Annotated Version

of

Boole's Algebra of Logic 1847

as presented in his booklet

THE MATHEMATICAL ANALYSIS OF LOGIC

BEING AN ESSAY TOWARDS

A CALCULUS OF DEDUCTIVE REASONING

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Preface

In his two books and one paper from the mid 1800s on reducing logical reasoning about classes to mathematical reasoning, Boole had a tendency to incompletely formulate definitions and explanations, or even omit them altogether. Some of his notions were vague and his proofs mysterious, and there were some dubious claims.

With all these problems nonetheless his algebraic methods, based on Common Algebra and idempotent variables, for deriving conclusions from 'primary' and 'secondary' propositions, seemed to work surprisingly well. They would provide the foundation and inspiration for others with mathematical experience to rapidly start developing what we now call Boolean Algebra, and even the more general subject of equational logic.

The original goal of this project was to recast Boole's work on logic in the style and language of modern mathematics in such a way that missing details would be filled in and the reader could easily compare the new versions with the originals to see that the new were reasonably faithful to the original. The optimism for being able to carry out this project was largely founded on reading the remarkable analysis of Boole's work in logic published in 1976/1986 by Theodore Hailperin [19]—he had found an explanation for why Boole's system worked.

This project turned out to be far more evasive than expected, requiring explanatory notes that often dwarfed the original text; it was eventually abandoned. Instead his original texts have been presented with marginal notes. A lengthy set of introductory notes has been added to the annotated version of his first book.

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Annotated Version of Boole's MAL

Notes on Boole's MAL

Following these Notes, which occupy pages i–lxvi, there is an annotated version of Boole's first book on logic, MAL, where the content of each page is the same as in Boole's original. Arabic page numbers in the Notes or in marginal comments will refer to pages in MAL unless specified otherwise.

The published work of George Boole (1815–1864) on logic consists of a small book (MAL), a paper (CL), and a second book (LT) (a substantial portion of which is for applications of his logic to probability):

MAL **(1847)** *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning.* Originally published in Cambridge by Macmillan, Barclay, & Macmillan, 1847. Reprinted in Oxford by Basil Blackwell, 1951.

CL **(1848)** *The Calculus of Logic*, The Cambridge and Dublin Mathematical Journal, **3** (1848), 183-198.

LT **(1854)** *An Investigation of The Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities.* Originally published by Macmillan, London, 1854. Reprint by Dover, 1958.

When Boole wrote MAL in 1847 he was a highly regarded 31-year old school master, running a private school in Lincolnshire to help support his parents and siblings.* By this time he was also a respected mathematician in the British Isles, having won a gold medal from the Royal Society three years earlier for the

*His father John Boole, a shoemaker, had instilled a great love of learning in his son, so much so that George went on to absorb higher mathematics and several languages on his own. After John's business failed, George, still a teenager, became the primary breadwinner for the family. For the life of George Boole, see [22] by Desmond MacHale and [23] by MacHale and Yvonne Cohen.

paper [3] on differential equations, using the rather new concepts of differential operators and symbolical algebra.

In the preface of LT Boole claimed to have completed writing MAL in a few weeks in 1847, but we have only scant information on how his ideas for this book developed. His sister MaryAnn wrote a biography of Boole that she never published, but now an edited version has appeared in [23], pp. 15–50. From this one finds that indeed Boole had a long-standing interest in a mathematical treatment of logic ([23], p. 41):

> He told me that from boyhood he had the conviction that Logic could be reduced to a mathematical science, and that he had often made himself ill on the attempt to prove it, but that it was not until 1847 that the true method flashed upon him.
>
> ⋮
>
> When in 1847 the true light flashed upon him and he entered upon the investigations that resulted in *The Laws of Thought*, he was literally like a man dazzled with excess of light, . . . If he could have communicated his thoughts and feelings to some sympathetic mind it would have been a relief to him, but his father was gone, and there was no one near to whom he could have made himself intelligible.

We do not know what aspect of MAL represents the "true method" that galvanized Boole's efforts to publish a mathematical treatment of logic, or more precisely, of deductive logic, which at the time meant a version of Aristotelian logic. From Boole's remarks on p. 45 about the contributions of Charles Graves (1812–1899) to his algebra of logic, it seems that Boole had been in contact with Graves while working out the details for Aristotelian logic, and perhaps that had been largely carried out before 1847. Unfortunately no correspondence between the two has been found. A plausible guess for Boole's 1847 breakthrough that his sister mentioned was his discovery of the properties of constituents along with the expansion of any term into a linear combination of constituents; this was the foundation of his general theory starting on p. 60.

When it came to the axiomatic foundations of mathematics, in particular for algebra, he had simply borrowed the wholly inadequate axioms from an 1839 paper [18] by his young Cambridge mentor Duncan Fahrquharson Gregory (1813-1844). These were the same axioms that Boole had used in his prize-winning paper of 1844. Boole, perhaps inspired by Gregory, had become a strong believer in symbolical algebra (which had provided his framework for studying differential operators), and he said it justified parts of his general results at the end of MAL, in particular his use of division to solve equations.

In the modern Boolean Algebra of classes one starts with the fundamental operations of *union* ($\cup$), *intersection* ($\cap$) and *complement* ($'$), and with the two constants *empty class* ($\emptyset$) and *universe* (U). Then one finds suitable equational axioms and rules of inference, and proves that every valid equation and equational inference can be obtained from these. A key result is that an equation or an equational inference is valid iff it holds in the two-element Boolean Algebra.*

Boole, on the other hand, evidently used the reverse of this process—he *started* with the equational algebra of numbers, with the operation symbols *addition* ($+$), *multiplication* ($\times$) and *subtraction* ($-$), and with the two constants 0 and 1, and found a way to make it work as an algebra of logic, but only after adding the *index law* $x^n = x$ (which it turns out only applied to variables). In LT the index law is replaced by $x^2 = x$, and in the form $x(1-x) = 0$ it is called the *law of duality.*

Forcing Aristotelian logic into the world of ordinary algebra, the algebra of numbers, came at a price—the operations of addition and subtraction of classes in Boole's Algebra are only partially defined! It turns out that an equation or equational argument is valid in Boole's Algebra iff it is valid in the integers $\mathbb{Z}$ when the variables are restricted to the values 0 and 1.

In MAL, for each class X he introduced an *elective operation* x that, when applied to a class Y, selected all the elements of Y that were in X. This captured Boole's understanding of one way the mind formed new concepts of classes from previous ones; he viewed MAL as a study of the science of the mind.†

His algebra appears, on first glance, to be an algebra of elec-

*The same procedure applies to developing the modern Boolean Algebra of propositions, with the fundamental operations being *or* ($\vee$), *and* ($\wedge$) and *not* ($\neg$), with the two constants *true* (T) and *false* (F). The two algebras are essentially the same as far as valid equations and equational inferences are concerned.

†Perhaps Boole was aiming to parallel what Newton, also a son of Lincolnshire, had done for the science of the physical world.

tive operations, combined into equations like $xy = yx$ and equational inferences like $2x = 0 \therefore x = 0$. But is it really just a disguised form of the algebra of classes that Boole uses in LT?

Aristotelian logic presented a ***catalog*** of the fundamental valid arguments (the conversions and syllogisms) for categorical and hypothetical propositions. As for the categorical propositions, the original goal of MAL was to provide an equational way to determine the catalog for a modestly extended version of Aristotelian logic, namely one that permitted contraries like not-X for the terms in categorical propositions. Such categorical propositions will be called BC-propositions.

In MAL Boole only considered syllogisms for which the premises were traditional Aristotelian categorical propositions, but the conclusion could be a BC-proposition. In his 1848 paper CL he offered a simple description of valid categorical syllogisms where the premises were also allowed to be BC-propositions.* He claimed to give an algebraic proof of this in Chap. XV of LT.

The algebra Boole used in MAL to analyze logical reasoning is quite elementary, at least until the general theory starts on p. 60. But there is much that needs to be clarified—on many a page one can ask "Exactly what did Boole mean to say here?" In his 1959 JSL review article [16] Michael Dummett (1925–2011) said

> anyone unacquainted with Boole's works will receive an unpleasant surprise when he discovers how ill constructed his theory actually was and how confused his explanations of it.

These Notes, along with the marginal comments, will hopefully provide plausible resolutions to some queries regarding MAL. Boole made definite progress in clarifying his algebra of logic with his second book LT, starting by getting rid of the annoying elective operations, but the major breakthrough for the modern reader is due to Theodore Hailperin (1916-2014) [19] in 1976/86.† His insights, plus subsequent investigations, form the basis for the explanations given in these Notes and the marginal comments to MAL. An important development that appeared after Hailperin's work has been a proper understanding of the role that the algebra of the integers $\mathbb{Z}$ played in Boole's Algebra.

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*For another simple determination of the valid BC-syllogisms see pp. xiv.

†This work has roots in the 1933 use of characteristic functions by Hassler Whitney [26] to convert Boolean Algebra into ordinary algebra.

Boole's Algebra of logic has two components:

A) the translation of propositions into equations, and vice-versa;

B) algebraic methods to derive best possible equational conclusions, with desired restrictions, from equational premises.

His process of going from propositional premises to propositional conclusion(s) is pictured in Tbl. 1. This process for the 1st Fig. AAA syllogism for categorical propositions is given in Tbl. 2.

Propositions	Translations	Equations
Premisses	$\xrightarrow{\text{Expression}}$	Premisses
$\vdots$ $\downarrow$		(apply $\big\downarrow$ algebra)
Conclusion(s)	$\xleftarrow{\text{Interpretation}}$	Conclusion(s)

Tbl. 1. Applying Algebra to Logic

Propositions	Translations	Equations
All Y is Z. All X is Y.	$\xrightarrow{\text{Expression}}$	$y = yz$ $x = xy$
$\vdots$ $\downarrow$		(apply $\big\downarrow$ algebra)
All X is Z.	$\xleftarrow{\text{Interpretation}}$	$x = xz$

Tbl. 2. Simple Example of Boole's Algebraic Method

Boole called the translation ($\rightarrow$) of a proposition into an equation *expression*, and the reverse translation ($\leftarrow$) was *interpretation*.

Expression and Interpretation

Boole's mathematical analysis of this extended Aristotelian logic was fairly detailed, taking place in pages 15–59, which is more than half of the content of MAL. Had he stopped on p. 59, with no further development of his algebra of logic, chances are that his work would have faded away into obscurity. It was the general theory being developed on pp. 60–82 that led to a significant expansion of Aristotelian logic, with algebraic methods to produce conclusions to complex propositional premises. This was reworked and extended in the logic portion of LT, published seven years later, where he focused primarily on algebraic ***algorithms*** for determining the most general conclusions one could obtain, under specified constraints, from any number of premises with any number of class or propositional variables. (This differs from modern treatments that have a significant focus on determining if a given argument is valid.) The fascination which Boole exhibited for Aristotelian logic in MAL gave way to a rather severe dislike for the same in LT (see Chap. XV of LT).

Topics in these Notes on MAL

This section reviews Aristotle's logic of categorical propositions, as commonly understood in the mid 19th century,* and summarizes Boole's extension of this logic.

Categorical Propositions. The structure of a categorical proposition is indicated in Tbl. 3.

Quantifier	*Subject*	*Copula*	*Predicate*

Tbl. 3. Structure of a Categorical Proposition

The four forms of **Aristotelian categorical propositions**, along with their classification as to *kind*, *quantity* and *quality* (see p. 20)† are listed in Tbl. 4.

Kind	Proposition	Quantity	Quality
A	All X is Y	universal	affirmative
E	No X is Y	universal	negative
I	Some X is Y	particular	affirmative
O	Some X is not Y	particular	negative

Tbl. 4. The Four Aristotelian Categorical Propositions
with X as Subject, Y as Predicate

The phrase *Aristotelian categorical proposition* will usually be abbreviated to *AC-proposition*. The phrase $\Phi(\mathrm{X},\mathrm{Y})$ *is an AC-proposition* does not specify which of X, Y is the subject, thus there are eight possibilities for an AC-proposition $\Phi(\mathrm{X},\mathrm{Y})$.

AC-proposition
Boole did not use function notation like $\Phi(\mathrm{X},\mathrm{Y})$ for propositions.

Conversion. Let Φ be an AC-proposition with subject X and predicate Y, and let Ψ be an AC-proposition with subject Y and predicate X. In Aristotelian logic, Ψ is a valid *conversion* of Φ if Φ implies Ψ. The forms of valid conversions in Aristotelian logic are given in Tbl. 5. Boole accepted the somewhat controversial *conversion by contraposition*, described in Tbl. 6. It was controversial only because there was lack of agreement as to

*A popular logic textbook of this period was *Elements of Logic* [25] by Richard Whately (1787-1863) of Oxford. First published in 1826, the 9th edition is cited in Boole's 1854 book LT, p. 239. It is not stated which edition Boole cited in MAL (see pp. 7, 20).

†Only on p. 44 are the words *quantity* and *quality* used as properties of categorical propositions.

Proposition	Conversion	Kind of Conversion
No X is Y	No Y is X	simple
Some X is Y	Some Y is X	simple
All X is Y	Some Y is X	by limitation
No X is Y	Some Y is not X	by limitation

Tbl. 5. Aristotelian Conversions

Proposition	Contrapositive
All X is Y	No not-Y is X
Some X is not Y	Some not-Y is X

Tbl. 6. Conversion by Contraposition

whether one should be allowed to use contraries, like not-Y, as the subject of a categorical proposition.*

Boole's categorical propositions are a modest extension of the AC-propositions, namely Tbl. 4 is increased by allowing X to be replaced by not-X, and independently, Y by not-Y, giving 32 forms of categorical propositions $\Phi(\mathrm{X}, \mathrm{Y})$. Such categorical propositions will be called *BC-propositions*, meaning *Boole's categorical propositions*. From the comment on p. 30, that negating a term (changing X to not-X, and vice-versa), does not change the kind of a proposition, **Boole used the A, E, I, O classifications for his categorical propositions.** Thus, for example, Some X is not not-Y would be of kind O, a particular negative proposition, even though it is clearly equivalent to Some X is Y, which is of kind I, a particular affirmative proposition.

BC-propositions

In the following let AXY denote the proposition All X is Y, let EXY denote the proposition No X is Y, etc. Let $\overline{\mathrm{X}}$ denote not-X, let $\overline{\mathrm{Y}}$ denote not-Y, etc.; also $\overline{\overline{\mathrm{X}}}$ is X, $\overline{\overline{\mathrm{Y}}}$ is Y, etc. Each of the four kinds A,E,I,O refers to four distinct forms of BC-propositions—see Tbl. 7.

The notations AXY, EXY, IXY, OXY, $\mathrm{A\overline{X}Y}$, etc.

Equivalent BC-Propositions. There are four classes of equivalent universal BC-propositions $\Phi(\mathrm{X}, \mathrm{Y})$, each with four members (see Tbl. 8); and likewise for the particular BC-propositions (see Tbl. 9). The AC-propositions are in boldface type.

*In the 4th item of the footnote on p. 44 Boole said "An Aristotelian proposition does not admit a term of the form not-Z in the subject, ...". This clearly excludes conversion by contraposition. Boole said that the six forms of conversions in Tbl. 5 and Tbl. 6 were the forms accepted by Whately in [25].

Kind					Quantity	Quality
A	AXY	$\mathrm{AX\overline{Y}}$	$\mathrm{A\overline{X}Y}$	$\mathrm{A\overline{X}\,\overline{Y}}$	universal	affirmative
E	EXY	$\mathrm{EX\overline{Y}}$	$\mathrm{E\overline{X}Y}$	$\mathrm{E\overline{X}\,\overline{Y}}$	universal	negative
I	IXY	$\mathrm{IX\overline{Y}}$	$\mathrm{I\overline{X}Y}$	$\mathrm{I\overline{X}\,\overline{Y}}$	particular	affirmative
O	OXY	$\mathrm{OX\overline{Y}}$	$\mathrm{O\overline{X}Y}$	$\mathrm{O\overline{X}\,\overline{Y}}$	particular	negative

Tbl. 7. The Sixteen Forms of Boole's Categorical Propositions with Subject X or $\overline{\mathrm{X}}$ and Predicate Y or $\overline{\mathrm{Y}}$

AXY	$\mathrm{A\overline{Y}\,\overline{X}}$	$\mathrm{EX\overline{Y}}$	$\mathrm{E\overline{Y}X}$
AYX	$\mathrm{A\overline{X}\,\overline{Y}}$	$\mathrm{E\overline{X}Y}$	$\mathrm{EY\overline{X}}$
$\mathrm{AX\overline{Y}}$	$\mathrm{AY\overline{X}}$	**EXY**	**EYX**
$\mathrm{A\overline{X}Y}$	$\mathrm{A\overline{Y}X}$	$\mathrm{E\overline{X}\,\overline{Y}}$	$\mathrm{E\overline{Y}\,\overline{X}}$

Tbl. 8. Rows give Equivalent Universal BC-propositions $\Phi(\mathrm{X}, \mathrm{Y})$

OXY	$\mathrm{O\overline{Y}\,\overline{X}}$	$\mathrm{IX\overline{Y}}$	$\mathrm{I\overline{Y}X}$
OYX	$\mathrm{O\overline{X}\,\overline{Y}}$	$\mathrm{I\overline{X}Y}$	$\mathrm{IY\overline{X}}$
$\mathrm{OX\overline{Y}}$	$\mathrm{OY\overline{X}}$	**IXY**	**IYX**
$\mathrm{O\overline{X}Y}$	$\mathrm{O\overline{Y}X}$	$\mathrm{I\overline{X}\,\overline{Y}}$	$\mathrm{I\overline{Y}\,\overline{X}}$

Tbl. 9. Rows give Equivalent Particular BC-Propositions $\Phi(\mathrm{X}, \mathrm{Y})$

The Semantics of Class Symbols. One of the more annoying issues that needs to be cleared up when reading MAL is to decide how to handle the semantics of class symbols—Boole seems to have been inconsistent. The **M-semantics** (or *modern semantics*) for class symbols says that a class symbol X can denote *any* subclass of the universe. With this semantics the universal categorical propositions have no existential import. Since Boole accepted the Aristotelian conversion by limitation, it is clear that he was not using M-semantics. He also accepted the four categorical syllogisms in Aristotelian logic with universal premises but only a particular conclusion (see Tbl. 14 on p. xiv).

M-semantics of class symbols

In his 1847 book ([15], pp. 110, 111) Augustus De Morgan (1806–1871) was quite clear about the Aristotelian semantics of class variables:

> On looking into any writer on logic, we shall see that existence is claimed for the significations of all the names. Never, in the statement of a proposition, do we find any room left for the alternative, suppose there should be no

> such things. Existence as objects, or existence as ideas, is tacitly claimed for the terms of every syllogism.

Following De Morgan's observation, let the **A-semantics** (for *Aristotelian semantics*) of class symbols be the requirement that **class symbols X can only denote non-empty classes.**

A-semantics of class symbols

The assumption that Boole was using the A-semantics of class symbols is consistent with his results on BC-propositions with one important exception, namely his second law of lawful transformations (see Tbl. 10) which includes: Some not-X is Y follows from All not-X is Y. Evidently Boole wanted the contrary not-X of a class symbol X to be non-empty, which is the same as requiring that X is not the universe. This suggests Boole was using something more restrictive than A-semantics, the **B-semantics** of class symbols, namely **class symbols X can only denote non-empty and non-universe classes.** However B-semantics clashes with exactly one claim of Boole (see Tbl. 18), namely that the premises AZY, AXY cannot be completed to a valid syllogism where the conclusion is a BC-proposition. With B-semantics one has the valid conclusion $\mathrm{I}\overline{\mathrm{X}}\,\overline{\mathrm{Z}}$ (simply replace the premises by their contrapositives). Another argument against Boole having used B-semantics is the fact that he said on p. 65 that a derivation of $x = 0$ needs contradictory propositions, but he made no claim that deriving $x = 1$ indicated a problem.

B-semantics of class symbols

The B-semantics of class symbols gives a simpler, more elegant version of Boole's categorical logic than the A-semantics, so **the discussion in these Notes of Boole's categorical logic will be based on B-semantics.** However changes that need to be made if one uses A-semantics or M-semantics will be indicated. For the application of his algebra to hypothetical syllogisms Boole evidently accepted M-semantics (since equations like $x = 0$ and $x = 1$ are used).

Lawful Transformations. After noting (p. 28) that Aristotelian logic omitted the valid argument

No X is Y therefore All Y is not-X,

Boole finished the chapter on conversions by considering the wider question of when a BC-proposition $\Psi(\mathrm{X}, \mathrm{Y})$ was a consequence of another BC-proposition $\Phi(\mathrm{X}, \mathrm{Y})$. If so, he said Ψ was a *lawful transformation* of Φ. From Tbl. 8 and Tbl. 9 one can readily read off all the lawful transformations:

- For $\Phi(X,Y)$ and $\Psi(X,Y)$ universal BC-propositions, one has:
  $\Psi(X,Y)$ is a lawful transformation of $\Phi(X,Y)$
  iff
  $\Phi(X,Y)$ and $\Psi(X,Y)$ are equivalent, that is, iff they are in the same row of Tbl. 8.

- Likewise with 'universal' replaced by 'particular', using Tbl. 9.

- If $\Phi(X,Y)$ is universal and $\Psi(X,Y)$ is particular, then:
  $\Psi(X,Y)$ is a lawful transformation of $\Phi(X,Y)$
  iff
  either $\Phi(X,Y)$ is in the first two rows of Tbl. 8 and $\Psi(X,Y)$ is in the last two rows of Tbl. 9, or $\Phi(X,Y)$ is in the last two rows of Tbl. 8 and $\Psi(X,Y)$ is in the first two rows of Tbl. 9. [With A-semantics, the subject of $\Phi(X,Y)$ must be a class symbol, that is, X or Y.]

