Review of:	``A Boole Anthology'', ed. by James Gasser. Synthese Library, **191**, Kluwer Academic Publishers, 2000
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This anthology contains the 17 papers listed below. The first seven, grouped under 'Classical Studies', are reprints. The last ten, under 'Recent Studies', appear for the first time—they are papers presented at the 1997 Conference in Lausanne to celebrate the sesquicentenary of the 1847 publication of Boole's *Mathematical Analysis of Logic*.

(1) Samuel Neil, The late George Boole, LLD, DCL, 1865.

(2) George Paxton Young, Remarks on Professor Boole's mathematical theory of the Laws of Thought, 1865

(3) Luis M. Laita, The influence of Boole's search for a universal method in analysis on the creation of his logic, 1977.

(4) Theodore Hailperin, Boole's algebra isn't Boolean algebra, 1981.

(5) Michael Dummett, Review of Boole, *Studies in Logic and Probability*, and of 'Celebration of the Centenary of *The Laws of Thought*', 1959.

(6) James W. van Evra, A reassessment of George Boole's theory of logic, 1977.

(7) John Corcoran and Susan Wood, Boole's criteria for validity and invalidity, 1980.

(8) Theodore Hailperin, Algebraical logic: Leibniz and Boole, 1997.

(9) Marie-José Durand-Richard, Logic versus algebra: English debates and Boole's mediation, 1997.

(10) Maria Panteki, The mathematical background of George Boole's *Mathematical Analysis of Logic (1847)*, 1997.

(11) Ivor Grattan-Guinness, On Boole's algebraic logic after *The Mathematical Analysis of Logic*, 1997.

(12) Sriram Nambiar, The influence of Aristotelian logic on Boole's philosophy of logic: the reduction of hypotheticals to categoricals, 1997.

(13) Béatrice Godart-Wendling, The conceptualization of time in Boole's algebraic logic, 1997.

(14) Gérard Bornet, George Boole and the science of logic, 1997.

(15) Volker Peckhaus, Was George Boole really the 'Father of Modern Logic'?, 1997.

(16) Shahid Rahman, Hugh MacColl and George Boole on hypotheticals, 1997.

(17) Nicla Vassallo, Psychologism in logic: some similarities between Boole and Frege, 1997.

Young (2) said in 1865 that Boole was the first to offer a comprehensive mathematical (algebraic) system that could be used to routinely establish the validity of any correct argument in logic. Although Boole's contemporaries and successors could see, by doing examples, that this system worked, many were troubled by the lack of a convincing explanation as to why it worked. A key portion of Boole's system was soon replaced by modern Boolean algebra, and other parts, like the equational treatment of particular statements and the use of division, have been abandoned. Nonetheless his original system has been the subject of numerous investigations over the past 150 years. Among the most popular themes are:

Q1: Why does Boole's system work?

Q2: What mathematical and cultural influences could have put Boole in the position of being interested in and able to develop such a novel system?

Q3: What was the impact of Boole's work on modern logic?

The information available on Boole's algebra of logic has changed dramatically in the last quarter century, but unfortunately it is far from having been properly absorbed by the mathematical and philosophical community. In particular the great confusion in the literature regarding Q1 needs to be cleared out. The following abbreviations will be used for Boole's two books on logic: MAL for *The Mathematical Analysis of Logic*, and LT for *Laws of Thought*.

For those who have (tried to) read Boole's work on logic, no doubt question Q1 looms large—how can Boole work with an $x + y$ that denotes the union of classes x and y when they are disjoint, but otherwise is simply uninterpretable? Or, why is it legitimate for Boole to deduce

$$x = yz + \frac{1}{0}(1-y)z + \frac{0}{0}(1-y)(1-z)$$

from $xy = z$?

To keep the review in bounds let us focus on the fragment of Boole's system that does not deal with particular statements, nor with the division operation. This fragment is, after all, the part that was reworked to give us modern Boolean algebra. It will be convenient to give it a name, say FRAG.

