner 1979 [8, Lemma 7.1 p. 288].

2.1.28 Burris & Werner 1979 [8, §7 p. 288].

2.1.29 Burris & Werner 1979 [8, Lemma 7.2(b) p. 288].

2.1.30 Burris & Werner 1979, this is in the proof of [8, Th. 7.3 p. 289].

2.1.31 Burris & Werner 1979 [8, Th. 7.3 p. 289].

2.1.32 – **2.1.33** Burris & Werner 1979, implicit in the proof of [8, Th. 9.13 p. 299].

Section 2.2

2.2.1 Burris & Werner 1979 [8, Lemma 8.3 p. 291]—here it is proved for π_n.

2.2.2 This is based on [8, Th. 8.4 p. 292] and [6, Th. 1.2 p. 69].

2.2.3(1) Burris & Werner 1979 [8, Cor. 8.5 p. 292].

2.2.3(2) Burris 1984 [6, Th. 1.2 p. 69] (except part (c) is missing).

2.2.3(3)(a) Burris 1984 [6, Lemma 1.6 p. 72].

2.2.3(3)(b) Burris 1984 [6, Th. 1.10 p. 72].

2.2.9 Burris & Werner 1979 [8, Prop. 8.6 p. 292].

Section 2.3

2.3.1 Werner 1970 (see [27, Cor. 1.9 p. 11]).

2.3.2 Grätzer 1964 (see [27, Lemma 1.10(2) p. 11]—Grätzer used the name "normal transform" instead of our "switching function."

2.3.3 See [8, §9 p. 293].

2.3.5 Burris & Werner 1979 [8, p. 294].

2.3.6 See [9, II.7–II.8].

2.3.7 Keimel & Werner 1974, Bulman-Fleming & Werner 1977 (see [9, IV.9.4 p. 165]).

2.3.8 Burris & Werner 1979 [8, Lemma 9.1 p. 294] (3 of the cases are done).

2.3.9 Some of these calculations are used in [9, IV §9].

2.3.10 Partly in [8, Lemma 9.6 p. 295].

2.3.11 Burris & Werner 1979 [8, Lemma 9.6(d) p. 296].

2.3.12(i) Burris & Werner 1979 [8, Lemma 9.9 p. 297].

2.3.12(ii) Burris & Werner 1979, The SCRP underlies the proof of [8, Lemma 9.4 p. 294], and the SCEP is [8, Lemma 9.10 p. 297].

2.3.13(ii) Werner 1978 [27, §2.5 Cor. 2 p. 27].

Section 2.4

2.4.1 This is in the proof of [8, Lemma 9.2 p. 294].

2.4.2 – 2.4.5 Burris & Werner 1979 [8, Lemma 9.2 p. 294].

2.4.6 – 2.4.7 See the proof of [8, Lemma 9.9 p. 297] for $\varphi_{(\subseteq)}$.

$\varphi_{nontrivial}$ is $\neg\chi$ in [8, Lemma 9.11 p. 298].

$\varphi_{atomless}$ expresses the same thing as $\prod$ in [8, Lemma 9.12 p. 298].

2.4.8(a) Burris & Werner 1979 [8, Cor. 9.5 p. 295].

2.4.8(b) Uses (a) above and [8, Lemma 9.12 p. 298].

Section 2.5

2.5.1 – 2.5.8 Based on [8, §9].

2.5.9 – 2.5.13 Burris & Werner 1979 [8, Th. 9.13 p. 298]—the proofs presented are somewhat different from that in [8].

Section 2.6

2.6.1 In [8, §10 p. 301] discriminator formulas are introduced in a more restrictive setting as primitive positive formulas.

2.6.4 – 2.6.5 Riedel 1984 [18, p. 65].

2.6.6 See [18, 2–7 p. 73] for a special case.

Section 2.7

2.7.1 See [10, p. 19].

2.7.5 See next item.

2.7.6 Burris & Werner 1979 [8, Lemma 10.1 p. 301].

2.7.7 Burris & Werner 1979, is essentially [8, Lemma 10.2 p. 301].

2.7.8 Point 1985 [17, Prop. 1 p. 606].

Section 2.8

2.8.1 Point 1985 [17, p. 605]—she uses the term $\mathcal{L}_t$-*faithful*. The notation we use is [18, p. 46].

2.8.2 McKenzie and Shelah 1971 [16, p. 55].

2.8.3 Riedel 1984, proved in [18, Th. 1.1–13 p. 47].

2.8.5 Burris & Werner 1979, is essentially [8, Lemma 10.6 p. 302].

2.8.7 Riedel 1984 [18, Lemma 1.1–12]—he requires K to be a universal class.

2.8.8 Riedel 1984 [18, Prop. 1.1–8 p. 43].

2.8.9 Inspired by Point 1985 [17].

2.8.10 Based on Point 1985 [17, Prop. 3].

2.8.11 The notion of "completing" is due to Robinson (see [3]).

2.8.12 Inspired by Point 1985 [17].

Section 2.9

2.9.1 – 2.9.8 Burris & Werner 1979, this is a rewrite and expansion of [8, §10 p. 301–306].

Section 2.10

2.10.1 The SCAP has been abstracted from [18, Lemma 1.1–10 p. 45].

Section 2.11

2.11.2 See [9, Def. IV.11.4 p. 176].

2.11.4 Riedel 1984 [18, Lemma 1.1–2 p. 37].

2.11.5 Riedel 1984 [18, p. 39].

2.11.7 Riedel 1984 [18, Lemma 1.1–7 p. 41] (is more general).

2.11.9 Generalizes Riedel 1984 [18, Prop. 1.1–11 p. 46], [18, Th. 1.1–13 p. 47]—he uses a universal class of FSI's.

2.11.10 Generalizes Riedel 1984 [18, Th. 1.1–17 p. 52].

2.11.11 Related to [18, §3].

2.11.12 This is more general than the usual definition.

2.11.13 This is a famous result of Mal'cev—see [9, Lemma 3.1 p. 221].

2.11.14 Burris [7].

2.11.15 Burris [7, Th. 3].

2.11.16 Riedel 1984 [18, Lemma 2–5 p. 70].

2.11.17 Riedel 1984 [18, Prop. 2–6 p. 71].

2.11.18 Riedel 1984 [18, Th. 2–15 p. 78].

2.11.19 Riedel 1984 [18, Prop. 2–18 p. 82].

2.11.20 Riedel 1984 [18, Th. 2–19 p. 84] (except that he uses FSI's instead of DI's).

Section 2.12

2.12.1 This is based on Jónsson's 1967 results on Mal'cev conditions for congruence distributive varieties as presented by Burris 1979—see [9, Lemma V.4.4 p. 229].

2.12.2 This is based on Baker's 1977 results on finitely based varieties—see [9, Lemma V.4.8 p. 230].

2.12.3 Burris 1984 as presented in [18, Lemma 2–3 p. 68].

2.12.4 Burris 1984 as presented in [18, Prop. 2–4 p. 69].

2.12.5 Riedel 1984 [18, Th. 2–19 & Remark p. 84].

2.12.6 Glass & Pierce 1980 [13, Th. B(1) p. 255].

Section 2.13

Based on Point 1985 [17]. Point uses a finite forcing argument whereas we have opted to work within the framework presented by Simmons [23].

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