Instead of an explicit description of lawful transformations, Boole gave a set of three laws (p. 30) for lawful transformations, and claimed that every lawful transformation could be obtained by applying a suitable sequence of these laws. Letting $\widehat{X}$ be either of X and not-X, and letting $\widehat{Y}$ be either of Y and not-Y, his **laws of lawful transformation** are in Tbl. 10. Conversions occur only in the Third Law; they are simple. [For A-semantics, change $\widehat{X}$ to X in the second law; for M-semantics delete the second law.]

	Proposition	Transform
First Law	All $\widehat{X}$ is $\widehat{Y}$	No $\widehat{X}$ is $\overline{\widehat{Y}}$
	No $\widehat{X}$ is $\widehat{Y}$	All $\widehat{X}$ is $\overline{\widehat{Y}}$
	Some $\widehat{X}$ is $\widehat{Y}$	Some $\widehat{X}$ is not $\overline{\widehat{Y}}$
	Some $\widehat{X}$ is not $\widehat{Y}$	Some $\widehat{X}$ is $\overline{\widehat{Y}}$
Second Law	All $\widehat{X}$ is $\widehat{Y}$	Some $\widehat{X}$ is $\widehat{Y}$
	No $\widehat{X}$ is $\widehat{Y}$	Some $\widehat{X}$ is not $\widehat{Y}$.
Third Law	No $\widehat{X}$ is $\widehat{Y}$	No $\widehat{Y}$ is $\widehat{X}$
	Some $\widehat{X}$ is $\widehat{Y}$	Some $\widehat{Y}$ is $\widehat{X}$.

Tbl. 10. Laws of Lawful Transformations
for Boole's Categorical Propositions

Tbl. 11 shows all possible instances $\Phi(X,Y) \Rightarrow \Psi(X,Y)$ of Boole's laws of lawful transformation stated in Tbl. 10. [For A-semantics remove all $\Rightarrow$ from a universal to a particular if the subject of the universal is not a class symbol; for M-semantics remove all $\Rightarrow$ from a universal to a particular.]

$$
\begin{array}{ccccccc}
\mathrm{AXY} & \Longleftrightarrow & \mathrm{EX\overline{Y}} & \Longleftrightarrow & \mathrm{E\overline{Y}X} & \Longleftrightarrow & \mathrm{A\overline{Y}\,\overline{X}} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\mathrm{IXY} & \Longleftrightarrow & \mathrm{OX\overline{Y}} & & \mathrm{O\overline{Y}X} & \Longleftrightarrow & \mathrm{I\overline{Y}\,\overline{X}} \\
\Updownarrow & & & & & & \Updownarrow \\
\mathrm{IYX} & \Longleftrightarrow & \mathrm{OY\overline{X}} & & \mathrm{O\overline{X}Y} & \Longleftrightarrow & \mathrm{I\overline{X}\,\overline{Y}} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\mathrm{AYX} & \Longleftrightarrow & \mathrm{EY\overline{X}} & \Longleftrightarrow & \mathrm{E\overline{X}Y} & \Longleftrightarrow & \mathrm{A\overline{X}\,\overline{Y}} \\
\\
\mathrm{AX\overline{Y}} & \Longleftrightarrow & \mathrm{EXY} & \Longleftrightarrow & \mathrm{EYX} & \Longleftrightarrow & \mathrm{AY\overline{X}} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\mathrm{IX\overline{Y}} & \Longleftrightarrow & \mathrm{OXY} & & \mathrm{OYX} & \Longleftrightarrow & \mathrm{IY\overline{X}} \\
\Updownarrow & & & & & & \Updownarrow \\
\mathrm{I\overline{Y}X} & \Longleftrightarrow & \mathrm{O\overline{Y}\,\overline{X}} & & \mathrm{O\overline{X}\,\overline{Y}} & \Longleftrightarrow & \mathrm{I\overline{X}Y} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\mathrm{A\overline{Y}X} & \Longleftrightarrow & \mathrm{E\overline{Y}\,\overline{X}} & \Longleftrightarrow & \mathrm{E\overline{X}\,\overline{Y}} & \Longleftrightarrow & \mathrm{A\overline{X}Y}
\end{array}
$$

Tbl. 11. Instances of Laws of Lawful Transformations

$\Psi(\mathrm{X},\mathrm{Y})$ is a lawful transformation of $\Phi(\mathrm{X},\mathrm{Y})$ iff there is a directed path in this table from $\Phi(\mathrm{X},\mathrm{Y})$ to $\Psi(\mathrm{X},\mathrm{Y})$. Thus, for example, $\mathrm{A\overline{Y}\,\overline{X}}$ and $\mathrm{O\overline{X}Y}$ are lawful transformations of AXY, the latter requiring B-semantics.

Boole's work on lawful transformations did not survive in LT, perhaps because there were difficulties in treating contraries not-X of class variables the same as class variables X. For example one has All not-Y is not-X implying Some not-Y is not-X, which leads to All X is Y implies Some not-Y is not-X.

Aristotelian Categorical Syllogisms. In Aristotelian logic a *categorical syllogism*, or more briefly an *AC-syllogism*, is an AC-propositional argument which has the form

AC-syllogism

$$\Phi_1(\mathrm{Y},\mathrm{Z}),\ \Phi_2(\mathrm{X},\mathrm{Y})\ \therefore\ \Phi(\mathrm{X},\mathrm{Z}), \tag{1}$$

with Z the predicate and X the subject of Φ, that is, Φ has the form

$$\text{(quantifier) X (copula) Z.}$$

The first premiss Φ_1 is the *major* premiss and Z the *major* term; the second premiss Φ_2 is the *minor* premiss and X the *minor* term. The two premises share a common term Y, called the *middle term*, which is eliminated in the conclusion.

Major premise/term

Minor premise/ term

Middle term

There are $8 \times 8 \times 4 = 256$ AC-syllogisms of the form (1) in Aristotelian logic, but only a few are *valid AC-syllogisms*, that is, valid AC-arguments. Among the valid AC-syllogisms, special

Valid AC-syllogism

attention has been paid to those where the conclusion is *most general*, that is, the conclusion cannot be strengthened; these will be called *valid AC_{mg}-syllogisms*. For example,

Valid AC_{mg}-syllogism

All Y is Z, All X is Y ∴ Some X is Z

is a valid AC-syllogism, but it is not a valid AC_{mg}-syllogism because the conclusion can be strengthened to give

All Y is Z, All X is Y ∴ All X is Z

which is a valid AC_{mg}-syllogism.

Mood and Figure. A simple classification system for AC-syllogisms using 4 figures and 64 moods has long been standard. The *mood* of an AC-syllogism is given by three consecutive letters, each being one of A, E, I, O, denoting the kinds of the first premiss, the second premiss and the conclusion. For example, the mood AEO means that the kind of the first premiss is A, of the second premiss is E, and of the conclusion is O.

Mood

The *figure* of an AC-syllogism is determined by the placement of the middle term in the premisses—see Tbl. 12 where the dashes are place-holders for the quantifier and the copula, and the middle term is in boldface type. Except for the choice of letters

Figure

First	Second	Third	Fourth
—**Y**—Z	—Z—**Y**	—**Y**—Z	—Z—**Y**
—X—**Y**	—X—**Y**	—**Y**—X	—**Y**—X

Tbl. 12. The Four Figures of AC-Syllogisms

X,Y,Z used for the three terms, an AC-syllogism is completely determined by its figure and mood. The figure plus mood will be called an *AC-specification*, for example, 3rd Fig. AAI is an AC-specification of a valid AC-syllogism. An AC-specification of a valid AC-syllogism will be called a *valid* AC-specification. There is a bijection between the 24 valid AC-specifications and the forms (1) of valid AC-syllogisms; thus to describe the forms of the valid AC-syllogisms it suffices to describe their specifications, which is done in Tbl. 13.

AC-specification

Valid AC-specification

1st Fig.	AAA	EAE	AAI	EAO	AII	EIO
2nd Fig.	AEE	EAE	AEO	EAO	AOO	EIO
3rd Fig.	AAI	EAO	AII	IAI	OAO	EIO
4th Fig.	AAI	AEE	EAO	AEO	IAI	EIO

Tbl. 13. The 24 Valid AC-Specifications

A valid AC-specification is a *valid* AC_{mg}-specification if it describes a valid AC_{mg}-syllogism, that is, where the conclusion is most general. The 19 valid AC_{mg}-specifications are given in Tbl. 14.

Valid AC_{mg}-specification

1st Fig.	AAA	EAE			AII	EIO
2nd Fig.	AEE	EAE			AOO	EIO
3rd Fig.	AAI	EAO	AII	IAI	OAO	EIO
4th Fig.	AAI	AEE	EAO		IAI	EIO

Tbl. 14. The 19 Valid AC_{mg}-Specifications

The modern version of Aristotelian logic does not recognize the existential import of universal propositions. This means that limitation is not valid in lawful transformations, and the number of valid AC-specifications of syllogisms is reduced to 15 (see Fig. 15), all of them being AC_{mg}-specifications.

1st Fig.	AAA	EAE			AII	EIO
2nd Fig.	AEE	EAE			AOO	EIO
3rd Fig.			AII	IAI	OAO	EIO
4th Fig.		AEE			IAI	EIO

Tbl. 15. The 15 Valid Modern AC-Specifications

BC-Syllogisms. Given two AC-premises $\Phi_1(\mathrm{Y},\mathrm{Z})$, $\Phi_2(\mathrm{X},\mathrm{Y})$, Boole noted that in some cases there was a conclusion relating X and Z which could be expressed by a BC-proposition $\Phi(\mathrm{X},\mathrm{Z})$, but not by an AC-proposition; or perhaps it could be expressed by an AC-proposition, but not with the subject and predicate in the required order to give an AC-syllogism. Boole captured these valid conclusions with his generalization of AC-propositions, allowing the subject and/or predicate to be contraries in the conclusion, and by dropping the Aristotelian restriction on the order of the terms in the conclusion.

BC-syllogisms are arguments of the form

BC-syllogism

$$\Phi_1(\mathrm{Y},\mathrm{Z}),\ \Phi_2(\mathrm{X},\mathrm{Y})\ \therefore\ \Phi(\mathrm{X},\mathrm{Z}), \tag{2}$$

where Φ_1, Φ_2 and Φ are BC-propositions. Unlike AC-syllogisms, it is not required that Z appear in the predicate of $\Phi(\mathrm{X},\mathrm{Z})$. If such a syllogism is a correct argument then it will be called a *valid* BC-syllogism. Thus given a valid BC-syllogism, switching the order of the premises always gives a valid BC-syllogism.

Valid BC-syllogism

Given a BC-syllogism (2), one can permute the premises and/or replace the premises and conclusion by any of the BC-propositions that are equivalent to them (as per Tbl. 8 and Tbl. 9) and have a BC-syllogism that is valid iff the original BC-syllogism is valid—two such BC-syllogisms will be said to be *equivalent*.

Equivalent BC-syllogisms

To determine the valid BC-syllogisms, first note that, from the first and third laws in Tbl. 10, every universal BC-proposition $\Phi(\mathrm{X},\mathrm{Y})$ is equivalent to one in the form All $\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{X}}$, where $\widehat{\mathrm{Y}}$ is Y or $\overline{\mathrm{Y}}$, and $\widehat{\mathrm{X}}$ is X or $\overline{\mathrm{X}}$. Likewise every particular BC-proposition $\Phi(\mathrm{X},\mathrm{Y})$ is equivalent to one of the form Some $\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{X}}$. Thus the premises $\Phi_1(\mathrm{Y},\mathrm{Z})$, $\Phi_2(\mathrm{X},\mathrm{Y})$ of a BC-syllogism can be put in the form

$$(\star)\quad \begin{array}{l} Q_1\ \widehat{\mathrm{Y}}\ \textsf{is}\ \widehat{\mathrm{Z}} \\ Q_2\ \widetilde{\mathrm{Y}}\ \textsf{is}\ \widehat{\mathrm{X}} \end{array}$$

where Q_1 and Q_2 are quantifiers; $\widehat{\mathrm{Y}}$, $\widetilde{\mathrm{Y}}$ are each Y or $\overline{\mathrm{Y}}$; $\widehat{\mathrm{X}}$ is X or $\overline{\mathrm{X}}$; and $\widehat{\mathrm{Z}}$ is Z or $\overline{\mathrm{Z}}$.

With the B-semantics of class symbols, the following two cases fully classify the premises in the form $(\star)$ above of valid BC-syllogisms:

CASE: $\widehat{\mathrm{Y}} = \widetilde{\mathrm{Y}}$

Q_1	Q_2	Most general conclusion $\Phi(\mathrm{X},\mathrm{Z})$
All	All	Some $\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Z}}$
All	Some	Some $\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Z}}$
Some	All	Some $\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Z}}$
Some	Some	$(None)$

[In this case, for M-semantics change the first Some $\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Z}}$ in the above table to $(None)$. For A-semantics, append *provided* $\widehat{\mathrm{Y}} = \widetilde{\mathrm{Y}} = \mathrm{Y}$ to the first Some $\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Z}}$.]

CASE: $\widehat{\mathrm{Y}} = \overline{\widetilde{\mathrm{Y}}}$

Q_1	Q_2	Most general conclusion $\Phi(\mathrm{X},\mathrm{Z})$
All	All	All $\overline{\widehat{\mathrm{X}}}$ is $\widehat{\mathrm{Z}}$
All	Some	$(None)$
Some	All	$(None)$
Some	Some	$(None)$

[No changes in this case for either M- or A-semantics.]

Boole's $\widehat{\mathrm{BC}}$-syllogisms. In MAL, ***Boole only considered the BC-syllogisms where the premises were AC-propositions.*** Such will be called $\widehat{\mathrm{BC}}$-syllogisms.*

Boole considered only $\widehat{\mathrm{BC}}$-syllogisms

Note that every BC-syllogism (2) with premises from the first three rows of Tbl. 8 and Tbl. 9 is equivalent to a $\widehat{\mathrm{BC}}$-syllogism. However a BC-syllogism with a premise from row 4 of Tbl. 8 or Tbl. 9 is not equivalent to a $\widehat{\mathrm{BC}}$-syllogism.

For a $\widehat{\mathrm{BC}}$-*specification* Boole used part of the AC-specification of 'figure plus mood', namely the $\widehat{\mathrm{BC}}$-specification was the 'figure plus the first two letters of the mood', that is, it had the form ith Fig. $\alpha\beta$, where $1 \le i \le 4$ and α, β are from $\{\mathrm{A}, \mathrm{E}, \mathrm{I}, \mathrm{O}\}$.

$\widehat{\mathrm{BC}}$-specification

There are 64 $\widehat{\mathrm{BC}}$-specifications, and each determines a single pair $\Phi_1(\mathrm{Y}, \mathrm{Z})$, $\Phi_2(\mathrm{X}, \mathrm{Y})$ of AC-premises. For example, the $\widehat{\mathrm{BC}}$-specification 4th Fig. AI determines the pair of AC-premises AZY, IYX. There is a bijection between the 64 $\widehat{\mathrm{BC}}$-specifications ith Fig. $\alpha\beta$ and the pairs $\Phi_1(\mathrm{Y}, \mathrm{Z})$, $\Phi_2(\mathrm{X}, \mathrm{Y})$ of AC-propositions.

A $\widehat{\mathrm{BC}}$-specification is *valid* if it specifies the AC-premises of a valid BC-syllogism. For such premises, if there is a valid universal conclusion, then it will be most general. Otherwise any valid particular conclusion will be most general.

Valid $\widehat{\mathrm{BC}}$-specification

If the pair $\Phi_1(\mathrm{Y}, \mathrm{Z})$, $\Phi_2(\mathrm{X}, \mathrm{Y})$ belongs to a valid $\widehat{\mathrm{BC}}$-specification ith Fig. $\alpha\beta$, then the pair is *completed* by a BC-proposition $\Phi(\mathrm{X}, \mathrm{Z})$ iff (2) is a valid syllogism.

Completion of a pair $\Phi_1(\mathrm{Y}, \mathrm{Z})$, $\Phi_2(\mathrm{X}, \mathrm{Y})$

Two $\widehat{\mathrm{BC}}$-specifications will be called *conjugates* if they fit one of the following three descriptions:

Conjugate $\widehat{\mathrm{BC}}$-specifications

- 1st Fig. $\alpha\beta$ and 4th Fig. $\beta\alpha$
- 2nd Fig. $\alpha\beta$ and 2nd Fig. $\beta\alpha$
- 3rd Fig. $\alpha\beta$ and 3rd Fig. $\beta\alpha$.

Thus, for example, 1st Fig. AO and 4th Fig. OA are conjugates, as are 2nd Fig. AO and 2nd Fig. OA. Given two conjugate $\widehat{\mathrm{BC}}$-specifications, either both are valid, or neither is valid.

A valid $\widehat{\mathrm{BC}}$-specification ith Fig. $\alpha\beta$ will be said to be *directly determinable* by the Aristotelian Rules if it is a $\widehat{\mathrm{BC}}$-specification of a pair of AC-premises $\Phi_1(\mathrm{Y}, \mathrm{Z})$, $\Phi_2(\mathrm{X}, \mathrm{Y})$ which can be completed to an AC-syllogism.

Directly determinable by Aristotelian Rules

*By not considering all BC-syllogisms Boole failed to recognize a simple classification. See p. xix for how he recovered from this oversight in CL.

A valid $\widehat{\mathrm{BC}}$-specification ith Fig. $\alpha\beta$ which is not directly determinable by the Aristotelian Rules will be said to be *indirectly determinable* by the Aristotelian Rules if it is conjugate to a $\widehat{\mathrm{BC}}$-specification that is directly determinable by the Aristotelian Rules.*

Indirectly determinable by Aristotelian Rules

From Tbl. 14 one immediately has 19 valid $\widehat{\mathrm{BC}}$-specifications in Boole's logic—they are displayed in Tbl. 16. These are valid

1st Fig.	AA	EA			AI	EI
2nd Fig.	AE	EA			AO	EI
3rd Fig.	AA	EA	AI	IA	OA	EI
4th Fig.	AA	AE		EA	IA	EI

Tbl. 16. Nineteen Valid $\widehat{\mathrm{BC}}$-Specifications Derived from the Valid Specifications in Aristotelian Logic

$\widehat{\mathrm{BC}}$-specifications that are directly determinable from Aristotle's Rules. Taking their conjugates adds 8 more valid $\widehat{\mathrm{BC}}$-specifications in Boole's logic, displayed in Tbl. 17—these are valid $\widehat{\mathrm{BC}}$-specifications

1st Fig.		AE		IE
2nd Fig.			OA	IE
3rd Fig.	AE		AO	IE
4th Fig.				IE

Tbl. 17. Eight more Valid Syllogisms Determined Using Conjugates

that are indirectly determinable from Aristotle's Rules. At this point one has 27 valid $\widehat{\mathrm{BC}}$-specifications of syllogisms; only a few more remain to complete the list.

Boole classified the valid $\widehat{\mathrm{BC}}$-specifications in the chapter OF SYLLOGISMS, pp. 31–47. His results are summarized in Tbl. 18. Those which are directly derived from the 19 valid Aristotelian syllogisms (see Tbl. 16) are indicated by enclosing the kinds of the two premises in a box, for example, 1st Fig. $\boxed{\mathrm{AA}}$.

Boole divided the $\widehat{\mathrm{BC}}$-specifications into four classes as follows:†

*Boole's definition of indirectly determinable specification requires that it be obtained from a directly determinable specification by changing the order of the premises or using conversion.

†See p. xxxvii for a discussion of how the four classes correspond to four ways of dealing with equational expressions of the premises.

Class	Figure	Valid by Aristotle's Rules Directly or Indirectly	Other Valid Specifications	Invalid Specifications
1st	1st	$\boxed{\text{AA}}$, $\boxed{\text{EA}}$	$\boxed{}$	$\boxed{}$
	2nd	$\boxed{\text{AE}}$, $\boxed{\text{EA}}$	$\boxed{}$	**AA**
	3rd	$\boxed{}$	$\boxed{}$	$\boxed{}$
	4th	$\boxed{\text{AA}}$, $\boxed{\text{AE}}$	$\boxed{}$	$\boxed{}$
2nd	1st	AE	EE	$\boxed{}$
	2nd	$\boxed{}$	EE	$\boxed{}$
	3rd	$\boxed{\text{AA}}$, AE, $\boxed{\text{EA}}$	EE	$\boxed{}$
	4th	$\boxed{\text{EA}}$	EE	$\boxed{}$
3rd	1st	$\boxed{\text{AI}}$, $\boxed{\text{EI}}$, IE	OE	AO, EO, IA, OA
	2nd	$\boxed{\text{AO}}$, OA, $\boxed{\text{EI}}$, IE	$\boxed{}$	AI, IA, EO, OE
	3rd	$\boxed{\text{AI}}$, $\boxed{\text{IA}}$, AO, **OE,**	$\boxed{}$	$\boxed{}$
		EO, $\boxed{\text{OA}}$, $\boxed{\text{EI}}$, IE	$\boxed{}$	$\boxed{}$
	4th	$\boxed{\text{IA}}$, IE	EO	OA, OE, AI, $\boxed{\textbf{EI}}$, AO
4th	1st	$\boxed{}$	$\boxed{}$	II, IO, OI, OO
	2nd	$\boxed{}$	$\boxed{}$	II, IO, OI, OO
	3rd	$\boxed{}$	$\boxed{}$	II, IO, OI, OO
	4th	$\boxed{}$	$\boxed{}$	II, IO, OI, OO

Tbl. 18. Summary of Boole's Classification of $\widehat{\mathrm{BC}}$-Specifications

- The 1st class has universal premises which can be completed to a universal conclusion either directly or indirectly using Aristotle's rules.
- The 2nd class has universal premises which can only be completed to a particular conclusion.
- The 3rd class is for one universal premiss and one particular premiss; the conclusion is always particular.
- The 4th class is for two particular premises, from which one never has a valid conclusion.