I know of no discussion of Boole's system written prior to 1976 that comes close to correctly explaining why FRAG works. And many after 1976 persist in trying to use union or symmetric difference to give an interpretation of Boole's $+$ as a total operation. Dummett's 1959 review (5) is particularly good in pointing out the difficulties encountered in trying to give a coherent account of Boole's system. Unfortunately his proof of the correctness of FRAG is seriously flawed because he uses symmetric difference to complete Boole's partial addition operation. Corcoran and Wood's 1980 article (7) does an excellent job of pointing out shortcomings of Boole's exposition in MAL, but their considerable discussion of what happens when one interprets $+$ as union and $-$ as relative complement is now mainly relevant to show why one should *not* use these interpretations.[1] Neither symmetric difference nor union are compatible with FRAG since Boole's system accepts the two arguments '$x + x = 0 \ \therefore x = 0$' and '$x + x = x \ \therefore x = 0$' as valid arguments.[2]

We are truly indebted to Hailperin for showing us, in 1976, why Boole's system works.[3] Rather than trying to complete the operations of $+$ and $-$ *on* the subclasses of the universe 1, he *embeds* this partial algebra of subclasses into a total algebra of *signed multisets* (perhaps one should say *multiclasses*), preserving all the laws that Boole uses.

With this understanding of Boole's system we can confidently describe FRAG as the **high school equational algebra of** $+, \cdot, -, 0, 1$ applied to **idempotent symbols**. By high school equational algebra I mean equational arguments '$s_1 = t_1, \ldots, s_n = t_n \ \therefore s = t$' that can be justified by the simple methods learned in high school. The equational arguments mentioned just two paragraphs above are good examples. However the argument '$x^2 = x \ \therefore x = 0$ or $x = 1$' is not acceptable. It is not an equational argument as one can

[1] For example, they say that Boole's conclusion of $ab' - a'b = 0$ from the two equations $ay + b = 0$ and $a'y + b' = 0$ actually follows from the second equation alone as it implies $b' = 0$.

[2] In the correspondence with Jevons in 1863, Boole emphatically denies that $x + x = x$ is a law.

[3] Hailperin's clarification of Boole's system is, surprisingly, still not widely known. I first heard of it in 1996.

see by writing it more formally: '$x^2 = x \;\therefore\; x = 0 \;\vee x = 1$'. The conclusion of this argument is a disjunction of equations, not an equation.

Here is Hailperin's description of FRAG in (4):

> While he [Boole] never made his algebra fully explicit, we inferred that what he did use was, if clarified, a commutative ring with unit, without nilpotents, and having idempotents which stood for classes. By thus hewing closely to "common algebra" Boole could use familiar procedures and techniques.

In other words, Boole uses high school equational algebra applied to idempotent symbols! Hailperin's contribution (4), *Boole's algebra isn't Boolean algebra*, is a must read for anyone interested in Q1. Or, even better, read the 1986 second edition of his revolutionary 1976 book, *Boole's Logic and Probability.* This will cast the Q1 struggles of some of the other papers in a completely new light.

Corcoran and Wood correctly note that Boole's laws

$$xy = yx \qquad x(u+v) = xu + xv \qquad x^n = x$$

are woefully inadequate for the equational arguments that Boole presents. So they add a number of other laws, including $x + x = x$ and the fact that $+$ distributes over $\cdot$, laws which are based on their previously mentioned interpretation. Then they give several detailed equational deductions (using this augmented system) to justify logical inferences found in Boole's MAL, and wonder why Boole did not do the same.