Four entries in Tbl. 18 are in bold-face type to indicate errors in Boole's classification:

Four errors in Boole's classification.

(1) On p. 35 Boole said that 2nd Fig. AA is invalid; but with B-semantics the premises All Z is Y, All X is Y can be completed by Some not-Z is not-X; just replace the premises by their contrapositives and use the fact that $\overline{\mathrm{Y}}$ is not empty.

The first error assumes B-semantics. With A-semantics it is not an error.

(2,3) The 3rd Fig. OE and EO specifications should be in the column Other Valid Specifications.

(4) The 4th Fig. EI can be completed to a valid AC-syllogism 4th Fig. EIO, so it is directly determinable.

Boole's treatment of BC-Syllogisms in CL. In CL Boole's only significant contribution was a simple description of the valid BC-syllogisms with most general conclusions.* First he changed the usage of the words *quantity* and *quality* so that they applied to the quantified subject and to the predicate of a categorical proposition, not to the whole proposition:

quantified subject	quantity	quality
All X	universal	affirmative
All not-X	universal	negative
Some X	particular	affirmative
Some not-X	particular	negative

predicate	quantity	quality
Z	particular	affirmative
not-Z	particular	negative

In the premises $\Phi_1(\mathrm{Y},\mathrm{Z}), \Phi_2(\mathrm{X},\mathrm{Y})$ of a BC-syllogism, each of the four indicated class variables can appear either affirmatively or negatively. The terms which involve X and Z are the *extremes*, the two terms which involve Y are the *middle* terms. The middle terms of the premises have *like* qualities if they are both affirmative or both negative, otherwise they have *opposite* qualities.

(1): $\Phi_1(\mathrm{Y},\mathrm{Z}), \Phi_2(\mathrm{X},\mathrm{Y})$ have middle terms of like qualities, and (at least) one is universal. The conclusion leaves the quality and quantity of both extremes unchanged.

(2): $\Phi_1(\mathrm{Y},\mathrm{Z}), \Phi_2(\mathrm{X},\mathrm{Y})$ have middle terms of opposite qualities. There are two cases to consider:

(2a): (At least) one extreme is universal. For the conclusion choose a universal extreme and change its quality and quantity but leave the other extreme unchanged.

(2b): The two middle terms are universal. For the conclusion change the quality and quantity of one of the extremes, leave the other extreme unchanged.

*In LT he claimed to justify this description in Chapter XV using two substantial applications of his strong Elimination Theorem.

Boole noted that this description of the most general valid B-syllogisms does away with the need for moods, figures and the order of the premises.

Here is a list is of valid syllogisms according to Boole's rules, where in each syllogism $\widehat{\mathrm{X}}$ is either X or not-X, etc.

Possibilities for (1):

All $\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{X}}$, All $\widehat{\mathrm{Z}}$ is $\widehat{\mathrm{Y}}$, therefore All $\widehat{\mathrm{Z}}$ is $\widehat{\mathrm{X}}$.

All $\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{X}}$, $\left\{\begin{array}{l}\text{Some } \widehat{\mathrm{Z}} \text{ is } \widehat{\mathrm{Y}} \\ \text{Some } \widehat{\mathrm{Y}} \text{ is } \widehat{\mathrm{Z}}\end{array}\right\}$ therefore $\left\{\begin{array}{l}\text{Some } \widehat{\mathrm{X}} \text{ is } \widehat{\mathrm{Z}} \\ \text{Some } \widehat{\mathrm{Z}} \text{ is } \widehat{\mathrm{X}}\end{array}\right\}$.

All $\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{X}}$, All $\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{Z}}$, therefore $\left\{\begin{array}{l}\text{Some } \widehat{\mathrm{X}} \text{ is } \widehat{\mathrm{Z}} \\ \text{Some } \widehat{\mathrm{Z}} \text{ is } \widehat{\mathrm{X}}\end{array}\right\}$.

Possibilities for (2a):

All $\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Y}}$, All $\widehat{\mathrm{Z}}$ is not-$\widehat{\mathrm{Y}}$, therefore $\left\{\begin{array}{l}\text{All } \widehat{\mathrm{X}} \text{ is not-} \widehat{\mathrm{Z}} \\ \text{All } \widehat{\mathrm{Z}} \text{ is not-} \widehat{\mathrm{X}}\end{array}\right\}$.

All $\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Y}}$, All not-$\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{Z}}$, therefore Some not-$\widehat{\mathrm{X}}$ is $\widehat{\mathrm{Z}}$.

Possibilities for (2b):

All $\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{X}}$, All not-$\widehat{\mathrm{Y}}$ is $\widehat{\mathrm{Z}}$, therefore $\left\{\begin{array}{l}\text{All not-} \widehat{\mathrm{X}} \text{ is } \widehat{\mathrm{Z}} \\ \text{All not-} \widehat{\mathrm{Z}} \text{ is } \widehat{\mathrm{X}}\end{array}\right\}$.

HYPOTHETICAL SYLLOGISMS

Boole adopted the accepted theory of hypothetical syllogisms. On p. 48 he defined a **hypothetical proposition** "to be two or more categoricals united by a copula (or conjunction)". The standard forms of hypothetical propositions were: (Conditional) If X then Y, and (Disjunctive) X or Y or ..., where X, Y, etc., denote categorical propositions. In modern logic X,Y, etc., are called propositional variables.

Certain arguments based on hypothetical propositions were generally accepted to be valid **hypothetical syllogisms**—Boole stated them on pp. 56–57, using his reduction of categorical propositions to propositional variables X, Y, Z, W. These are listed below.*

1. **Disjunctive Syllogism** (two versions)

	X or Y (exclusive 'or')		X or Y (inclusive 'or')
	X		Not X
∴	Not Y.	∴	Y.

2. **Constructive Conditional Syllogism**
   - If X then Y
   - X
   - ∴ Y.

3. **Destructive Conditional Syllogism**
   - If X then Y
   - Not Y
   - ∴ Not X.

4. **Simple Constructive Dilemma**
   - If X then Y
   - If Z then Y
   - X or Z (exclusive 'or')
   - ∴ Y.

5. **Complex Constructive Dilemma**
   - If X then Y
   - If W then Z
   - X or W (inclusive 'or')
   - ∴ Y or Z (inclusive 'or').

*To convert Boole's versions of the hypothetical syllogisms back to the forms based on categorical propositions that Whately would have used, replace X by 'A is B', Y by 'C is D', Z by 'E is F', and W by 'G is H'.

6. **Complex Destructive Dilemma** (two versions)

If X then Y	If X then Y
If W then Z	If W then Z
Not Y or not Z (exclusive 'or')	Not Y or not Z (inclusive 'or')
∴ Not X or not W (inclusive 'or').	∴ Not X or not W (inclusive 'or').

Boole noted on p. 57 that one could easily extend the list, for example, by using propositions that were blends of the conditional and disjunctive forms such as

If X is true, then either Y is true, or Z is true.

But he did not describe the concept of an arbitrary propositional formula, that is, an arbitrary Boolean combination of propositional variables. Instead he limited his work to the simple forms one might encounter in ordinary discourse.

BOOLE'S EQUATIONAL ALGEBRA

Today the word "Boolean" has a meaning in algebra that was established in the first half of the 20th century by the Harvard philosopher Josiah Royce (1855–1916) and his students, following the lead of Charles Sanders Peirce (1839–1914)—see [11]. It refers to an important simplification of Boole's approach to the algebra of logic that started with the work [21] of William Stanley Jevons (1835–1882) in 1864.* Consequently, **when referring to what Boole actually did, the word Boolean will be avoided.** *Boole's Algebra* will be used for what Boole did instead of *Boolean algebra*. When the word Boolean is used, it will be in the modern sense.

The modern approach to creating an algebra of logic for classes is to start with the three fundamental operations—union ($\cup$), intersection ($\cap$) and complement ($'$)—and determine the basic laws and rules of inference. This leads to Boolean algebra.

Boole's approach was essentially the reverse—he started with the laws and rules of inference of common algebra, defined multiplication, then realized he needed to add another law (the index law). Addition and subtraction were then forced to be partially defined operations (see [9]). When studying the equational theory of a partial algebra, or a collection of partial algebras, in general one cannot proceed as with total algebras, but Boole was fortunate in that his partial algebra approach to the logic of classes could be developed as though one were working with total algebras (where the fundamental operations are totally defined). At times Boole would start with equational premises that were totally defined, use equational inferences that gave only partially defined equations in some of the intermediate steps, and end up with totally defined equations which he claimed were valid conclusions. This puzzled those who wanted to understand *why* his algebra of logic gave correct results.

*In 1863–1864 Jevons corresponded with Boole describing his alternate approach to the algebra of logic, where $+$ is totally defined and one has $x + x = x$. Boole flatly rejected this approach, saying that $x + x = x$ was not true in the algebra of logic. For details see the 1991 article [17] by Ivor Grattan-Guinness (1941–2014). It is easy to understand Boole's opposition to adding $x + x = x$ as a law of his algebra—it would have obstructed his approach, based on Common Algebra and the index law, since $x + x = x$ implies $x = 0$ in Common Algebra.

Common Algebra. Boole said his algebra was *common algebra* augmented by the *index law* $x^n = x$ (for variables only as it turns out). **By *Common Algebra* he meant the equations and equational reasoning that hold in *numerical algebra*, the algebra of numbers.** It suffices to restrict ones attention to the modern algebraic structure $\mathbf{Z}$ consisting of the set of integers $\mathbb{Z}$ equipped with the three binary operations addition $(+)$, multiplication $(\cdot)$, subtraction $(-)$ and the two numerical constants 0 and 1.* This structure is written more briefly as $\mathbf{Z} = \langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$.†

Common Algebra had been brought to a refined state by Boole's time, and Boole was expert in using it. Readers of these notes will likely have had an introductory algebra course and consequently will be able to understand and use the many conventions that have been developed to expedite carrying out algebraic manipulations. First there are conventions to avoid using parentheses as much as possible, for example when writing sums $p_1 + p_2 + \cdots + p_n$ and products $p_1 \cdot p_2 \cdots p_n$. One justifies this by saying that addition and multiplication are associative, but surprisingly Boole never mentioned the associative laws.

Subtraction on the other hand is not associative as one sees from the example $(3-2)-1 \neq 3-(2-1)$, so a convention is needed for an expression like $p_1 - p_2 - \cdots - p_n$. The standard approach is to use the minus sign "$-$" in two ways, one as the binary operation of subtraction as in $p - q$, and the other is as the unary operation of taking the negative of, as in $-p$. They are intimately connected by the equations $-p = 0 - p$ and $p - q = p + (-q)$.Then the modern convention is to read $p_1 - p_2 - \cdots - p_n$ as $p_1 + (-p_2) + \cdots + (-p_n)$, but in Boole's algebra it would be read as $(\cdots(p_1 - p_2) - \cdots) - p_n$.

Given a mixture of sums, differences and products, it is understood that multiplication precedes addition, subtraction and taking the negative of; taking the negative of precedes addition and subtraction, so $-pq - r + s$ means $-(pq) + (-r) + s$. Then of course there are all the rules for handling the unary mi-

*Boole used the symbol '$\times$' instead of '$\cdot$' for multiplication. One frequently abbreviates $p \cdot q$ to pq.

†In modern algebra the unary minus is taken as a fundamental operation, and subtraction is defined. Then the algebraic structure $\mathbf{Z} = \langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$ is called *the ring of integers*. However Boole took subtraction as fundamental, and simply used the unary minus as in common algebra. Boole's choice will be followed in these notes because it is essential to expressing propositions in his partial algebra—see p. xxxi .

nus such as $-(-p) = p$, $(-p)q = -(pq)$, $(-p)(-q) = pq$, and $-(p-q) = (-p) + q$. p added to itself n times is np; p multiplied by itself n times is expressed by p^n, introducing exponential notation. One has the familiar laws for exponentiation: $p^m \cdot p^n = p^{m+n}$, $p^n \cdot q^n = (pq)^n$ and $(p^m)^n = p^{mn}$.

Using these laws and conventions, along with the distributive law and commutative laws, complex algebraic expressions can be put in polynomial form, for example, $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$. Polynomials are highly regarded by algebraists because they have no parentheses, making them easier to read.

Boole was counting on his readers being familiar with these aspects of common algebra, enabling him to carry out algebraic manipulations without detailed explanations. Furthermore, being quite fluent with common algebra he would have been able to rapidly work through lots of examples (such as the catalog of syllogisms) to check the merits of various translations between propositions and equations.

Algebraic Terms versus Logical Terms. In modern logic, terms for the language of $\mathbb{Z}$ are defined recursively by:

- variables* are terms
- 0 and 1 are terms
- if p and q are terms then so are $(p+q)$, $(p \cdot q)$ and $(p-q)$.

Such terms will be called *logical terms*—as they get larger, the accumulation of parentheses can make them difficult to read. In these notes *terms*, or *algebraic terms*, will refer not only to logical terms, but to any of the algebraic expressions obtained by the above-mentioned conventions and abbreviations. Thus, for example, $(x + (y \cdot z))$ is a logical term, and $x + yz$ is a term.

logical terms

(algebraic) terms

Given a term p, in modern notation one writes $p(x_1, \ldots, x_n)$ for p to indicate that the variables occurring in p are *among* $x_1, \ldots, x_n$. Each term† $p(x_1, \ldots, x_n)$, more briefly $p(\vec{x})$, is naturally associated with a function $p^{\mathbb{Z}}(\vec{x}) : \mathbb{Z}^n \to \mathbb{Z}$, called the *term-function* defined on $\mathbb{Z}$ by $p(\vec{x})$. Two terms p and q are $\mathbb{Z}$-*equivalent* if $p^{\mathbb{Z}}(\vec{x}) = q^{\mathbb{Z}}(\vec{x})$ for some/any list $\vec{x}$ of variables containing the variables of both p and q. Every term is $\mathbb{Z}$-equivalent

term-function

$\mathbb{Z}$-equivalent terms

*Boole used the word *variable* sparingly in MAL, and not at all in LT, preferring instead *symbol* or *letter*.

†$p(x_1, \ldots, x_n)$ is simply called a *term* instead of the more accurate phrase: *term p with a specified list of variables containing the variables of p.*

to a logical term as well as to a polynomial (with integer coefficients).

Starting on p. 60 Boole used the notations $\phi(x)$, $\phi(x,y)$, etc., to denote terms with a partial, perhaps complete list of the variables appearing in the term. In this commentary on Boole's algebra, modern notation such as $p(\vec{x},\vec{y})$ will be used for a term which Boole might have expressed as $\phi(\vec{x})$.

Laws and Valid Inferences of Common Algebra. A $\mathbb{Z}$-*equation* is an expression $p = q$ where p and q are terms. $p = q$ *holds in* $\mathbb{Z}$ if p and q are $\mathbb{Z}$-equivalent; then $p = q$ is a $\mathbb{Z}$-*law*. For example, $(x+y)^2 = x^2 + 2xy + y^2$ is a $\mathbb{Z}$-law.

$\mathbb{Z}$-equation

$\mathbb{Z}$-law

It is a standard result that two terms are $\mathbb{Z}$-equivalent iff there is a polynomial to which they are both $\mathbb{Z}$-equivalent. This gives an algorithm to determine if a $\mathbb{Z}$-equation is a $\mathbb{Z}$-law.

A $\mathbb{Z}$-equational inference $p_1 = q_1, \ldots, p_k = q_k \ \therefore\ p = q$ is *valid for* (or *holds in*) $\mathbb{Z}$ if every assignment of values to the variables which make all the ${p_i}^{\mathbb{Z}} = {q_i}^{\mathbb{Z}}$ true also makes $p^{\mathbb{Z}} = q^{\mathbb{Z}}$ true. This includes simple inferences like $2p = 0 \ \therefore\ p = 0$, and difficult to prove inferences like the even power instances of Fermat's Last Theorem, namely $x^{2n} + y^{2n} = z^{2n} \ \therefore\ xyz = 0$, for $n = 2, 3, \ldots$ In contrast to the laws of $\mathbb{Z}$, the collection of $\mathbb{Z}$-equational inferences that hold in $\mathbb{Z}$ is not decidable.

Valid inference for $\mathbb{Z}$

Laws and Valid Inferences of Boole's Algebra. A $\mathbb{Z}$-equation is a *law of* Boole's Algebra iff it can be derived from* the $\mathbb{Z}$-laws, the index law $x^n = x$, and the $\mathbb{Z}$-equational inferences that hold in $\mathbb{Z}$. Thus one has $(x+y)^2 = x^2+2xy+y^2 = x+2xy+y$ in Boole's Algebra. Two terms p and q are *equivalent* in Boole's Algebra if $p = q$ is a law of Boole's Algebra.

Law of Boole's Algebra

Equivalent terms in Boole's Algebra

A $\mathbb{Z}$-equational inference is *valid* in Boole's Algebra iff the conclusion can be derived from the premises using the $\mathbb{Z}$-laws, the index law and valid $\mathbb{Z}$-equational inferences. Thus in Boole's Algebra one has the valid inference $(x+y)^2 = x + y \ \therefore\ xy = 0$.

Valid inferences of Boole's Algebra

From now on the $\mathbb{Z}$- prefix to various concepts mentioned above will be assumed understood, and omitted.

Dropping $\mathbb{Z}$- prefix

The Rule of 0 and 1. In recent years (see [8], [12]) it has been noted that Boole stated a simple algorithm in LT to determine the laws and valid inferences of his algebra.† For φ a

*For a precise definition of what is meant by a *derivation* see p. XXX of these notes.

†For the laws see the discussion of Prop. 1 on pp. 61, 62 of MAL; for the laws and valid inferences see item 15, p. 37 of LT.

first-order formula let $\mathbb{Z} \models_{01} \varphi$ mean that φ holds in $\mathbb{Z}$ provided the variables, bound and unbound, are restricted to 0,1. Then one has the *Rule of 0 and 1*, or simply **R01**:*

Defn. of $\mathbb{Z} \models_{01} \varphi(\vec{x})$

The Rule of 0 and 1

I. An equation ε holds in Boole's Algebra iff $\mathbb{Z} \models_{01} \varepsilon$.

II. An equational argument $\varepsilon_1, \ldots, \varepsilon_k \therefore \varepsilon$ is valid in Boole's Algebra iff $\mathbb{Z} \models_{01} \left(\bigwedge_{i=1}^{k} \varepsilon_i\right) \rightarrow \varepsilon$.

Although the algorithmic approach to Boole's Algebra using $\mathbb{Z} \models_{01}$ is easier to develop than the axiomatic approach, it is the latter, using derivations of equations, that one mainly sees in Boole's work on logic.†

General Solutions. In the mathematical analysis of his expansion of Aristotelian logic in pp. 15–59, Boole used the *general solution* of very simple equations, namely $\widehat{y} = v(1 - \widehat{x})$ is the general solution of $\widehat{x}\,\widehat{y} = 0$ for $\widehat{x}$ either x or $1 - x$ and $\widehat{y}$ either y or $1-y$.‡ Boole used this in the form: if $\widehat{x}\,\widehat{y} = 0$ then $\widehat{y} = v(1-\widehat{x})$ for some v. (This is not an equational inference because of the phrase "for some v".) This evidently inspired Boole to examine the general solution of an equation $p(\vec{x}, y) = 0$, for y in terms of $\vec{x}$.

General solution

Elimination of one Variable from two Equations. Using $\mathbb{Z} \models_{01}$ gives an algorithm to check if an equational argument

*Dummett [16], p. 206, claimed that the interpretation in the integers, using $\mathbb{Z} \models_{01}$, gave a consistency proof for Boole's Algebra. But he did not realize that the same could be used, under M-semantics of class variables, to justify the application of Boole's Algebra to logic.

With M-semantics for class variables one can strengthen R01 to

III. A Horn formula φ holds in Boole's Algebra iff $\mathbb{Z} \models_{01} \varphi$.

This goes well beyond Boole's equational logic; it is used in [12] to prove precise versions of his main theorems (Expansion, Reduction, Elimination and Solution) using the M-semantics of class variables.

†In the chapter FIRST PRINCIPLES Boole claimed that the *three equational laws* $\left[\begin{array}{rcl} x(y+z) & = & xy+xz \\ xy & = & yx \\ x^n & = & x \quad (\text{for variables}) \end{array}\right]$ and the *inference rule* (which he called an *axiom*) "equivalent operations performed upon equivalent subjects produce equivalent results" gave a foundation for his algebra of logic (see pp. 17, 18). On p. 18 he said, quite incorrectly, that the first two laws—the distributive law and the commutative law—justify the use of all the processes of Common Algebra. Boole's axiomatic foundation of Common Algebra was just window-dressing for the use of Common Algebra that follows.

‡With M-semantics this can be justified using R01, namely by showing that

$$\mathbb{Z} \models_{01} \widehat{x}\,\widehat{y} = 0 \;\leftrightarrow\; (\exists v)\big[\widehat{y} = v(1-\widehat{x})\big].$$

is valid in Boole's Algebra. However, given equational premises $\varepsilon_1, \ldots, \varepsilon_k$, Boole was mainly interested in algorithms for finding the most general equational conclusion ε, subject to various restrictions. For example, given the equations $\varepsilon_1(y,z)$ and $\varepsilon_2(x,y)$ for the premises of a syllogism, he wanted an algorithm to find the most general equation $\varepsilon(x,z)$ that followed from the two premises. Such an algorithm would be an *elimination* procedure (since y has been eliminated in the conclusion).

In MAL, Boole's primary mathematical tool to eliminate the middle term from the premises of a syllogism was a simple *Elimination Theorem* from Common Algebra (see p. 32):

$$ay + b = 0,\ cy + d = 0\ \therefore\ ad - bc = 0. \tag{3}$$

Weak Elimination Theorem

However this is not the most general elimination result for his algebra, consequently it will at times be referred to as the *weak* elimination theorem. The strong version, using the reduction and elimination theorems of LT, is*

$$ay + b = 0,\ cy + d = 0\ \therefore\ (b^2 + d^2)\big[(a+b)^2 + (c+d)^2\big] = 0. \tag{4}$$

It is a nice exercise to show that the conclusion of (4) implies the conclusion of (3) in Boole's Algebra.

The Semantics of Boole's Algebra. In MAL Boole used $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \ldots$ to denote classes in the logic of categorical propositions, and to denote categorical propositions in the logic of hypothetical propositions.

Elective operation

For a class X he let x be the *elective operation* which, for any class Z, selects the elements of Z that are in X, giving the class $\mathrm{X} \cap \mathrm{Z}$ (see p. 15).† Nowadays we would write $x(\mathrm{Z}) = \mathrm{X} \cap \mathrm{Z}$, but Boole never used the notation $x(\mathrm{Z})$.

Furthermore (see pp. 15, 20), Boole used the symbol 1 to denote *the Universe*, and the same symbol to denote the identity elective operation, that is, $1(\mathrm{X}) = \mathrm{X}$.

It is very tempting to assume Boole was working with an algebra of elective operations, defining the operations of multiplication, addition, and subtraction on the collection of elective

*To derive (4) by the methods of LT, first reduce the two premise equations $ay + b = 0,\ cy + d = 0$ to the single equation $(ay+b)^2 + (cy+d)^2 = 0$. Letting $f(y)$ denote the left side of this equation, the result in LT of eliminating y is $f(0)f(1) = 0$, giving (4).

†His decision to work with elective operations acting on classes, rather than directly with classes (as he would in LT), was likely inspired by his previous successes with differential operators.

operations. But his comments on pages 15 and 16 cast doubt on this view. He said that when no subject is expressed then one is to assume the universe 1 is the subject, and gives the example

$$x = x\ (1),$$

meaning what we would write as

$$x(1) = x(1),$$

that is, $\mathrm{X} = \mathrm{X}$. Thus it appears that by the equation $x = x$ he really means the equation $\mathrm{X} = \mathrm{X}$.

Then he goes into some detail on the nature of the product xy:

> From these premises it will follow, that the product xy will represent, in succession, the selection of the class Y, and the selection from the class Y of such individuals of the class X as are contained in it, the result being the class whose members are both Xs and Ys. And in like manner the product xyz will represent a compound operation of which the successive elements are the selection of the class Z, the selection from it of such individuals of the class Y as are contained in it, and the selection from the result thus obtained of all the individuals of the class X which it contains, the final result being the class common to X, Y, and Z.

Thus the product xy is not an elective operation but a class, namely the class $x(y(1))$, in modern notation $\mathrm{X} \cap \mathrm{Y}$. This definition of multiplication as an intersection is the one real impact of his use of elective operations; otherwise it is the proverbial albatross hanging around the neck, adding an unnecessary layer of complication to his algebra. One is left to wonder why Boole did not simply define his algebraic operations on classes and completely omit elective operations as he would in LT. Actually one sees that he was moving in this direction in his 1848 paper CL where he wrote

If Boole were working with an algebra of elective operations then xy would be the elective operation determined by the class $\mathrm{X} \cap \mathrm{Y}$, and not simply the class $\mathrm{X} \cap \mathrm{Y}$.

$$x1 = x = \mathrm{X},$$

but still that paper used elective expressions. In LT the letters $x, y, etc.$ denote classes.

His operation of addition is not clearly formulated in MAL, and subtraction is explained only for the expression $1 - x$. Based on the algebra of logic in LT, one can make reasonable assumptions regarding how to define addition and subtraction in MAL:*

Definitions of multiplication, addition and subtraction of elective operations.

1) *multiplication:* $xy = \mathrm{X} \cap \mathrm{Y}$, $xyz = \mathrm{X} \cap \mathrm{Y} \cap \mathrm{Z}$, etc.

2) *addition:* $x + y = \mathrm{X} \cup \mathrm{Y}$ provided X and Y are disjoint classes, and

3) *subtraction:* $x - y = \mathrm{X} \setminus \mathrm{Y}$ provided Y is a subclass of X.

Addition and subtraction are *undefined* if the stated conditions on X and Y do not hold; thus these two operations are not total. This seemed to be of little concern to Boole—he carried out equational reasoning just as one would with totally defined operations.†

Boole's goal regarding Aristotelian logic was to show that his algebra, and his translations between propositions and equations, were such that one could apply the procedure in Tbl. 1 to derive precisely the lawful transformations of BC-propositions as well as the valid $\widehat{\mathrm{BC}}$-syllogisms. For the hypothetical syllogisms he relied on a standard claim that such could be reduced to reasoning about categorical propositions. However Boole did not note the fact that the semantics of class variables was not the same in the two situations—for the logic of hypothetical propositions he appeared to be using M-semantics, permitting the equations $x = 0$ and $x = 1$.

An Alternative to using Elective Operations. When reading MAL, if one is not comfortable with elective operations, one can replace the variables $x, y, \ldots$ by the corresponding class symbols $\mathrm{X}, \mathrm{Y}, \ldots$, with the operations being defined on classes by

1) *multiplication* XY is the intersection $\mathrm{X} \cap \mathrm{Y}$;

2) *addition* $\mathrm{X} + \mathrm{Y}$ is the union $\mathrm{X} \cup \mathrm{Y}$, provided X and Y are disjoint classes, and undefined otherwise;

3) *subtraction* $\mathrm{X} - \mathrm{Y}$ is the difference $\mathrm{X} \setminus \mathrm{Y}$, provided Y is a subclass of X; and undefined otherwise.

*In a previous version of these Notes the operations of multiplication, addition and subtraction were defined to map pairs of elective operations to an elective operation. That effort is now regarded as misguided.

†In LT Boole tried to justify this in his discussion of the Principles of Symbolical Reasoning. The analysis was not correct (see [14]), but nevertheless one can justify the use of ordinary equational reasoning (for total algebras) when working with Boole's Algebra.

This is the semantics of the operations of addition, subtraction and multiplication in LT, but the notation is somewhat different—usually lower case letters $x, y, \ldots$ represent classes in LT. 1 still denotes the universe. The empty class is introduced and denoted by 0. Thus the equations of MAL look much the same as those in LT, but the variables represent different objects—elective operations in the former, classes in the latter.

The somewhat strange looking definitions of addition and subtraction, as partial operations, are actually forced by Common Algebra and the index law once multiplication has been defined as intersection (see [9]). To see how Boole ended up in this awkward situation, for classes X and Y suppose $\mathrm{X}+\mathrm{Y}$ is defined (as a class). Then $(\mathrm{X}+\mathrm{Y})^2 = \mathrm{X}+\mathrm{Y}$ by the index law. By Common Algebra one has $\mathrm{X}^2+2\mathrm{XY}+\mathrm{Y}^2 = \mathrm{X}+\mathrm{Y}$, and from the index law this gives $\mathrm{X}+2\mathrm{XY}+\mathrm{Y} = \mathrm{X}+\mathrm{Y}$. Common Algebra then gives $\mathrm{XY} = 0$. Thus if $\mathrm{X}+\mathrm{Y}$ is defined then X and Y are disjoint classes. A similar argument applies to subtraction.

Boole's Algebra has more in common with Boolean Rings than with Boolean Algebra. In Boole's Algebra one can express familiar operations on classes by totally defined terms:*

$$\begin{array}{rcl}
\mathrm{X}' & = & 1-\mathrm{X} \\
\mathrm{X}\cap\mathrm{Y} & = & \mathrm{X}\cdot\mathrm{Y} \\
\mathrm{X}\cup\mathrm{Y} & = & \mathrm{X}+(1-\mathrm{X})\cdot\mathrm{Y} \\
\mathrm{X}\smallsetminus\mathrm{Y} & = & \mathrm{X}\cdot(1-\mathrm{Y}) \\
\mathrm{X}\triangle\mathrm{Y} & = & \mathrm{X}\cdot(1-\mathrm{Y})+(1-\mathrm{X})\cdot\mathrm{Y}
\end{array}$$

The same results hold in the Boolean ring of classes, where multiplication is intersection and addition is symmetric difference.

By the Rule of 0 and 1 it is seen that Boole's Algebra is closer than Boolean Rings to the algebra of the integers in that *all* the equations *and* equational arguments valid in the integers are valid in Boole's Algebra. Boolean Rings satisfy all the equations valid in the integers but not all the equational arguments, for example, $x+x=0$ implies $x=0$ fails in Boolean Rings.

*For such expressions it was essential that Boole's partial algebra took the binary operation of subtraction as fundamental instead of the unary additive inverse that is fundamental in ring theory. The latter choice would have made it impossible to find a totally defined term to express the complement of a class since $-\mathrm{X}$ is only defined for $\mathrm{X}=0$ in Boole's partial algebra.

Boole's Equational Expressions of Categorical Propositions. The existential import of universal categorical propositions in Aristotelian logic led Boole to introduce in MAL multiple equational expressions for categorical propositions in order to handle the various valid arguments that drew particular conclusions from universal premises. Tbl. 19 states the most important of Boole's equational expressions for AC-propositions.*

		Equation	
Type	Proposition	Primary	Secondary
A	All X is Y	$x = xy$	$x = vy$
E	No X is Y	$xy = 0$	$x = v(1-y)$
I	Some X is Y	$v = xy$	$vx = vy$
O	Some X is not Y	$v = x(1-y)$	$vx = v(1-y)$.

Tbl. 19. Equational Expressions of Aristotelian Propositions when using Aristotelian Semantics for Class Variables

For each of the universal propositions (A, E), the secondary equation implies the primary equation (in Boole's Algebra). Boole justifies the converses, for example $x = xy \therefore x = vy$, by claiming that the secondary is the general solution of the primary.†

For the particular propositions (I, O), the secondary equations follow from the primary equations in Boole's Algebra, but not conversely. What is most notable about the particular propositions is that Boole required a new variable v to express them as equations. (Boole frequently uses the letter v for this, but it could be any letter that has not been used so far in the equations.)

The same equational expressions are used for BC-propositions $\Phi(\mathrm{X}, \mathrm{Y})$, provided not-X is expressed by $1-x$ and not-Y by $1-y$. For example, the equational expressions for No not-X is not-Y are: primary $(1-x)(1-y) = 0$, and secondary $1-x = v(1-(1-y))$, which simplifies to $1 - x = vy$.

---

*The four secondary equations in Tbl. 19 became the primary equations of LT.

†When using the modern semantics of class variables, this can be handled using $\mathbb{Z} \models_{01}$ by putting an existential quantifier on v, namely

$$\mathbb{Z} \models_{01} x = xy \longleftrightarrow (\exists v)(x = vy).$$

An important thing to remember is that each introduction of a v in an algebraic derivation requires a new v, for otherwise one can run into false deductions in the propositional interpretation. For example, if one has the premises All Z is Y and All X is Y, and expresses them as $z = vy$ and $x = vy$, using the same v, then one immediately has $z = x$, and thus X = Z, which is not a valid conclusion.

Another key item is knowing when v can be interpreted as 'Some' since Boole is far from clear on this. If v is introduced in a derivation by any of the expressions $x = vy$, $v = xy$ or $vx = vy$, then one has the interpretations 'Some X' for vx and 'Some Y' for vy. Furthermore, if vx can be read as 'Some X', and one can derive $vx = y$ or $vx = vy$, then vy can be read as 'Some Y'. Actually if one can derive $x = ty$ or, given that vx can be interpreted as 'Some X', $vx = ty$, for any term t, then this equation can be interpreted as 'Some X is Y'. Similar statements hold if we replace y by $1 - y$ and Y by not-Y.*

Although v may have been introduced into a derivation with the interpretation 'Some', algebraic manipulations can lead to results that are false if v is taken to mean 'Some' in the wrong context. For example, Boole permits one to express 'Some X is Y' by $vx = vy$. From this equation one can easily derive $v(1 - x) = v(1 - y)$. If one interprets v as 'Some' in this equation then one has 'Some not-X is not-Y', which is not a valid conclusion from 'Some X is Y'—one does not want to interpret v as 'Some' in this last equation.

When carrying out an equational derivation in Boole's Algebra, it can be useful to make marginal notes whenever an expression like vx or $v(1 - x)$ occurs that can be interpreted as 'Some X', respectively 'Some not-X'.

Syllogisms and the Elimination Theorem. Boole regarded syllogisms as arguments based on eliminating the middle term from the premises. He believed that the elimination theorem (3) on p. xxviii from Common Algebra was the perfect tool to determine which syllogism premises led to valid syllogisms, and if so, to find the most general conclusion.

The first step in applying the elimination theorem is to note that the equational expressions, primary or secondary, for any

*There is one exception to this discussion, namely on p. 43 Boole expresses Some X is not Y by $vy = v(1 - x)$ when it is only the case that v can be interpreted as 'Some' for the terms vx and $v(1 - y)$.

pair $\Phi_1(\mathrm{Z},\mathrm{Y})$, $\Phi_2(\mathrm{Y},\mathrm{X})$ of categorical premises can be put in the form $ay+b=0,\ cy+d=0$, where y does not appear in any of the coefficients a,b,c,d.

Boole's method (for AC-premises) is as follows:

(a) If the premises are both universal

> (a1) use the primary expressions for the premises and apply the elimination theorem from Common Algebra.
>
> If that fails to give an equation that can be interpreted as a categorical proposition, then
>
> (a2) use the secondary expression for one of the premises, and carry out the elimination.
>
> If that fails to give an equation that can be interpreted as a categorical proposition, then the premises do not belong to a valid syllogism.

(b) If one premiss is universal and one is particular, use the primary expression for the universal premiss and the secondary expression for the particular premiss, and apply the elimination theorem.

If that fails to produce an equation that can be interpreted as a categorical proposition, then the premises do not belong to a valid syllogism.

(c) If both premises are particular propositions use the secondary expressions for them. This will always fail to give a conclusion equation that can be interpreted as a categorical proposition.

Examples from MAL.

Here are some examples of Boole's method to analyze syllogisms:

Example *where (a1) succeeds.* The 1st Fig. AA premises on p. v are expressed by the primary equations $y(1-z)=0$ and $x(1-y)=0$. Putting them in the form of the premises of the elimination theorem one has $(1-z)y+0=0$ and $xy-x=0$. Thus $a=1-z$, $b=0$, $c=x$ and $d=-x$. By the elimination theorem one has

$$(1-z)y=0,\ xy-x=0\ \therefore\ x(1-z)=0.$$

The conclusion equation is interpreted as All X is Z, giving the well-known valid 1st Fig. AAA AC-syllogism on p. v.

$\boxed{\text{Example}}$ *where (a1) fails but (a2) suceeds.* Applying the elimination theorem to the 3rd Fig. AA primary premiss equations gives

$$(1-z)y = 0,\ (1-x)y = 0\ \therefore\ 0 = 0.$$

Thus (a1) fails to yield a valid syllogism.

Use the secondary equation for one of the premises, say the second premiss. Then the elimination theorem gives

$$(1-z)y = 0,\ y = vx\ \therefore\ vx(1-z) = 0.$$

Boole interprets this as Some Xs are Zs, showing that the 3rd Fig. AAI AC-syllogism is valid.

However $vx(1-z) = 0$ is not equivalent to either of the equations that Boole gave to express Some X is Z, namely $v = xz$ and $vx = vz$. Another translation rule is needed: if vx can be interpreted as Some X, then $vx(1-z) = 0$ can be interpreted as Some X is Z.

$\boxed{\text{Example}}$ *where (a) shows that the premises do not belong to a valid syllogism.* For the 1st Fig. EE specification one has the primary expressions $zy = 0$ and $xy = 0$. Applying elimination gives 0=0, so (a1) fails.

Using the secondary expression for the second premiss one has $zy = 0$ and $x = v(1-y)$. Applying elimination gives $(v-x)z = 0$. This does not lead to an equation that is interpretable as a categorical proposition.

$\boxed{\text{Example}}$ *where (a) shows that the premises do not belong to a valid syllogism.* Applying the elimination theorem to the 2nd Fig. AA primary premiss equations gives

$$z(1-y) = 0,\ x(1-y) = 0\ \therefore\ zx = zx.$$

The conclusion $zx = zx$ is equivalent to $0 = 0$.

Next, using the secondary equation for the second premiss, the elimination theorem gives

$$z(1-y) = 0,\ x = vy\ \therefore\ xz = vz$$

This does not lead to an equation that is interpretable as a categorical proposition.

For premises that are particular, Boole used only secondary translations.

$\boxed{\text{Example}}$ Eliminating y in the 1st Fig. EI premiss equations gives

$$zy = 0,\ vy = vx\ \therefore\ vxz = 0.$$

This is interpreted as Some X is not Z.

Example Eliminating y in the 1st Fig. II premiss equations gives

$$vy = vz,\ wx = wy\ \therefore\ vwx = vwz.$$

This does not lead to an equation that is interpretable as a categorical proposition.

Secondary expressions suffice. In a long footnote, starting on p. 42, Boole claimed that one could obtain all the valid $\widehat{\mathrm{BC}}$-specifications ith Fig. $\alpha\beta$ by using only the secondary expressions. Boole essentially noted that each AC-proposition $\Phi(\mathrm{X}, \mathrm{Y})$ can be expressed in exactly one of the following two forms (p. 44):

$$ay = bx \quad \text{(affirmative)} \quad \text{or} \quad ay = b(1-x) \quad \text{(negative)},$$

where the coefficients a, b belonged to $\{1, v\}$. This follows from an enumeration of all possible cases (see p. 43):

AXY	$vy = x$	AYX	$y = vx$
EXY	$y = v(1-x)$	EYX	$y = v(1-x)$
IXY	$vy = vx$	IYX	$vy = vx$
OXY	$vy = v(1-x)$	OYX	$vy = v(1-x)$

It is important to note that in each of these expressions the variable v can be interpreted as 'Some' except in the expression for OXY. In this expression one actually has vx interpreted as 'Some X', and $v(1-y)$ as 'Some not-Y'.*

Given two AC-premises $\Phi_1(\mathrm{Y}, \mathrm{Z})$ and $\Phi_2(\mathrm{X}, \mathrm{Y})$, let them be expressed as above by $ay = b\widehat{z}$, and $cy = d\widehat{x}$, where $\widehat{x} = x$ or $1-x$, $\widehat{z} = z$ or $1-z$, and the coefficients $a, b \in \{1, v\}$ and $c, d \in \{1, v'\}$. Then the result of (weakly) eliminating y is $ad\widehat{x} = bc\widehat{z}$, and the coefficients ad and bc are in $\{1, v, v', vv'\}$.

He gave details for only a single example of analyzing a syllogism, namely for the premises AYZ, AXY (see pp. 42,43). The premises are expressed as $y = vz$ and $x = v'y$; the elimination of y yields $x = vv'z$, which interprets as AXZ.

The advantage of this way of expressing the AC-propositions is that as long as one avoids syllogisms with either OZY or OXY in the premisses then the elimination of y immediately gives the correct result, that is, an equation that either immediately interprets into a BC-proposition that is the desired valid conclusion,

*A sketch of an analysis of syllogisms using secondary expressions, but with the improved Elimination Theorem, would be given in Chap. XV of LT. OXY would be expressed by $v(1-y) = vx$; and OZY by $v(1-y) = vz$, restoring the 'Some' interpretation of v.

or it cannot be so interpreted, in which case there is no valid conclusion. Dealing with OZY and OXY require further details.

After Boole's single example of a syllogism he presented five general results on syllogisms, noting in conclusion that "many other general theorems may in like manner be proved."

1st result, p. 43: *A valid* $\widehat{\mathrm{BC}}$*-syllogism with affirmative premises has an affirmative AC-conclusion.* The affirmative premisses $\Phi_1(\mathrm{Y},\mathrm{Z})$ and $\Phi_2(\mathrm{X},\mathrm{Y})$ can be expressed as $ay = bz$ and $cy = dx$. Eliminating y gives $adx = bcz$, which expresses an affirmative AC-conclusion.

2nd result, p. 44: *A valid* $\widehat{\mathrm{BC}}$*-syllogism with an affirmative premise and a negative premise has a negative AC-conclusion.* The premises are expressed by equations $ay = b\widehat{z}$ and $cy = d\widehat{x}$ with $\{\widehat{x},\widehat{z}\} = \{x, 1-z\}$ or $\{\widehat{x},\widehat{z}\} = \{1-x, z\}$. Elimination gives either $adx = bc(1-z)$ or $ad(1-x) = bcz$, and each of these interprets as a negative AC-proposition.

3rd result, p. 44: *A valid* $\widehat{\mathrm{BC}}$*-syllogism with both premises negative has a conclusion with both subject and predicate negated.* The premises are expressed by equations $ay = b(1-z)$ and $cy = d(1-x)$. Elimination gives $ad(1-x) = bc(1-z)$.