For a proper understanding of Boole's use of algebra we need answers to Q2, so we turn to Laita (3) and Panteki (10) for discussions of Boole's work in analysis prior to 1847. One of the major influences on Boole was Duncan Gregory. Gregory stated three laws in 1839,

$$xy = yx \qquad x(u+v) = xu + xv \qquad x^m \cdot x^n = x^{m+n},$$

and said that these were sufficient to derive the binomial theorem (with fractional exponents).[4] He then used the three laws to justify applying the binomial theorem in his work with differential operators. In Boole's 1841 paper on linear DEs he says, without proof, that the three laws of Gregory

[4]He based this assertion on the proof found in Euler's *Algebra*, a popular textbook from the late 1700s which has notorious gaps by modern standards, but was considered quite acceptable in the first half of the nineteenth century.

also suffice to obtain the partial fraction decomposition of a rational function, and he applies this to the inverse of a linear differential operator with constant coefficients. Next, in Boole's prize winning paper of 1844, on differential and difference equations, he states the three laws as a basis for his work, inquiring if the third law might not be merely a necessity of notation.

Replacing Gregory's third law with $x^n = x$ gives Boole's three laws in MAL. However, in MAL Boole goes further by adding a *rule of inference*[5] (that we now call the *replacement rule* in equational logic), and by claiming that, in view of the first two of these laws, "all the processes of common algebra are applicable to the present system". What Boole actually uses, as regards FRAG, is (with rare exceptions) just the equational part of common algebra. The laws of MAL are clearly inadequate for what Boole does. But the point is that Boole (incorrectly) believes that the two laws give him the right to use the common (equational) algebra, and he immediately invokes that right. Thus his system really is, as mentioned above, common (equational) algebra with idempotent symbols, so there is no reason for him to derive results just from the initial laws. Van Evra's warning in (6), to be cautious about judging Boole's work by modern standards, certainly applies here. It is only from the historical context that one can understand why, in 1847, Boole did not find it necessary to develop his system from first principles. A modest schoolteacher, Boole evidently just accepted what the Cambridge scholar Gregory claimed to be true.[6]

Laita and Panteki conclude that Boole's expertise with the algebra of differential operators may well have been the real secret of his successful attack on the algebra of logic. Personally I would put a different spin on this, namely Boole's work with the algebra of differential operators and the algebra of logic were successful applications of ordinary algebra to non-numerical

[5] By adding this rule of inference Boole is very close, as noted by Corcoran and Wood, to a modern formal equational logic system. One needs the reflexive, symmetric and transitive properties of equality, and one needs to treat the law $x^n = x$ as a scheme of quantifier free equations. The first two laws are assumed to be universally quantified. Boole's system really studies quantifier free deductions, whereas modern Boolean algebra focuses on deductions involving universally quantified equations.

[6] Indeed, from 1839 to 1847 there were apparently no serious challenges to these laws being adequate for the purposes claimed. However, shortly after receiving a copy of MAL, Cayley objected to Boole's claim that 'all the processes of common algebra' were justified. In 1854, when Boole published LT, we see him listing a number of additional laws (but still not enough). But then he discovers his famous 'algebra of 0 and 1', and abandons the idea of basing his algebra on a few laws.

situations. No doubt his early success with applying ordinary algebra to differential operators fueled his confidence to apply ordinary algebra to logic.

Regarding Q3, Dummett (5) and Peckhaus (15) discuss whether or not Boole should be called the father of modern logic. Dummett says 'no' because Boole made no inroads into the discovery of notation for quantifiers nor did he present a formal system. Peckhaus gives a detailed discussion of who influenced whom in the development of logic, and comes to the conclusion that there were really many seeds that led to modern logic, and that it has no 'father'.

Hopefully enough has been said so the reader can see that the 17 papers offer a fascinating insight into the struggle to understand Boole's development of an algebra of logic. Patient scholarship, at times going down a wrong path, has led to a much clearer picture. Amazingly, Boole was able to use ordinary high school equational algebra, applied to idempotent symbols, as an algebra that actually worked for the logic of classes. De Morgan expressed surprise that an algebra created for numbers could apply to logic. We are still surprised.

Errata: on page 35, in the third line after equation (8), change 'by making $v = 0$' to 'by making $u = 0$'. Also, equation (9) should be: $f(x) = ux + v(1 - x)$. page 95, line 8: change 'of admitting logic' to 'of admitting division'.