4th result, p. 44: *Let* $\Phi_1(\mathrm{Y},\mathrm{Z})$*,* $\Phi_2(\mathrm{X},\mathrm{Y})$ *be the premises of a valid AC-syllogism. If* $\Phi_2(\mathrm{X},\mathrm{Y})$ *is changed in quality (from affirmative to negative, or vice-versa), and perhaps in quantity, one can no longer complete the premises to a valid syllogism.*

5th result, p. 44: *Let* $\Phi_1(\mathrm{Y},\mathrm{Z})$*,* $\Phi_2(\mathrm{X},\mathrm{Y})$ *be the premises of a valid AC-syllogism. If* $\Phi_2(\mathrm{X},\mathrm{Y})$ *is replaced by its contradictory, one can no longer complete the premises to a valid syllogism.*

The four classes of $\widehat{\mathrm{BC}}$ **syllogisms.** Boole said his subdivision of the $\widehat{\mathrm{BC}}$-specifications into four Classes in Tbl. 18 was motivated by his mathematical analysis.

- **1st Class.** Valid $\widehat{\mathrm{BC}}$-specifications ith Fig. $\alpha\beta$ with α and β universal (that is, each is A or E), and whose Aristotelian instance $\Phi_1(\mathrm{Y},\mathrm{Z})$, $\Phi_2(\mathrm{X},\mathrm{Y})$ can be completed by a universal conclusion $\Phi(\mathrm{X},\mathrm{Z})$—all but the 4th Fig. AA are directly determinable by the Aristotelian Rules. The 4th Fig. AA can be completed to an Aristotelian syllogism with a particular conclusion; its conjugate can be completed using a universal BC-proposition. **Boole claimed that all cases in the 1st class can be mathematically analyzed using the primary equational expressions for the premises.** The details of two cases are presented: 1st

Fig. AA, 2nd Fig. AE.

- **2nd Class.** Valid $\widehat{\mathrm{BC}}$-specifications ith Fig. $\alpha\beta$ of two universal premises that can only be completed by a particular conclusion. He subdivided the cases into those that were directly or indirectly determinable by the Aristotelian Rules, and those cases not so determinable. **Boole claimed that all 2nd Class cases can be mathematically analyzed using the primary equational expression for one of the premises, the secondary equational expression for the other.** Details of three cases are presented: 1st Fig. AE, 3rd Fig. AA and 1st Fig. EE.

- **3rd Class.** Valid $\widehat{\mathrm{BC}}$-specifications ith Fig. $\alpha\beta$ of one universal premiss and one particular premiss. Such premises can only be completed by a particular conclusion. **For the mathematical analysis, the universal premiss is expressed by its primary equation, the particular premiss by its secondary equation.** The cases are subdivided into those that are directly or indirectly determinable by the Aristotelian Rules, and those cases not so determinable. The non-valid specifications of one universal premiss and one particular premiss are also listed, with the claim that in such cases one always ends up with the equation $0 = 0$ when eliminating y, using auxiliary equations from p. 25 where necessary. Details of four cases are presented: 1st Fig. AI, 2nd Fig. AO, 1st Fig. AO and 2nd Fig. AI.

- **4th Class.** $\widehat{\mathrm{BC}}$-specifications ith Fig. $\alpha\beta$ for two particular premises, **both of which are expressed by their secondary equations.** These specifications are never valid, but in some cases Boole noted that he was not able to achieve $0 = 0$ by elimination and the use of his auxiliary equations. Details of two cases are presented: 3rd Fig. II and 1st Fig. IO.

Adapting Boole's Translations of Categorical propositions to the Modern Semantics for Class Variables. The existential import of universal categorical propositions in Aristotelian logic created complications for Boole to handle with equational reasoning. Now a quick summary is given of how elegant his approach becomes when one uses the modern semantics of class symbols (introduced by Peirce and reinforced by

Schröder in the last two decades of the 19th century). Tbl. 20 gives translations for the four types of categorical propositions, in both directions (expression and interpretation).

Type	Proposition	Equation
A	All X is Y	$x = xy$
E	No X is Y	$xy = 0$
I	Some X is Y	$v = vxy$
O	Some X is not Y	$v = vx(1-y)$.

Tbl. 20. Equational Expressions of Aristotelian Propositions when using Modern Semantics for Class Variables

Versions of the variable v, like v, v' or v_1, are to be used only when expressing particular propositions. If there are several premiss propositions that are particular, then the equational expressions for them must have distinct v's.

Let $\Phi_1, \ldots, \Phi_k \ \therefore \ \Phi$ be an argument with all propositions categorical, and let $\varepsilon_1, \ldots, \varepsilon_k$ be equational expressions of the premises as per the above. Then the propositional argument is valid iff there is an equation ε expressing Φ that can be derived from the premiss equations using Boole's Algebra, that is, using the laws and valid inferences of the integers along with the index law $x^n = x$; or equivalently, such that $\mathbb{Z} \models_{01} \bigwedge_{i=1}^{k} \varepsilon_i \to \varepsilon$. Unlike Boole's system, there is no need to keep track of when a variable v can be read as Some. Details are presented in the context of the modern Boolean algebra of classes in [13].

Consequences of using Common Algebra. Did Boole find an optimal way of expressing categorical propositions using an algebra that includes Common Algebra, that is, the equational reasoning for the integers $\mathbb{Z}$? It seems unlikely that one can find definitions of the three binary operations $+, \cdot, -$ that are totally defined on classes such that Common Algebra holds, and for which one can find equations to express the categorical propositions such that the validity of propositional arguments can be determined by algebraic means.

If one defines multiplication to correspond to intersection, that is, $xy = z$ iff $\mathrm{X} \cap \mathrm{Y} = \mathrm{Z}$, then one has the index law and one can prove that addition and subtraction with the largest possible domains of definition must be the partial operations used by Boole.* (See [9].) Thus an interesting alternative to Boole's definitions of the three binary operations requires a different definition of multiplication—no such algebra is known.

*Actually one just needs a part of Boole's subtraction, namely one only needs $1 - x$ to be totally defined.

At the time that Boole was writing MAL, it was well-known that hypothetical propositions could be converted into "categorical propositions".* For example, in Whately's book [25] one sees examples of the hypothetical If X then Y changed into examples of The case that X is the case that Y.† Boole had an algebra of logic for categorical propositions, so it was only natural to use it with hypothetical propositions converted into categorical propositions.

He started by defining on p. 49 the **hypothetical universe;** it was denoted by 1 and made up of all *cases and conjunctures of circumstances*, which was usually abbreviated to just *cases*. Given a hypothetical proposition X he defined the **elective operation** x to select all the cases in which the proposition X was true.

Although not clearly stated, Boole's goal was to express hypothetical propositions $\Psi(\vec{\mathrm{X}})$ as elective equations $\psi(\vec{x}) = 1$ such that a hypothetical inference

$$\Phi_1(\vec{\mathrm{X}}), \ldots, \Phi_k(\vec{\mathrm{X}}) \ \therefore \Phi(\vec{\mathrm{X}})$$

was valid iff the corresponding equational inference

$$\phi_1(\vec{x}) = 1, \ldots, \phi_k(\vec{x}) = 1 \ \therefore \phi(\vec{x}) = 1$$

was valid in Boole's Algebra.

Although Boole correctly translates several hypothetical propositions into elective equations, his explanation of why this is correct is not convincing.

Rather than work solely with the hypothetical universe, he introduced a switch from the plural 'cases' to the singular 'case', reading If X then Y as 'the case where X is true is the case where Y is true', and introduced the **universe of a proposition** with two cases, 'the proposition is true' and 'the proposition is false'. Then he presented his method to find $\psi(\vec{x})$ given $\Psi(\vec{\mathrm{X}})$; it looks rather like a fragmentary relative of modern propositional logic with truth tables.‡

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*This undefined notion of categorical proposition is not exactly the same as that stated earlier. It is more like Boole's notion of a *primary* proposition in LT, that is, a proposition about classes.

†In LT (p. 176) Boole credited Wallis's *Institutio Logicae* of 1687 for giving him the idea to use *cases* to convert hypothetical propositions into categorical propositions.

‡Boole abandoned the True/False table approach to propositional logic in LT, replacing it by a time based approach, namely each hypothetical proposition was assigned the (class of) times for which it was true.

On pp. 49, 50 he stated, for various assignments of truth-values to three propositional variables, the associated algebraic expressions. For two variables X, Y one has Tbl. 21. In terms

Cases		Expression
X	Y	
T	T	xy
T	F	$x(1-y)$
F	T	$(1-x)y$
F	F	$(1-x)(1-y)$

Tbl. 21. Algebraic Expressions

of the hypothetical universe this is easily understood as saying that xy expresses the proposition 'X and Y', that is, it selects the cases for which X is true and Y is true. Likewise, $x(1-y)$ expresses the proposition 'X and not-Y', etc.

His general method for finding an algebraic expression $\psi(\vec{x})$ for a propositional formula $\Psi(\vec{\mathrm{X}})$ can be found on p. 52, namely the assertion of $\Psi(\vec{\mathrm{X}})$ says that a disjunction of certain truth-valued assignments to the variables must hold. (Such truth-valued assignments will be said to *belong to* $\Psi(\vec{\mathrm{X}})$.)* For example, the assertion of If X then Y is viewed as saying that if X is assigned T then Y must be assigned T, so the assignments belonging to If X then Y are TT, FT, FF. Boole does not give an algorithm to determine if a truth-valued assignment belongs to a proposition.

The expression $\psi(\vec{x})$ for $\Psi(\vec{\mathrm{X}})$ is just the sum of the expressions associated to the truth-valued assignments belonging to the assertion of $\Psi(\vec{\mathrm{X}})$. Then the assertion $\Psi(\vec{\mathrm{X}})$ is expressed by the equation $\psi(\vec{x}) = 1$. For example, the truth-valued assignments belonging to the proposition If X then Y are the first, third and fourth rows of Tbl. 21, so the expression of If X then Y is

$$\begin{aligned} \psi(x,y) &= xy + (1-x)y + (1-x)(1-y) \\ &= 1 - x(1-y), \end{aligned}$$

Setting this equal to 1 gives the equational expression $1 - x(1 - y) = 1$ for If X then Y, which reduces to $x(1-y) = 0$.

*In modern terminology they are the assignments of truth values to the variables X_i which make $\Psi(\vec{\mathrm{X}})$ true—but apparently Boole did not view $\Psi(\vec{\mathrm{X}})$ as a truth-valued function.

Here are Boole's examples expressing hypothetical propositions by equations (pp. 51–54):*

- 'X is true' is expressed by $x = 1$.
- 'X is false' is expressed by $x = 0$.
- 'X is true and Y is true' is expressed by $xy = 1$.
- 'X is true or Y is true', with 'or' inclusive, is expressed by $xy + x(1 - y) + (1 - x)y = 1$, and thus by $x + y - xy = 1$.
- 'X is true or Y is true', with 'or' exclusive, is expressed by $x(1 - y) + (1 - x)y = 1$, and thus by $x + y - 2xy = 1$.
- 'X is false or Y is false', with 'or' inclusive, is expressed by $x(1-y)+(1-x)y+(1-x)(1-y) = 1$, and thus by $xy = 0$.
- 'X is false or Y is false', with 'or' exclusive, is expressed by $x(1 - y) + (1 - x)y = 1$, and thus by $x + y - 2xy = 1$.
- 'X is true or Y is true or Z is true', with 'or' exclusive, is expressed by $x(1-y)(1-z)+(1-x)y(1-z)+(1-x)(1-y)z = 1$, and thus by $x + y + z - 2(xy + xz + yz) + 3xyz = 1$.
- 'X is true or Y is true or Z is true', with 'or' inclusive, is expressed by $(1 - x)(1 - y)(1 - z) = 0$, and thus by $x + y + z - (xy + xz + yz) + xyz = 0$.
- 'If X is true then Y is true' is expressed by $xy + (1 - x)y + (1 - x)(1 - y) = 1$, and thus by $x(1 - y) = 0$.
- 'If X is true then Y is false' is expressed by $x(1 - y) + (1 - x)y + (1 - x)(1 - y) = 1$, and thus by $xy = 0$.
- 'If X is false then Y is false' is expressed by $xy + x(1 - y) + (1 - x)(1 - y) = 1$, and thus by $(1 - x)y = 0$.

Since Boole includes equations like $x = 0$ and $x = 1$, the categorical logic that is needed evidently uses M-semantics for class variables; this is not the semantics that he needed for his version of Aristotelian categorical logic.

Certain arguments using hypothetical propositions were standard examples of valid **hypothetical syllogisms**—Boole stated them on pp. 56–57, using his reduction of categorical propositions

*It is in this chapter, on p. 53, that one first sees expressions with numerical coefficients other than $0, \pm 1$.

to propositional variables X, Y, Z, W. These are listed below, along with their equational expressions.*

1. **Disjunctive Syllogism** (two versions)

	X or Y (exclusive 'or')		$x+y-2xy=1$
	X		$x=1$
$\therefore$	Not Y	$\therefore$	$y=0.$

	X or Y (inclusive 'or')		$x+y-xy=1$
	Not X		$x=0$
$\therefore$	Y	$\therefore$	$y=1.$

2. **Constructive Conditional Syllogism**

	If X then Y		$x(1-y)=0$
	X		$x=1$
$\therefore$	Y	$\therefore$	$y=1.$

3. **Destructive Conditional Syllogism**

	If X then Y		$x(1-y)=0$
	Not Y		$y=0$
$\therefore$	Not X	$\therefore$	$x=0.$

4. **Simple Constructive Dilemma**

	If X then Y		$x(1-y)=0$
	If Z then Y		$z(1-y)=0$
	X or Z (exclusive 'or')		$x+z-2xz=1$
$\therefore$	Y	$\therefore$	$y=1.$

5. **Complex Constructive Dilemma**

	If X then Y		$x(1-y)=0$
	If W then Z		$w(1-z)=0$
	X or W (inclusive 'or')		$x+w-xw=1$
$\therefore$	Y or Z (inclusive 'or')	$\therefore$	$y+z-yz=1.$

6. **Complex Destructive Dilemma** (two versions)

	If X then Y		$x(1-y)=0$
	If W then Z		$w(1-z)=0$
	Not Y or not Z (exclusive 'or')		$y+z-2yz=1$
$\therefore$	Not X or not W (inclusive 'or')	$\therefore$	$xw=0.$

*Boole noted that these 'hypothetical syllogisms' did not have the form of a syllogism, namely two premises with three variables, with one variable common to the two premises.

To convert Boole's versions of the hypothetical syllogisms back to the forms based on categorical propositions that Whately would have used, replace X by 'A is B', Y by 'C is D', Z by 'E is F', and W by 'G is H'.

	If X then Y		$x(1-y)=0$
	If W then Z		$w(1-z)=0$
	Not Y or not Z (inclusive 'or')		$yz=0$
$\therefore$	Not X or not W (inclusive 'or')	$\therefore$	$xw=0.$

The equational arguments given for these hypothetical syllogisms are easily seen to be valid in Boole's Algebra.*

Boole noted on p. 57 that one could easily extend the list of hypothetical syllogisms by using propositions that were blends of the conditional and disjunctive forms such as "If X is true, then either Y is true, or Z is true". But he did not seem to have had the general idea of a propositional formula, that is, a Boolean combination of propositional variables.

The hypothetical propositions $\Psi(\vec{\mathrm{X}})$ that Boole worked with were quite simple, and the assertion of such could easily be viewed as asserting that certain assignments of truth-values to the propositional variables must hold. There is no indication that an assignment of truth-values to the propositional variables implied an assignment of a truth value to $\Psi(\vec{\mathrm{X}})$, much less a recursive procedure for assigning truth-values to arbitrary propositional formulas. Otherwise Boole might have been credited with introducing truth-tables into propositional logic.

At the end of the chapter on hypotheticals (p. 59) Boole claimed that "Every Proposition which language can express may be represented by elective symbols . . . ". However the examples he gave in MAL stayed close to the standard simple categorical and hypothetical propositions of the time. LT would have more complex examples.

A Rigorous Version. Now a version of Boole's treatment of hypotheticals will be given that meets modern standards. In the context of hypothetical propositions, 1 will be used to denote True, and 0 to denote False.

A propositional formula $\Psi(\vec{\mathrm{X}})$ can be viewed as a mapping (defined in the usual recursive fashion) on $\{0,1\}$ so that given a list $\sigma = \sigma_1, \ldots, \sigma_m$ of truth-values, $\Psi(\sigma)$ is a truth-value. Define a hypothetical inference

$$\Phi_1(\vec{\mathrm{X}}), \ldots, \Phi_k(\vec{\mathrm{X}}) \;\therefore\; \Phi(\vec{\mathrm{X}}) \tag{5}$$

*It seems Boole had discovered the switching functions of ordinary algebra—the $\psi(\vec{x})$ are idempotent when the variables are restricted to $\{0,1\}$. These would be used, credited to Boole, in the late 1940s by Howard Aiken [1] at the Harvard Computing Laboratory.

to be valid iff any assignment σ of truth values to the X_i that makes the premisses true also makes the conclusion true, that is

$$\Phi_1(\sigma) = 1, \ldots, \Phi_k(\sigma) = 1 \ \therefore \Phi(\sigma) = 1 \tag{6}$$

is valid for all σ.

For any $\Psi(\vec{\mathrm{X}})$ **define** $\psi(\vec{x})$ to be the sum of the constituents $\mathrm{C}_\sigma(\vec{x})$ (see p. lv) such that $\Psi(\sigma) = 1$. Then for all σ,

$$\Psi(\sigma) = 1 \ \text{ iff } \ \psi(\sigma) = 1. \tag{7}$$

By (6) and (7) the validity of inference (5) is equivalent to saying that

$$\phi_1(\sigma) = 1, \ldots, \phi_k(\sigma) = 1 \ \therefore \phi(\sigma) = 1 \tag{8}$$

is valid for all σ. By R01 this is equivalent to the inference

$$\phi_1(\vec{x}) = 1, \ldots, \phi_k(\vec{x}) = 1 \ \therefore \phi(\vec{x}) = 1 \tag{9}$$

being valid in Boole's Algebra.

THE RULE OF 0 AND 1

This is a good place to present a detailed *equational proof system* for Boole's Algebra, and use it to prove the Rule of 0 and 1.[*] There are three binary operation symbols $(+, \cdot, -)$ and two constants $(0, 1)$.

Definition 1 *A* derivation of an equation ε from equations $\varepsilon_1, \ldots, \varepsilon_k$ *is a list $\delta_1, \ldots, \delta_t$ of equations such that δ_t is ε, and each δ_i satisfies one of the following:*

0. δ_i is one of the premises $\varepsilon_1, \ldots, \varepsilon_k$;

1. δ_i is a law of Common Algebra (a law of $\mathbb{Z}$);

2. δ_i is an instance of the index law ($x^n = x$);

3. there are $i_1, \ldots, i_j$, all less than i, such that $\delta_{i_1}, \ldots, \delta_{i_j} \therefore \delta_i$ is a valid argument in Common Algebra (valid for $\mathbb{Z}$).

One has a derivation of an equation ε *if the above holds with condition 0 removed (so there are no premises to be considered).*

The notation $\varepsilon_1, \ldots, \varepsilon_k \vdash \varepsilon$ means that there is a derivation of ε from $\varepsilon_1, \ldots, \varepsilon_k$. An equational argument $\varepsilon_1, \ldots, \varepsilon_k \therefore \varepsilon$ is said to be valid *in Boole's Algebra if there is a derivation of the conclusion from the premises.*

The notation $\vdash \varepsilon$ means that there is a derivation of ε. In this case one says ε is a law *of Boole's Algebra.*

In the following let $\vec{x}$ be the list $x_1, \ldots, x_m$ of variables, let $\vec{y}$ be the list $y_1, \ldots, y_n$ of variables, and for j a positive integer let S_j be the set of sequences σ of 0s and 1s of length j.

Lemma 2 *If*

$$(\star) \quad \varepsilon_1(\vec{x}), \ldots, \varepsilon_k(\vec{x}) \;\vdash\; \varepsilon(\vec{x})$$

---

[*]One can reduce the equational axioms in item 1 to the collection of all substitution instances of the usual axioms for commutative rings with unity, but with $t + (-t) = 0$ replaced by $t - t = 0$ and $t + (0 - t) = 0$ for t a term. One can replace $x^n = x$ in item 2 by $x^2 = x$ and the inference rules of item 3 by the familiar reflexive, symmetric and transitive rules, the replacement rule, and the rule $nt = 0 \therefore t = 0$, for t a term. Note that unlike the standard Birkhoff equational logic, substitution is not a valid rule of inference. However the laws and valid arguments for $\mathbb{Z}$ are closed under substitution—only the index law makes problems for the substitution rule.

then there is a derivation $\delta_1(\vec{x}),\ldots,\delta_t(\vec{x})$ *of* $(\star)$ *that does not use any variables besides those attached to the premisses and conclusion.*

PROOF: Let $\delta_1(\vec{x},\vec{y}),\ldots,\delta_t(\vec{x},\vec{y})$ be a derivation for $(\star)$. Let τ be the sequence of 0s in S_n. The following proves that $\delta_1(\vec{x},\tau),\ldots,\delta_t(\vec{x},\tau)$ is also a derivation of $(\star)$:

0. If $\delta_i(\vec{x},\vec{y})$ is one of the premises $\varepsilon_1(\vec{x}),\ldots,\varepsilon_k(\vec{x})$ then so is $\delta_i(\vec{x},\tau)$.
1. If $\delta_i(\vec{x},\vec{y})$ is a law of Common Algebra (a law of $\mathbb{Z}$), then so is $\delta_i(\vec{x},\tau)$.
2. If $\delta_i(\vec{x},\vec{y})$ is ${x_j}^n = x_j$, for some j and n, then so is $\delta_i(\vec{x},\tau)$.

   If $\delta_i(\vec{x},\vec{y})$ is ${y_j}^n = y_j$, for some j and n, then $\delta_i(\vec{x},\tau)$ is $0^n = 0$, a law of $\mathbb{Z}$.
3. If there are $i_1,\ldots,i_j$, all less than i, such that $\delta_{i_1}(\vec{x},\vec{y}),\ldots,\delta_{i_j}(\vec{x},\vec{y}) \therefore \delta_i(\vec{x},\vec{y})$ is a valid argument in Common Algebra (valid for $\mathbb{Z}$), then so is $\delta_{i_1}(\vec{x},\tau),\ldots,\delta_{i_j}(\vec{x},\tau) \therefore \delta_i(\vec{x},\tau)$.

Lemma 3 *(*SUBSTITUTION*) If*

$$(\star)\ \varepsilon_1(\vec{x},\vec{y}),\ldots,\varepsilon_k(\vec{x},\vec{y})\ \vdash\ \varepsilon(\vec{x},\vec{y})$$

then for $\sigma \in \mathrm{S}_m$ *one has*

$$(\star\star)\ \varepsilon_1(\sigma,\vec{y}),\ldots,\varepsilon_k(\sigma,\vec{y})\ \vdash\ \varepsilon(\sigma,\vec{y})$$

PROOF: Given a derivation $\delta_1(\vec{x},\vec{y}),\ldots,\delta_t(\vec{x},\vec{y})$ for $(\star)$, one can (as in the previous lemma) readily verify that $\delta_1(\sigma,\vec{y}),\ldots,\delta_t(\sigma,\vec{y})$ is a derivation for $(\star\star)$.

Lemma 4 *For ground equations* $\varepsilon_1,\ldots,\varepsilon_k,\varepsilon$*,*

$$\varepsilon_1,\ldots,\varepsilon_k \vdash \varepsilon \quad \textit{iff} \quad \mathbb{Z} \models \Big(\bigwedge_i \varepsilon_i\Big) \to \varepsilon \quad \textit{iff} \quad \mathbb{Z} \models_{01} \Big(\bigwedge_i \varepsilon_i\Big) \to \varepsilon.$$

PROOF: There are no variables in ground equations, thus there is no difference between $\mathbb{Z} \models$ and $\mathbb{Z} \models_{01}$. This means the second "iff" holds.

Suppose $\varepsilon_1,\ldots,\varepsilon_k \ \vdash\ \varepsilon$. In view of Lemma 2 there is a ground derivation $\delta_1,\ldots,\delta_t$. The proof that $\mathbb{Z}\ \models\ \left(\bigwedge_i \varepsilon_i\right) \to \varepsilon$ follows from proving by induction on s that $\mathbb{Z}\ \models\ \left(\bigwedge_i \varepsilon_i\right) \to \delta_s$.

One has δ_1 being either a law of $\mathbb{Z}$ or one of the ε_i. Thus $\mathbb{Z} \models \left(\bigwedge_i \varepsilon_i\right) \rightarrow \delta_1$.

For the induction hypothesis suppose $\mathbb{Z} \models \left(\bigwedge_i \varepsilon_i\right) \rightarrow \delta_r$ for $1 \leq r < s \leq t$. The goal is to prove $\mathbb{Z} \models \left(\bigwedge_i \varepsilon_i\right) \rightarrow \delta_s$. If δ_s is one of the ε_i or a law of $\mathbb{Z}$, then $\mathbb{Z} \models \delta_s$. The only other possibility is that there are $s_1, \ldots, s_j$, all less than s, such that $\delta_{s_1}, \ldots, \delta_{s_j} \therefore \delta_s$ is valid in $\mathbb{Z}$. By the induction hypothesis it follows that $\mathbb{Z} \models \left(\bigwedge_i \varepsilon_i\right) \rightarrow \delta_s$.

For the converse assume that $\mathbb{Z} \models \left(\bigwedge_i \varepsilon_i\right) \rightarrow \varepsilon$. This says that $\varepsilon_1, \ldots, \varepsilon_k \therefore \varepsilon$ is a valid inference for $\mathbb{Z}$, thus it follows that $\varepsilon_1, \ldots, \varepsilon_k, \varepsilon$ is a derivation of ε from $\varepsilon_1, \ldots, \varepsilon_k$. This means that $\varepsilon_1, \ldots, \varepsilon_k \vdash \varepsilon$.

Definition 5 *A term p is* idempotent *if* $\vdash p^2 = p$.

Lemma 6 *x and $1-x$ are idempotent. A product of idempotent terms is again idempotent.*

Proof: Clearly $\vdash x^2 = x$. The following gives a derivation for $\vdash (1-x)^2 = 1-x$:

1.	$(1-x)^2 = 1-2x+x^2$	Common Algebra
2.	$x^2 = x$	index law
3.	$(1-x)^2 = 1-2x+x$	$1,2 \therefore 3$ Common Algebra
4.	$1-2x+x = 1-x$	Common Algebra
5.	$(1-x)^2 = 1-x$	$3,4 \therefore 5$ Common Algebra.

Definition 7 *Let* $\mathrm{C}_1(x) = x$ *and* $\mathrm{C}_0(x) = 1-x$. *For* $\sigma \in \mathrm{S}_m$ *define the* constituent $\mathrm{C}_\sigma(\vec{x})$ *of* $\vec{x}$ *by*

$$\mathrm{C}_\sigma(\vec{x}) \;=\; \prod_i \mathrm{C}_{\sigma(i)}(x_i).$$

The key properties of constituents are in the next lemma.

Lemma 8 *For σ and τ in S_m,*

1. $\vdash \mathrm{C}_\sigma(\vec{x})^2 = \mathrm{C}_\sigma(\vec{x})$
2. $\vdash \mathrm{C}_\sigma(\vec{x})\mathrm{C}_\tau(\vec{x}) = 0$ *if* $\sigma \neq \tau$
3. $\vdash \sum_\sigma \mathrm{C}_\sigma(\vec{x}) \;=\; 1$
4. $\vdash \mathrm{C}_\sigma(\sigma) = 1$, *and for* $\sigma \neq \tau$, $\vdash \mathrm{C}_\sigma(\tau) = 0$.

PROOF:
(1) By Lemma 6, $\mathrm{C}_\sigma(\vec{x})$ is a product of idempotent terms, hence it is also idempotent.
(2) If $\sigma \neq \tau$ then for some i one has $\sigma(i) \neq \tau(i)$, and thus $\vdash\ \mathrm{C}_{\sigma(i)}(x_i) \cdot \mathrm{C}_{\tau(i)}(x_i) = x_i - x_i^2$. Then $\vdash\ \mathrm{C}_{\sigma(i)}(x_i) \cdot \mathrm{C}_{\tau(i)}(x_i) = 0$, which leads to $\vdash\ \mathrm{C}_\sigma(\vec{x})\mathrm{C}_\tau(\vec{x}) = 0$.
(3) Starting with $\vdash\ \prod_i \big(\mathrm{C}_0(x_i) + \mathrm{C}_1(x_i)\big) = 1$, multiplying out gives the desired result.
(4) This follows from the definition of $\mathrm{C}_i(j)$ for $i, j \in \{0,1\}$.

Theorem 9 (EXPANSION) *For any term $p(\vec{x},\vec{y})$, the expansion of the term about $\vec{x}$ is give by*

$$\vdash\ p(\vec{x},\vec{y}) = \sum_\sigma p(\sigma,\vec{y})\mathrm{C}_\sigma(\vec{x}).$$

The complete *expansion of a term $p(\vec{x})$ is given by*

$$\vdash\ p(\vec{x}) = \sum_\sigma p(\sigma)\mathrm{C}_\sigma(\vec{x}).$$

PROOF: By induction on the length m of $\vec{x}$. For $m = 1$ first use Common Algebra to write $p(x,\vec{y})$ as a polynomial in x:

$$\vdash\ p(x,\vec{y}) \ = \ a(\vec{y}) + b_1(\vec{y})x + \cdots + b_n(\vec{y})x^n.$$

Next use the index law to obtain

$$\vdash\ p(x,\vec{y}) \ = \ a(\vec{y}) + b_1(\vec{y})x + \cdots + b_n(\vec{y})x.$$

Setting $b(\vec{y})$ equal to $b_1(\vec{y}) + \cdots + b_n(\vec{y})$ one has, by Common Algebra,

$$\vdash\ p(x,\vec{y}) \ = \ a(\vec{y}) + b(\vec{y})x.$$

Using the Substitution Lemma 3 yields

$$\begin{aligned} \vdash\ p(0,\vec{y}) &= a(\vec{y}) \\ \vdash\ p(1,\vec{y}) &= a(\vec{y}) + b(\vec{y}), \end{aligned}$$

and thus, by Common Algebra

$$\begin{aligned} \vdash\ a(\vec{y}) &= p(0,\vec{y}) \\ \vdash\ b(\vec{y}) &= p(1,\vec{y}) - p(0,\vec{y}), \end{aligned}$$

which by Common Algebra leads to

$$\begin{aligned} \vdash\ p(x,\vec{y}) &= p(0,\vec{y}) + \big(p(1,\vec{y}) - p(0,\vec{y})\big)x \\ &= p(1,\vec{y})x + p(0,\vec{y})(1-x) \\ &= p(1,\vec{y})\mathrm{C}_1(x) + p(0,\vec{y})\mathrm{C}_0(x). \end{aligned}$$

For the induction step, assume the expansion theorem holds for $\vec{x}$ of length m. Then by the induction hypothesis and the ground step, for $\sigma \in \mathrm{S}_m$

$$
\begin{aligned}
\vdash\ p(\vec{x}, x_{m+1}, \vec{y}) &= \sum_\sigma p(\sigma, x_{m+1}, \vec{y})\mathrm{C}_\sigma(\vec{x}) \\
&= \sum_\sigma \Big(p(\sigma, 1, \vec{y})x_{m+1} + p(\sigma, 0, \vec{y})(1 - x_{m+1})\Big)\mathrm{C}_\sigma(\vec{x}) \\
&= \sum_\sigma p(\sigma, 1, \vec{y})\mathrm{C}_\sigma(\vec{x})x_{m+1} + \sum_\sigma p(\sigma, 0, \vec{y})\mathrm{C}_\sigma(\vec{x})(1 - x_{m+1}) \\
&= \sum_\tau p(\tau, \vec{y})\mathrm{C}_\tau(\vec{x}, x_{m+1}),
\end{aligned}
$$

where $\tau \in \mathrm{S}_{m+1}$.

Corollary 10

$$
\begin{aligned}
\Big\{p(\sigma)\mathrm{C}_\sigma(\vec{x}) = 0\ :\ \sigma \in \mathrm{S}_m\Big\} &\ \vdash\ p(\vec{x}) = 0 \\
p(\vec{x}) = 0 &\ \vdash\ p(\sigma)\mathrm{C}_\sigma(\vec{x}) = 0 \quad \textit{for } \sigma \in \mathrm{S}_m.
\end{aligned}
$$

PROOF: The complete expansion of $p(\vec{x})$ easily yields the first result. For the second, by the same expansion one has

$$p(\vec{x}) = 0\ \vdash\ \Big(\sum_\tau p(\tau)\mathrm{C}_\tau(\vec{x})\Big) = 0.$$

Multiplying both sides of the conclusion of the last assertion by $\mathrm{C}_\sigma(\vec{x})$, where $\sigma \in \mathrm{S}_m$, one has, by properties of the constituents of $\vec{x}$,

$$\Big(\sum_\tau p(\tau)\mathrm{C}_\tau(\vec{x})\Big) = 0\ \vdash\ p(\sigma)\mathrm{C}_\sigma(\vec{x}) = 0,$$

and thus $p(\vec{x}) = 0\ \vdash\ p(\sigma)\mathrm{C}_\sigma(\vec{x}) = 0$ for $\sigma \in \mathrm{S}_m$.

Theorem 11 (REDUCTION) *Given terms $p_1, \ldots, p_k$,*

$$
\begin{aligned}
p_1 = 0, \ldots, p_k = 0 &\ \vdash\ \Big(\sum_i {p_i}^2\Big) = 0 \\
\Big(\sum_i {p_i}^2\Big) = 0 &\ \vdash\ p_j = 0 \quad \textit{for } 1 \le j \le k.
\end{aligned}
$$

PROOF: From

$$\mathbb{Z}\ \models\ \Big(\bigwedge_i\ p_i = 0\Big) \leftrightarrow \Big(\sum_i {p_i}^2\Big) = 0$$

Lemma 4 gives the desired conclusions.

Theorem 12 (RULE OF 0 AND 1)

$$\varepsilon_1(\vec{x}), \ldots, \varepsilon_k(\vec{x}) \ \vdash \ \varepsilon(\vec{x}) \quad \textit{iff} \quad \mathbb{Z} \ \models_{01} \ \Big(\bigwedge_i \varepsilon_i(\vec{x})\Big) \rightarrow \varepsilon(\vec{x}).$$

PROOF: Use Common Algebra to write the $\varepsilon_i(\vec{x})$ as $p_i(\vec{x}) = 0$, and $\varepsilon(\vec{x})$ as $q(\vec{x}) = 0$. Let $p(\vec{x}) = \sum_i p_i(\vec{x})^2$. Then by the Reduction Theorem it suffices to prove

$$p(\vec{x}) = 0 \ \vdash \ q(\vec{x}) = 0 \quad \text{iff} \quad \mathbb{Z} \ \models_{01} \ p(\vec{x}) = 0 \rightarrow q(\vec{x}) = 0.$$

Suppose $p(\vec{x}) = 0 \vdash q(\vec{x}) = 0$. Then by the Substitution Lemma one has for $\sigma \in \mathrm{S}_m$, $p(\sigma) = 0 \vdash q(\sigma) = 0$. By Lemma 4 one has, for $\sigma \in \mathrm{S}_m$, $\mathbb{Z} \models p(\sigma) = 0 \rightarrow q(\sigma) = 0$, that is, $\mathbb{Z} \models_{01} p(\vec{x}) = 0 \rightarrow q(\vec{x}) = 0$.

For the converse suppose $\mathbb{Z} \models_{01} p(\vec{x}) = 0 \rightarrow q(\vec{x}) = 0$. Then for any $\tau \in \mathrm{S}_m$, $\mathbb{Z} \models p(\tau) = 0 \rightarrow q(\tau) = 0$, and from this it follows that

$$\mathbb{Z} \models \Big(\bigwedge_\tau \ p(\tau)\mathrm{C}_\tau(\vec{x}) = 0\Big) \rightarrow \Big(\bigwedge_\tau \ q(\tau)\mathrm{C}_\tau(\vec{x}) = 0\Big)$$

and thus, for $\sigma \in \mathrm{S}_m$,

$$\Big\{p(\tau)\mathrm{C}_\tau(\vec{x}) = 0 \ : \ \tau \in \mathrm{S}_m\Big\} \ \vdash \ q(\sigma)\mathrm{C}_\sigma(\vec{x}) = 0.$$

By Corollary 10, for $\tau \in \mathrm{S}_m$,

$$p(\vec{x}) = 0 \ \vdash \ p(\tau)\mathrm{C}_\tau(\vec{x}) = 0,$$

and thus, for $\sigma \in \mathrm{S}_m$,

$$p(\vec{x}) = 0 \ \vdash \ q(\sigma)\mathrm{C}_\sigma(\vec{x}) = 0.$$

Again by the Expansion Theorem,

$$\Big\{q(\sigma)\mathrm{C}_\sigma(\vec{x}) = 0 \ : \ \sigma \in \mathrm{S}_m\Big\} \ \vdash \ q(\vec{x}) = 0,$$

and thus $p(\vec{x}) = 0 \ \vdash \ q(\vec{x}) = 0$.

Boole's definition of an elective function on p. 16 is simply an expression involving elective symbols. It surely includes what are here called terms, and likely the quotients of terms—in Common Algebra such quotients can be written as quotients of polynomials, and these are called rational functions. Boole made a modest attempt to include the division operation in his algebra of logic in MAL, but in LT he gave up on the possibility of doing this in general, with the only application of division being in his solution theorem where formal division, followed by formal expansion, gave a useful mnemonic. Consequently, except for the discussion of the solution theorem, the comments on elective functions will be restricted to terms.

There is an important difference between Boole's notation for an elective function and modern terminology, namely when he writes $\phi(\vec{x})$ for a term, there is the possibility that some variables of ϕ do not appear in the list $\vec{x}$. Writing a term p as $p(\vec{x})$ in modern notation means that all the variables of p appear in the list $\vec{x}$, and perhaps other variables. Terms in Boole's notation will be expressed using Greek letters, like $\phi(\vec{x})$, whereas in the comments latin letters like $p(\vec{x})$ will be used with the understanding that this is modern notation.

The Expansion Theorem. The chapter PROPERTIES OF ELECTIVE FUNCTIONS is the first of two chapters on general results for Boole's Algebra. It starts off on p. 60 with a power series expansion of an elective function,

$$\phi(x) \;=\; \sum_n (1/n!)\, \phi^{(n)}(0)\, x^n,$$

from which he proved the **Expansion Theorem** in one variable:

$$\phi(x) \;=\; \phi(1)x + \phi(0)(1-x).$$

Boole's reason for bringing in power series is not known; perhaps it was in the hopes of extending his results to include rational functions, to justify his work with division. When working with terms the power series expansion just gives a polynomial; in such cases perhaps Boole viewed the power series expansion as a convenient way to describe the polynomial that is equivalent to $\phi(x)$.

Boole's above expansion theorem in modern notation is

$$\vdash\; p(x,\vec{y}) \;=\; p(1,\vec{y})x + p(0,\vec{y})(1-x), \qquad (10)$$

where $p(x,\vec{y})$ is a term. His proof started with expanding $p(x,\vec{y})$ as a polynomial in x, say

$$\vdash\ p(x,\vec{y}) \ = \ p_0(\vec{y}) + p_1(\vec{y})\cdot x + \cdots + p_n(\vec{y})\cdot x^n.$$

This step belongs to Common Algebra; however Boole preferred to make the coefficients explicit by bringing in the formula for a power series expansion. Then he used the index law to obtain

$$\vdash\ p(x,\vec{y}) \ = \ p_0(\vec{y}) + \big[p_1(\vec{y}) \ + \cdots + p_n(\vec{y})\big]\cdot x.$$

Thus one has, for suitable terms $a(\vec{y})$ and $b(\vec{y})$,

$$\vdash\ p(x,\vec{y}) \ = \ a(\vec{y}) + b(\vec{y})\cdot x.$$

Boole's next step is to substitute the idempotents 1 then 0 into this equation to obtain the following, giving (10):

$$\begin{aligned}\vdash\ p(1,\vec{y}) \ &= \ a(\vec{y}) + b(\vec{y})\\ \vdash\ p(0,\vec{y}) \ &= \ a(\vec{y}).\end{aligned}$$

The expansion (10) about one variable is easily proved using R01, as is the general **Expansion Theorem**

$$\vdash\ p(\vec{x},\vec{y}) \ = \ \sum_\sigma p(\sigma,\vec{y})\mathrm{C}_\sigma(\vec{x}), \tag{11}$$

Boole used the less expressive notation (see p. 58)

$$a_1t_1 + a_2t_2 \cdots + a_rt_r$$

for the expansion, where a_i is the coefficient of the constituent t_i.

for $\vec{x}$ the list of variables $x_1,\ldots,x_m$, where σ runs over all sequences of 0s and 1s of length m, and $\mathrm{C}_\sigma(\vec{x})$ is defined by

$$\mathrm{C}_\sigma(\vec{x}) \ = \ \prod_i \mathrm{C}_{\sigma(i)}(x_i)$$

where $\mathrm{C}_1(x_i) = x_i$, $\mathrm{C}_0(x_i) = 1 - x_i$.

Equation (11) gives the expansion of $p(\vec{x},\vec{y})$ about $\vec{x}$. If there is no $\vec{y}$, so that all the variables in p are included in $\vec{x}$, then one has a *complete* expansion

$$\vdash\ p(\vec{x}) \ = \ \sum_\sigma p(\sigma)\mathrm{C}_\sigma(\vec{x}), \tag{12}$$

and the coefficients are integers.

Moduli

Constituents

The $p(\sigma,\vec{y})$ are the *moduli* of $p(\vec{x},\vec{y})$ with respect to the variables $\vec{x}$, and in MAL the $\mathrm{C}_\sigma(\vec{x})$ are the *constituents* of $p(\vec{x})$; it would be more appropriate to call the $\mathrm{C}_\sigma(\vec{x})$ the constituents of the list of variables $\vec{x}$.

For example letting $p(x,y)$ be $x+y$ one has the expansion

$$\vdash\ x+y \ = \ (1+y)x + y(1-x)$$

about x, and the complete expansion

$$\vdash\ x+y = 2xy + x(1-y) + (1-x)y.$$

Regarding constituents, the following hold in Boole's Algebra (pp. 63, 64):

Fundamental laws governing constituents

(a) $\vdash\ \mathrm{C}_\sigma(\vec{x})^n = \mathrm{C}_\sigma(\vec{x})$,

(b) $\vdash\ \mathrm{C}_\sigma(\vec{x})\,\mathrm{C}_\tau(\vec{x}) = 0$ if $\sigma \neq \tau$,

(c) $\vdash\ 1\ =\ \sum_\sigma \mathrm{C}_\sigma(\vec{x})$.

One needs to add the obvious laws

(d) $\vdash\ \mathrm{C}_\sigma(\tau) = \begin{cases} 1 & \text{if } \sigma = \tau \\ 0 & \text{otherwise,} \end{cases}$
where τ is also a sequence of 0s and 1s of length m.

Using the expansion theorem, Boole quickly arrived at a number of results:*

Boole claimed that items (P1)–(P5) are consequences of the expansion theorem

(P1) (Prop. 1, p. 61) $\vdash\ p(\vec{x},\vec{y}) = q(\vec{x},\vec{y})$ holds in Boole's Algebra iff the corresponding moduli with respect to $\vec{x}$ are equal, that is, for all σ, $\vdash\ p(\sigma,\vec{y}) = q(\sigma,\vec{y})$. When the expansion is complete, that is, there is no $\vec{y}$, (P1) says that $\vdash\ p(\vec{x}) = q(\vec{x})$ iff $\mathbb{Z} \models_{01} p(\vec{x}) = q(\vec{x})$. Thus Boole had R01 for laws in MAL.

Two corollaries are given:
$\vdash\ p(\vec{x},\vec{y})^n = p(\vec{x},\vec{y})\ \leftrightarrow\ \bigwedge_\sigma \big(p(\sigma,\vec{y})^n = p(\sigma,\vec{y})\big)$
$\vdash\ p(\vec{x},\vec{y})q(\vec{x},\vec{y}) = r(\vec{x},\vec{y})\ \leftrightarrow\ \bigwedge_\sigma \big(p(\sigma,\vec{y})q(\sigma,\vec{y}) = r(\sigma,\vec{y})\big)$.

(P2) (Prop. 2, p. 64) $p(\vec{x}) = 0$ is equivalent to the collection of *constituent equations* $\mathrm{C}_\sigma(\vec{x}) = 0$ where $p(\sigma) \neq 0$. Thus *every* equation is equivalent to a collection of totally interpretable equations, since every constituent equation $\mathrm{C}_\sigma(\vec{x}) = 0$ is totally interpretable.†

(P3) (Prop. 3, p. 66) An equation $w = p(\vec{x})$ is equivalent to the collection of equations $w = \sum\{\mathrm{C}_\sigma(\vec{x}) : p(\sigma) = 1\}$ and $\mathrm{C}_\sigma(\vec{x}) = 0$ whenever $p(\sigma) \notin \{0, 1\}$.

(P4) (Prop. 4, p. 67) Given $p(\vec{x},\vec{y},z)$ and arbitrary terms a_σ,

$$\vdash\ p\Big(\vec{x},\vec{y},\sum_\sigma a_\sigma \mathrm{C}_\sigma(\vec{x})\Big)\ =\ \sum_\sigma p(\vec{x},\vec{y},a_\sigma)\mathrm{C}_\sigma(\vec{x}).$$

*For item (P5) one might prefer that MAL be developed using the algebra of the rationals $\mathbb{Q}$.

†Given a term $p(\vec{x})$, define the *idempotent reduct* $p^\star(\vec{x})$ of $p(\vec{x})$ to be $\sum\{\mathrm{C}_\sigma(\vec{x}) : p(\sigma) \neq 0\}$. Then $p^\star(\vec{x})$ is idempotent and totally interpretable. By P2, $\vdash\ p(\vec{x}) = 0 \leftrightarrow p^\star(\vec{x}) = 0$; hence every equation is equivalent to a totally interpretable equation. (On p. 65 the preferred equivalent form of $p = 0$ is $p^\star = 0$.)

(P5) (Prop. 5, p. 67) Given propositions Φ and Ψ, Boole's Prop. 5 says that if Φ implies Ψ then Ψ is either "equivalent" to Φ or a "limitation" of Φ. His proof is as follows: let Φ and Ψ be expressed by equations $p_\Phi(\vec{x}) = 0$ and $p_\Psi(\vec{x}, \vec{y}) = 0$. He seemed to say that there is $q(\vec{x}, \vec{y})$ such that $p_\Psi(\vec{x}, \vec{y}) = q(\vec{x}, \vec{y}) \cdot p_\Phi(\vec{x})$. Then $p_\Phi(\sigma) = 0$ implies $p_\Psi(\sigma, \vec{y}) = 0$, from which Boole concluded that Ψ is a limitation of Φ.

In view of what looks like an important link to LT, consider the special case of P5 when Ψ is expressed by $p_\Psi(\vec{x}) = 0$. Then the following statements are equivalent:

(a) $\Phi \therefore \Psi$ is valid in propositional logic.

(b) $p_\Phi(\vec{x}) = 0 \therefore p_\Psi(\vec{x}) = 0$ is valid in Boole's Algebra.

(c) $p_\Psi(\vec{x}) = q(\vec{x}) \cdot p_\Phi(\vec{x})$, for some $q(\vec{x})$ with rational coefficients.

(d) $p_\Phi(\sigma) = 0$ implies $p_\Psi(\sigma) = 0$ for all σ.

(e) $\mathbb{Z} \models_{01} \; p_\Phi(\vec{x}) = 0 \;\rightarrow\; p_\Psi(\vec{x}) = 0$.

The equivalence of these statements can be proved as follows. A key tenet of Boole's logic is that (a) and (b) are equivalent. The fact that (b) implies (c) is in Boole's proof of Prop. 5, p. 67. This step evidently requires that one be permitted to choose $q(\vec{x}) \in \mathbb{Q}[x]$ (for example, to deduce $x = 0$ from $2x = 0$).

(c) implies (d) is clear, as is the equivalence of (d) and (e). The expansion theorem and (P4) above show that (d) implies (b).

Then from (a) one has (d), consequently (a) implies

$$\big\{\mathrm{C}_\sigma(\vec{x}) = 0 : p_\Psi(\sigma) \neq 0\big\} \;\subseteq\; \big\{\mathrm{C}_\sigma(\vec{x}) = 0 : p_\Phi(\sigma) \neq 0\big\}.$$

This says, in view of (P2), that if $\Phi \therefore \Psi$ is valid then Ψ is either equivalent to Φ, or it is a limitation of Φ.

The equivalence of (b) and (d) is a special case of R01.*

*The Rule of 0 and 1 appears, in words only, on pp. 37–38 of LT. The item (P1) and its corollaries are not explicitly stated in LT, but they clearly fall under the Rule of 0 and 1. (P2) appears on p. 83 of LT as a RULE, and (P3) is essentially stated on p. 90 of LT. Neither (P4) nor (P5) appear in LT; perhaps the Rule of 0 and 1 absorbed all that Boole wanted from these two items.

The proof of Prop. 5, in particular of (c) above, suggests an alternative to Hailperin's axiomatization of Boole's Algebra, which was as non-trivial com-

The Reduction and Solution Theorems. In the final chapter, OF THE SOLUTION OF ELECTIVE EQUATIONS, Boole gave two main results.

1) **Reduction Theorem:** (p. 78)
A system of equations
$$p_1(\vec{x}) = 0, \ldots, p_k(\vec{x}) = 0$$
can be reduced to a single equation
$$p_1(\vec{x}) + \lambda_2 p_2(\vec{x}) + \cdots + \lambda_k p_k(\vec{x}) \;=\; 0.$$
The *Lagrange multipliers* λ_i are arbitrary parameters.* Lagrange multipliers

2) **Solution Theorem (for one equation):** (pp. 70-74)
For the algebraic analysis of his version of Aristotelian logic, Boole only needed to solve some very simple equations like $x(1-y) = 0$. Evidently this motivated him to consider finding the general solution to an arbitrary equation $p(\vec{x}, w) = 0$ for w in terms of the other variables $\vec{x}$. He was no doubt pleased to announce (p. 7) that finding the general solution was always possible.

To find the general solution of
$$p(\vec{x}, w) = 0 \tag{13}$$
for w, let
$$\begin{aligned} J_0 &= \{\sigma : p(\sigma,1) \neq 0 = p(\sigma,0)\} \\ J_1 &= \{\sigma : p(\sigma,1) = 0 \neq p(\sigma,0)\} \\ J_v &= \{\sigma : p(\sigma,1) = 0 = p(\sigma,0)\} \\ J_\infty &= \{\sigma : p(\sigma,1) \neq 0 \neq p(\sigma,0)\}. \end{aligned}$$

mutative rings with unity with idempotent variables and no additive nilpotents, that is, the additive group is torsion-free. "No additive nilpotents" is expressed by the quasi-equations $ns(\vec{x}) = 0 \rightarrow s(\vec{x}) = 0$, for $n = 1, 2, \ldots$.

One could also axiomatize Boole's Algebra as $\mathbb{Q}$-algebra with idempotent variables, the rational numbers being viewed as unary functions. This makes it possible to give a purely equational axiomatization of Boole's Algebra. Then one would work with polynomials with rational coefficients, and the Rule of 0 and 1 would have $\mathbb{Z}$ replaced by $\mathbb{Q}$. One would also have $p_1(\vec{x}) = 0, \ldots, p_k(\vec{x}) = 0 \therefore q(\vec{x}) = 0$ is valid iff $q(\vec{x}) = \sum r_i(\vec{x}) \cdot p_i(\vec{x})$ is valid for some choice of terms $r_i(\vec{x})$.

*This reduction theorem is stated in LT (replacing λ_i by c_i), but the main reduction used there is
$$p_1(\vec{x})^2 + \cdots + p_k(\vec{x})^2 \;=\; 0,$$
avoiding the introduction of parameters.

Then (13) is equivalent to the general solution (14) plus the constraint equations (15) on the variables $\vec{x}$:

$$\begin{aligned} w &= \Big(\sum_{\sigma\in J_1} \mathrm{C}_\sigma(\vec{x})\Big) + \Big(\sum_{\sigma\in J_v} v_\sigma \mathrm{C}_\sigma(\vec{x})\Big) && (14) \\ \mathrm{C}_\sigma(\vec{x}) &= 0 \quad \text{for } \sigma \in J_\infty. && (15) \end{aligned}$$

The v_σ are arbitrary parameters.*

Remark: From the basic properties of constituents, the constituent equations (15) can be replaced by

$$p(\vec{x},0)\cdot p(\vec{x},1) \;=\; 0. \qquad (15')$$

Boole did not present his solution theorem as above, but rather by a simple heuristic. First note that equation (13) is equivalent to

$$\big(p(\vec{x},0) - p(\vec{x},1)\big)\cdot w \;=\; p(\vec{x},0). \qquad (16)$$

Apply formal division to obtain

$$w \;=\; \frac{p(\vec{x},0)}{p(\vec{x},0) - p(\vec{x},1)}. \qquad (17)$$

Next apply formal expansion to the right side of (17):

$$\frac{p(\vec{x},0)}{p(\vec{x},0) - p(\vec{x},1)} \;=\; \sum_{\sigma} \frac{p(\sigma,0)}{p(\sigma,0) - p(\sigma,1)}\mathrm{C}_\sigma(\vec{x}). \qquad (18)$$

The following table gives the value Boole assigned to the coefficients of the $\mathrm{C}_\sigma(\vec{x})$ in (18), for σ in the various J's:

	Coeff. of $\mathrm{C}_\sigma(\vec{x})$		Coeff. of $\mathrm{C}_\sigma(\vec{x})$
$\sigma \in J_0$	0	$\sigma \in J_1$	1
$\sigma \in J_v$	$\frac{0}{0}$	$\sigma \in J_\infty$	not $0, 1, \frac{0}{0}$

By remembering that $0/0$ is to be thought of as an arbitrary parameter v, and that $\sigma \in J_\infty$ signifies that $\mathrm{C}_\sigma(\vec{x})$ must vanish, by evaluating the coefficients of the right side of (18) one easily arrives at the solution and constraint equations.

Boole believed his solution theorem offered a new and powerful tool for studying the consequences of a collection of propositions. If there is more than one proposition in the collection, he first used the Reduction Theorem to convert the equations of

*Boole said that the form of the solution reminded him of the theory of solving linear differential equations (see pp. 70,72), a subject in which he was well-versed.

the collection into a single equation. Some of the discussion of solving equations in MAL is more complicated than in LT because he was solving equations $p(\vec{x}, \vec{\lambda}, w) = 0$ which had Lagrange multipliers.*

Solution Theorem (for a system of equations). The main result on solving systems of equations in MAL is Boole's attempted general solution of three equations $\phi(x, y, z) = 0$, $\psi(x, y, z) = 0$, $\chi(x, y, z) = 0$ in three variables for the variable z (pp. 78–81). He said that this result contains all the key steps for solving any number of equations in any number of variables for one of the variables. From this claim one can deduce that, given a system of equations $p_i(\vec{x}, z) = 0$, his constraint equations, when solving for z, would be $p_i(\vec{x}, 0) \cdot p_j(\vec{x}, 1) = p_i(\vec{x}, 1) \cdot p_j(\vec{x}, 0)$ for $i \neq j$. However the correct constraints, which one can find using the sum of squares reduction theorem and the elimination theorem in LT, are $p_i(\vec{x}, 0) \cdot p_j(\vec{x}, 1) = 0$ for all i, j.†

Rather than trying to extend the algebra of logic to rational expressions $p(\vec{x})/q(\vec{x})$, as Boole attempted to do for his solution theorem, it seems best to regard his 'formal division followed by formal expansion and interpretation' as a clever and convenient mnemonic device for solving an equation.

*This complication does not enter into the discussion in LT because, except for a single demonstration example, the reductions in LT are obtained by summing squares.

†The proof is as follows (for M-semantics). First reduce the system of equations to the single equation $p(\vec{x}, y) = 0$ where $p(\vec{x}, y) = \sum_i p_i(\vec{x}, y)^2$. Then the constraint condition on $\vec{x}$ to guarantee that one can solve for y is $p(\vec{x}, 0)p(\vec{x}, 1) = 0$, that is, $\left(\sum_i p_i(\vec{x}, 0)^2\right) \cdot \left(\sum_j p_j(\vec{x}, 1)^2\right) = 0$, which is equivalent to the collection of equations $p_i(\vec{x}, 0)p_j(\vec{x}, 1) = 0$ for all i, j.

Connections to LT (1854)

Seven years after publishing MAL, Boole gave a better organized account of his algebraic approach to logic in LT. Instead of x denoting an elective operation determined by a class denoted by X, the class would now be denoted by x, and the operations of addition, subtraction and multiplication would be defined for classes. Boole would prove, modulo certain unstated assumptions, that only two numbers could be used as names of classes, namely 0 and 1; furthermore 0 had to denote Nothing (which is called the empty class today) and 1 the Universe. In 1847 De Morgan had introduced the concept of a *limited universe* ([15], p. 41); Boole adopted this in LT, but used the name *universe of discourse* (LT, p. 42), which also included the 'actual' universe. Constituents would continue to play a central role. The elective operations would be employed only in his analysis of the operations of the mind (which has met with little favor), and the infinite series proof of the elimination theorem only appears in a footnote, as an alternative for functions that have infinite series expansions. (Perhaps he hoped to apply this to rational functions.)

Most of the key ideas of Boole's Algebra in his masterwork on logic, LT, were present in some form in MAL, and LT is in good part concerned with clarifying (and correcting) what was said in MAL. More laws would be given for the Common Algebra, but still not enough. His rules of inference were that the addition, subtraction or multiplication of equals gives equals.

The validity of working with mathematical expressions that were (at least partly) uninterpretable as classes became a central issue in LT. Boole's justification would be based upon his Principles of Symbolical Reasoning, which was incorrect—one cannot, in general, take a collection of partial algebras with some laws that hold where defined and assume that the sort of equational deductions that one makes for total algebras will always yield equations that hold in the partial algebras when defined. However his equational algebra of logic is one of the cases where his symbolical method has been justified (by Hailperin [19], using M-semantics for class variables).

The focus of Boole's Algebra in LT was on four theorems already mentioned:

(EXPANSION) Essentially as in MAL:

$$p(\vec{x},\vec{y}) = \sum_{\tau} p(\vec{x},\tau)\mathrm{C}_{\tau}(\vec{y}).$$

(REDUCTION) Preference for sums of squares method instead of Lagrangian multipliers:

$$p_1 = \cdots = p_k = 0 \ \text{ iff } \ \sum_i p_i^2 = 0.$$

The Expansion and Reduction Theorems hold for the M-, A- and B-semantics.

(ELIMINATION) This is greatly improved over the version in MAL. Eliminating $\vec{y}$ in an equation $p(\vec{x},\vec{y}) = 0$ is described by the following inference:

$$p(\vec{x},\vec{y}) = 0 \ \therefore \ \prod_{\tau} p(\vec{x},\tau) \ = \ 0.$$

The modern strengthening of this result, for M-semantics, is:

$$(\exists\vec{y})\big(p(\vec{x},\vec{y}) = 0\big) \ \text{ iff } \ \prod_{\tau} p(\vec{x},\tau) \ = \ 0.$$

Thus given $p(\vec{x},y) = 0$, for M-semantics one has:

$$(\exists y)\big(p(\vec{x},y) = 0\big) \ \text{ iff } \ p(\vec{x},0)\cdot p(\vec{x},1) \ = \ 0.$$

To include the A- and B-semantics in this result let $\varphi(\vec{x})$ be the formula ($p^\star$ is defined in a footnote on p. lvi)

$$\begin{array}{ll} p(\vec{x},0)\cdot p(\vec{x},1) = 0 & \\ \quad \text{and } p^\star(\vec{x},1) \neq 1 & \text{for A- or B-semantics} \\ \quad \text{and } p^\star(\vec{x},0) \neq 1 & \text{for B-semantics.} \end{array}$$

Then one has $(\exists y)\big(p(\vec{x},y) = 0\big)$ iff $\varphi(\vec{x})$.

(SOLUTION) For a single equation $p(\vec{x},y) = 0$, essentially as in MAL, as described on p. lviii, valid for M-semantics. Ernst Schröder (1841–1902) gave an elegant formulation of the solution theorem for Boolean algebra on p. 447 of the first volume of his masterwork [24], and this readily translates into Boole's Algebra. With $\overline{q}$ meaning $1-q$, using the formula $\varphi(\vec{x})$ above one has for A-, B- or M-semantics:

$p(\vec{x}, y) = 0$ if and only if

$$\varphi(\vec{x}) \text{ and } (\exists v)\big[y = p^\star(\vec{x}, 0)\cdot \overline{v} \ + \ \overline{p^\star(\vec{x}, 1)}\cdot v\big],$$

which is equivalent to

$$\varphi(\vec{x}) \text{ and } (\exists v)\big[y = p^\star(\vec{x}, 0) \ + \ v\cdot \overline{p^\star(\vec{x}, 0)}\cdot \overline{p^\star(\vec{x}, 1)}\big],$$

and also to

$$\varphi(\vec{x}) \text{ and } (\exists v)\big[y = p^\star(\vec{x}, 0) \ + \ v\cdot \big(1 - p^\star(\vec{x}, 0) - p^\star(\vec{x}, 1)\big)\big].$$

In the case of M-semantics, this can be proved using R01.

LT has numerous examples worked out in detail—some are likely too long and involve subject matter that is not so interesting to a modern audience. Boole regarded his main contribution in LT, aside from his improved version of the algebra of logic, to be the application of this algebra to probability theory. History has not supported this view—in this regard he is remembered for being a pioneer in the algebra of logic, not in probability theory.

Hailperin concluded that the proper setting for Boole's work was non-trivial commutative rings with unity without additively nilpotent elements, with the relevant models being algebras of signed multi-sets. This passes over the subtlety that Boole was actually working with partial algebras of classes for which there were totally defined terms that defined union, intersection and complement.

Boole's idea of proving equational theorems for these partial algebras by working with equations in a manner that is usually reserved for total algebras turns out to be valid, although his justification of this fact was wrong. Hailperin gave a proof based on the fact that Boole's partial algebras are isomorphic to the restriction of powers of the integers to their idempotents.*

Acknowledgements. The remarkable work done by Project Gutenberg to make public domain books available in modern digital form has greatly expedited this work. In particular the author is grateful for the essentially error free transcription of MAL into LaTeX by Andrew D. Hwang, an outstanding contributor to Project Gutenberg.

*Hassler Whitney [26] had this partial algebra of idempotents in front of him in 1933 without realizing its connection to Boole's work. His goal was to translate equations and equational arguments in modern Boolean algebra into equations and equational arguments in the integers.

Numerous modifications have been made to the Project Gutenberg version in what follows, including making the page content agree with that of the original MAL, highlighting text that I found more relevant to understanding Boole's work, the addition of margin notes, and reverting the text to its original typographical and mathematical errors (except that the corrections of the errata from the Stanford list have been made), pointing out the errors in the margin.

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S.N.B.
Waterloo, Ontario
November 20, 2022

THE MATHEMATICAL ANALYSIS

OF LOGIC,

BEING AN ESSAY TOWARDS A CALCULUS
OF DEDUCTIVE REASONING.

BY GEORGE BOOLE.

Ἐπικοινωνοῦσι δὲ πᾶσαι αἱ ἐπιστῆμαι ἀλλήλαις κατὰ τὰ κοινά. Κοινά δὲ λέγω, οἷς χρῶνται ὡς ἐκ τούτων ἀποδεικνύντες· ἀλλ' οὐ περὶ ὧν δεικνύουσιν, οὐδε ὃ δεικνύουσι.

Aristotle, *Anal. Post.*, lib. I. cap. XI.

CAMBRIDGE:
MACMILLAN, BARCLAY, & MACMILLAN;
LONDON: GEORGE BELL.

1847

A copy of MAL in the Stanford Library (with a handwritten note stating that this was a gift from a father to his son in 1871) says regarding the printing:

CAMBRIDGE:
Printed by Metcalfe and Palmer, Trinity-Street

PRINTED IN ENGLAND BY
HENDERSON & SPALDING
LONDON. W.I

ERRATA

Page	5,	note,		*for* VI. *read* IV.
,,	8,	line	31,	*for* first *read* fact.
,,	17,	,,	12,	*for* abstraction *read* election.
,,	53,	,,	12,	*for* numbers *read* members.
,,	66,	,,	12 and 31,	*for* z *read* w.
,,	80,	,,	27,	*for* 3 *read* 2.

These are errata from a slip of paper in the Stanford copy mentioned on the previous page; the last two were noted by Boole in CL. These corrections have been made in this annotated version—other corrections have been made only in the margins.

The page and line numbers in the Errata refer to Boole's original publication, and subsequent facsimile copies. The content of the pages in this annotated version agree with the original, but not necessarily the formatting of the individual lines of text.

Andrew D. Hwang's excellent digitization of MAL for Project Gutenberg was evidently based on a facsimile copy from the University of Toronto Libraries, a copy that was published in 1948 by the Philosophical Library, Inc. of New York, NY; it included six of the seven corrections mentioned in the Stanford errata.

PREFACE.

In presenting this Work to public notice, I deem it not irrelevant to observe, that speculations similar to those which it records have, at different periods, occupied my thoughts. **In the spring of the present year my attention was directed to the question then moved between Sir W. Hamilton and Professor De Morgan; and I was induced by the interest which it inspired, to resume the almost-forgotten thread of former inquiries.** It appeared to me that, although Logic might be viewed with reference to the idea of quantity,* it had also another and a deeper system of relations. If it was lawful to regard it from *without*, as connecting itself through the medium of Number with the intuitions of Space and Time, it was lawful also to regard it from *within*, as based upon facts of another order which have their abode in the constitution of the Mind. The results of this view, and of the inquiries which it suggested, are embodied in the following Treatise.

See p. 9 regarding the Hamilton–De Morgan conflict. This Sir W. Hamilton was the Scottish philosopher, not the Irish mathematician.

It is not generally permitted to an Author to prescribe the mode in which his production shall be judged; but there are two conditions which I may venture to require of those who shall undertake to estimate the merits of this performance. The first is, that no preconceived notion of the impossibility of its objects shall be permitted to interfere with that candour and impartiality which the investigation of Truth demands; the second is, that their judgment of the system as a whole shall not be founded either upon the examination of only .... (pagebreak in MAL)

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*See p. 42. [Note: Should this have been p. 4?]

a part of it, or upon the measure of its conformity with any received system, considered as a standard of reference from which appeal is denied. **It is in the general theorems which occupy the latter chapters of this work,—results to which there is no existing counterpart,—that the claims of the method, as a Calculus of Deductive Reasoning, are most fully set forth.**

The general results start on p. 60.

What may be the final estimate of the value of the system, I have neither the wish nor the right to anticipate. The estimation of a theory is not simply determined by its truth It also depends upon the importance of its subject, and the extent of its applications; beyond which something must still be left to the arbitrariness of human Opinion. If the utility of the application of Mathematical forms to the science of Logic were solely a question of Notation, I should be content to rest the defence of this attempt upon a principle which has been stated by an able living writer: "Whenever the nature of the subject permits the reasoning process to be without danger carried on mechanically, the language should be constructed on as mechanical principles as possible; while in the contrary case it should be so constructed, that there shall be the greatest possible obstacle to a mere mechanical use of it."* In one respect, the science of Logic differs from all others; the perfection of its method is chiefly valuable as an evidence of the speculative truth of its principles. To supersede the employment of common reason, or to subject it to the rigour of technical forms, would be the last desire of one who knows the value of that intellectual toil and warfare which imparts to the mind an athletic vigour, and teaches it to contend with difficulties and to rely upon itself in emergencies.

TYPO: its truth. It also

LINCOLN, *Oct.* 29, 1847.

*Mill's *System of Logic, Ratiocinative and Inductive*, Vol. II. p. 292.

MATHEMATICAL ANALYSIS OF LOGIC.

INTRODUCTION.

THEY who are acquainted with the present state of the theory of **Symbolical Algebra,** are aware, that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon **the laws of their combination.** Every system of interpretation which does not affect the truth of the relations supposed, is equally admissible, and it is thus that the same process may, under one scheme of interpretation, represent the solution of a question on the properties of numbers, under another, that of a geometrical problem, and under a third, that of a problem of dynamics or optics. This principle is indeed of fundamental importance; and it may with safety be affirmed, that **the recent advances of pure analysis** have been much assisted by the influence which it has exerted in directing the current of investigation.

But the full recognition of the consequences of this important doctrine has been, in some measure, retarded by accidental circumstances. It has happened in every known form of analysis, that the elements to be determined have been conceived as measurable by comparison with some fixed standard. The predominant idea has been that of magnitude, or more strictly, of numerical ratio. The expression of magnitude, or (pagebreak in MAL)

By Symbolical Algebra Boole meant equational reasoning, that is, given equational axioms (which he called *laws*) and rules of inference (which he called *axioms*), deriving equations from equations without reference to models of the laws. He never succeeded in properly formalizing this subject.

By "the laws of their combination" he meant the equational axioms (laws), each side of an equation being a combination of symbols (called *terms* in modern logic).

The "recent advances" seems to be a reference to his own work in differential equations, using differential operators.

of operations upon magnitude, has been the express object for which the symbols of Analysis have been invented, and for which their laws have been investigated. Thus the abstractions of the modern Analysis, not less than the ostensive diagrams of the ancient Geometry, have encouraged the notion, that Mathematics are essentially, as well as actually, the Science of Magnitude.

The consideration of that view which has already been stated, as embodying the true principle of the Algebra of Symbols, would, however, lead us to infer that this conclusion is by no means necessary. If every existing interpretation is shewn to involve the idea of magnitude, it is only by induction that we can assert that no other interpretation is possible. And it may be doubted whether our experience is sufficient to render such an induction legitimate. The history of pure Analysis is, it may be said, too recent to permit us to set limits to the extent of its applications. Should we grant to the inference a high degree of probability, we might still, and with reason, maintain the sufficiency of the definition to which the principle already stated would lead us. **We might justly assign it as the definitive character of a true Calculus, that it is a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation.** That to the existing forms of Analysis a quantitative interpretation is assigned, is the result of the circumstances by which those forms were determined, and is not to be construed into a universal condition of Analysis. **It is upon the foundation of this general principle, that I purpose to establish the Calculus of Logic, and that I claim for it a place among the acknowledged forms of Mathematical Analysis, regardless that in its object and in its instruments it must at present stand alone.**

That which renders Logic possible, is the existence in our minds of general notions,—our ability to conceive of a class, and to designate its individual members by a common name.

The theory of Logic is thus intimately connected with that of Language. A successful attempt to express logical propositions by symbols, the laws of whose combinations should be founded upon **the laws of the mental processes which they represent,** would, so far, be a step toward a philosophical language. But this is a view which we need not here follow into detail.* **Assuming the notion of a class, we are able, from any conceivable collection of objects, to separate by a mental act, those which belong to the given class, and to contemplate them apart from the rest.** Such, or a similar act of election, we may conceive to be repeated. The group of individuals left under consideration may be still further limited, by mentally selecting those among them which belong to some other recognised class, as well as to the one before contemplated. **And this process may be repeated with other elements of distinction,** until we arrive at an individual possessing all the distinctive characters which we have taken into account, and a member, at the same time, of every class which we have enumerated. **It is in fact a method similar to this which we employ whenever, in common language, we accumulate descriptive epithets for the sake of more precise definition.**

Now **the several mental operations** which in the above case we have supposed to be performed, **are subject to peculiar laws.** It is possible to assign relations among them, whether as respects the repetition of a given operation or the succession of different ones, or some other particular, which are never violated. **It is, for example, true that the result of two successive acts is** .... (pagebreak in MAL)

It was the tradition to ground the subject of logic in mental faculties and processes, like Conception, Judgement and Reasoning. Boole focused on conceptions of classes and their elective operations; and the operations of multiplication, addition and subtraction of elective operations (in LT these three operations would be applied directly to conceptions of classes).

Given a class X, let x be the operation on classes Y defined by $x(\mathrm{Y}) := \mathrm{X} \cap \mathrm{Y}$. x will be called the *elective operation* determined by X.

Given elective operations $x, y, \ldots$, one can compose them to form an elective operation $xy \cdots$. This parallels ordinary language, e.g., one can think of the phrase "big green giants" as first selecting green objects from the class of giants, and then selecting big objects from the resulting class. Thus "big green" is a composition of big and green.

The phrase "subject to special laws" sounds better than "subject to peculiar laws".

*This view is well expressed in one of Blanco White's Letters:—"Logic is for the most part a collection of technical rules founded on classification. The Syllogism is nothing but a result of the classification of things, which the mind naturally and necessarily forms, in forming a language. All abstract terms are classifications; or rather the labels of the classes which the mind has settled."—*Memoirs of the Rev. Joseph Blanco White*, vol. II. p. 163. See also, for a very lucid introduction, Dr. Latham's *First Outlines of Logic applied to Language*, Becker's *German Grammar, &c.* Extreme Nominalists make Logic entirely dependent upon language. For the opposite view, see Cudworth's *Eternal and Immutable Morality*, Book IV. Chap. III.

unaffected by the order in which they are performed; and there are at least two other laws which will be pointed out in the proper place. These will perhaps to some appear so obvious as to be ranked among necessary truths, and so little important as to be undeserving of special notice. **And probably they are noticed for the first time in this Essay.** Yet it may with confidence be asserted, that if they were other than they are, the entire mechanism of reasoning, nay the very laws and constitution of the human intellect, would be vitally changed. A Logic might indeed exist, but it would no longer be the Logic we possess.

This is the commutative law: $xy = yx$. The other two laws are the index law $x^n = x$ and the distributive law $x(y+z) = xy + xz$. In LT the index law is replaced by the idempotent law $x^2 = x$.

Such are the elementary laws upon the existence of which, and upon their capability of exact symbolical expression, the method of the following Essay is founded; and it is presumed that the object which it seeks to attain will be thought to have been very fully accomplished. **Every logical proposition, whether categorical or hypothetical, will be found to be capable of exact and rigorous expression, and not only will the laws of conversion and of syllogism be thence deducible, but the resolution of the most complex systems of propositions, the separation of any proposed element, and the expression of its value in terms of the remaining elements, with every subsidiary relation involved. Every process will represent deduction, every mathematical consequence will express a logical inference. The generality of the method will even permit us to express arbitrary operations of the intellect, and thus lead to the demonstration of general theorems in logic analogous, in no slight degree, to the general theorems of ordinary mathematics.** No inconsiderable part of the pleasure which we derive from the application of analysis to the interpretation of external nature, arises from the conceptions which it enables us to form of the universality of the dominion of law. The general formulæ to which we are conducted seem to give to that element a visible presence, and the multitude of particular cases to which they apply, demonstrate the extent of its sway. Even the symmetry. . . . . (pagebreak in MAL)

Boole's glowing account of what his algebraic approach to logic achieves is tempered by how limited the subject of logic was in the mid 19th century.

of their analytical expression may in no fanciful sense be deemed indicative of its harmony and its consistency. Now I do not presume to say to what extent the same sources of pleasure are opened in the following Essay. The measure of that extent may be left to the estimate of those who shall think the subject worthy of their study. But I may venture to assert that such occasions of intellectual gratification are not here wanting. **The laws we have to examine are the laws of one of the most important of our mental faculties. The mathematics we have to construct are the mathematics of the human intellect.** Nor are the form and character of the method, apart from all regard to its interpretation, undeserving of notice. There is even a remarkable exemplification, in its general theorems, of **that species of excellence which consists in freedom from exception.** And this is observed where, in the corresponding cases of the received mathematics, such a character is by no means apparent. The few who think that there is that in analysis which renders it deserving of attention for its own sake, may find it worth while to study it under a form in which **every equation can be solved and every solution interpreted.** Nor will it lessen the interest of this study to reflect that every peculiarity which they will notice in the form of the Calculus represents a corresponding feature in the constitution of their own minds.

Those working in symbolic logic have long given up claims that logic gives valuable insights into how the mind works. It is simply the study of what we accept as correct reasoning. Unfortunately in LT Boole would follow up in greater detail on the connections of his system with mental processes.

Actually, given an equation, Boole states an equational condition that is necessary and sufficient for the given equation to have a solution; and if there is a solution then Boole shows how to find the general solution.

It would be premature to speak of the value which this method may possess as an instrument of scientific investigation. I speak here with reference to the theory of reasoning, and to the principle of a true classification of the forms and cases of Logic considered as a Science.* **The aim of these investigations was in the first instance confined to the expression of the received logic, and to the forms of the Aristotelian arrangement,** . . . . (pagebreak in MAL)

Treating Aristotelian logic, slightly generalized by Boole, by algebraic means is covered in pages 20–59. Pages 60–81 present some general theorems of Boole's algebra—examples of how to apply these theorems, in some cases improvements of these theorems, to logic will mainly appear in LT.

* "Strictly a Science"; also "an Art."—*Whately's Elements of Logic.* Indeed ought we not to regard all Art as applied Science; unless we are willing, with "the multitude," to consider Art as "guessing and aiming well"?—*Plato, Philebus.*

but it soon became apparent that restrictions were thus introduced, which were purely arbitrary and had no foundation in the nature of things. These were noted as they occurred, and will be discussed in the proper place. When it became necessary to consider the subject of hypothetical propositions (in which comparatively less has been done), and still more, when an interpretation was demanded for the general theorems of the Calculus, **it was found to be imperative to dismiss all regard for precedent and authority,** and to interrogate the method itself for an expression of the just limits of its application. Still, however, there was no special effort to arrive at novel results. But among those which at the time of their discovery appeared to be such, it may be proper to notice the following.

A logical proposition is, according to the method of this Essay, expressible by an equation the form of which determines the rules of conversion and of transformation, to which the given proposition is subject. Thus the law of what logicians term simple conversion, is determined by the fact, that the corresponding equations are symmetrical, that they are unaffected by a mutual change of place, in those symbols which correspond to the convertible classes. **The received laws of conversion were thus determined, and afterwards another system, which is thought to be more elementary, and more general. See Chapter, *On the Conversion of Propositions*.**

The premises of a syllogism being expressed by equations, the elimination of a common symbol between them leads to a third equation which expresses the conclusion, this conclusion being always the most general possible, whether Aristotelian or not. Among the cases in which no inference was possible, it was found, that there were two distinct forms of the final equation. It was a considerable time before the explanation of this fact was discovered, but it was at length seen to depend upon the presence or absence of a true medium of comparison between the premises. The distinction which is thought to be new is illustrated in the Chapter, *On Syllogisms*.

The categorical propositions were expressed by simple equations. Boole believed that conversions and syllogisms, the bread and butter of Aristotelian logic, could be presented in a more coherent fashion when viewed from the equational perspective.

The general theory that starts on page 60 would evolve into the main part of his algebra of logic in LT. In LT Aristotelian logic receives only one small chapter, Chap. XV.

In the categorical propositions

A	All X is Y
E	No X is Y
I	Some X is Y
O	Some X is not Y

Boole allowed X to be replaced by not-X as well as Y by not-Y.

This led to his three laws on p. 30 for transforming a categorical proposition Φ into any other categorical proposition Ψ that logically follows from Φ.

TYPO: *Of the Conversion*

The elimination theorem Boole used in MAL (see p. 32) was rather limited—the full-strength elimination theorem would appear in LT. In the chapter on categorical syllogisms in MAL, Boole chose the two premiss propositions from among the four types (A,E,I,O) in Aristotelian logic, but the most general conclusion might not be Aristotelian.

TYPO: *Of Syllogisms.*

The nonexclusive character of the disjunctive conclusion of a hypothetical syllogism, is very clearly pointed out in the examples of this species of argument. **The class of logical problems illustrated in the chapter, *On the Solution of Elective Equations*, is conceived to be new: and it is believed that the method of that chapter affords the means of a perfect analysis of any conceivable system of propositions, an end toward which the rules for the conversion of a single categorical proposition are but the first step.**

TYPO: *Of the Solution*

Given a collection of premises $\Phi_1(\vec{X}, W)$, $\ldots$, $\Phi_k(\vec{X}, W)$ about classes $\vec{X}$, W, there is a method to find the general solution for W in terms of $\vec{X}$. If $k > 1$ this involves a reduction of the equational versions $\varepsilon_i(\vec{x}, w)$ of the premises to a single equation $\varepsilon(\vec{x}, w, \vec{\lambda})$ by "Lagrange multipliers" $\vec{\lambda}$. A single equation is solved for w by formal division and formal expansion.

However, upon the originality of these or any of these views, I am conscious that I possess too slight an acquaintance with the literature of logical science, and especially with its older literature, to permit me to speak with confidence.

It may not be inappropriate, before concluding these observations, to offer a few remarks upon the general question of the use of symbolical language in the mathematics. Objections have lately been very strongly urged against this practice, on the ground, that by obviating the necessity of thought, and substituting a reference to general formulæ in the room of personal effort, it tends to weaken the reasoning faculties.

These objections concern the Hamilton–De Morgan dispute that motivated Boole to write MAL. This discussion will continue through p. 13.

Now the question of the use of symbols may be considered in two distinct points of view. First, it may be considered with reference to the progress of scientific discovery, and secondly, with reference to its bearing upon the discipline of the intellect.

And with respect to the first view, it may be observed that as it is one fruit of an accomplished labour, that it sets us at liberty to engage in more arduous toils, so it is a necessary result of an advanced state of science, that we are permitted, and even called upon, to proceed to higher problems, than those which we before contemplated. The practical inference is obvious. If through the advancing power of scientific methods, we find that the pursuits on which we were once engaged, afford no longer a sufficiently ample field for intellectual effort, the remedy is, to proceed to higher inquiries, and, in new tracks, to seek for difficulties yet unsubdued. And such is, ... (pagebreak in MAL) indeed, the actual law of scientific progress. We must be content, either to abandon the hope of further conquest, or to employ such aids of symbolical language, as are proper to the stage of progress, at which we have arrived. Nor need we fear to commit ourselves to such a course. We have not yet arrived so near to the boundaries of possible knowledge, as to suggest the apprehension, that scope will fail for the exercise of the inventive faculties.

In discussing the second, and scarcely less momentous question of the influence of the use of symbols upon the discipline of the intellect, an important distinction ought to be made. It is of most material consequence, whether those symbols are used with a full understanding of their meaning, with a perfect comprehension of that which renders their use lawful, and an ability to expand the abbreviated forms of reasoning which they induce, into their full syllogistic devolopment; or whether they are mere unsuggestive characters, the use of which is suffered to rest upon authority.

TYPO: development

The answer which must be given to the question proposed, will differ according as the one or the other of these suppositions is admitted. In the former case an intellectual discipline of a high order is provided, an exercise not only of reason, but of the faculty of generalization. In the latter case there is no mental discipline whatever. It were perhaps the best security against the danger of an unreasoning reliance upon symbols, on the one hand, and a neglect of their just claims on the other, that each subject of applied mathematics should be treated in the spirit of the methods which were known at the time when the application was made, but in the best form which those methods have assumed. The order of attainment in the individual mind would thus bear some relation to the actual order of scientific discovery, and the more abstract methods of the higher analysis would be offered to such minds only, as were prepared to receive them.

The relation in which this Essay stands at once to Logic and to Mathematics, may further justify some notice of the question which has lately been revived, as to the relative value of the two studies in a liberal education. One of the chief objections which have been urged against the study of Mathematics in general, is but another form of that which has been already considered with respect to the use of symbols in particular. And it need not here be further dwelt upon, than to notice, that if it avails anything, it applies with an equal force against the study of Logic. **The canonical forms of the Aristotelian syllogism are really symbolical; only the symbols are less perfect of their kind than those of mathematics.** If they are employed to test the validity of an argument, they as truly supersede the exercise of reason, as does a reference to a formula of analysis. Whether men do, in the present day, make this use of the Aristotelian canons, except as a special illustration of the rules of Logic, may be doubted; yet it cannot be questioned that when the authority of Aristotle was dominant in the schools of Europe, such applications were habitually made. And our argument only requires the admission, that the case is possible.

But the question before us has been argued upon higher grounds. Regarding Logic as a branch of Philosophy, and defining Philosophy as the "science of a real existence," and "the research of causes," and assigning as its *main* business the investigation of the "why, (τὸ δίοτι)," while Mathematics display only the "that, (τὸ ὁτὶ)," **Sir W. Hamilton has contended, not simply, that the superiority rests with the study of Logic, but that the study of Mathematics is at once dangerous and useless**.* The pursuits of the mathematician "have not only not trained him to that acute scent, to that delicate, almost instinctive, tact which, in the twilight of probability, the search and discrimination of its finer facts demand; they have gone to cloud his vision, to indurate his touch, to all but the blazing light, the iron chain of demonstration, and left him out of the narrow confines of his science, to a passive *credulity* in any premises, or to

Sir W. Hamilton's views on Logic and Mathematics.

(pagebreak in MAL)

* *Edinburgh Review*, vol. LXII. p. 409, and *Letter to A. De Morgan, Esq.*

an absolute *incredulity* in all." In support of these and of other charges, both argument and copious authority are adduced.*

I shall not attempt a complete discussion of the topics which are suggested by these remarks. My object is not controversy, and the observations which follow are offered not in the spirit of antagonism, but in the hope of contributing to the formation of just views upon an important subject. Of Sir W. Hamilton it is impossible to speak otherwise than with that respect which is due to genius and learning.

Philosophy is then described as the *science of a real existence* and *the research of causes.* And that no doubt may rest upon the meaning of the word *cause*, it is further said, that philosophy "mainly investigates the *why.*" These definitions are common among the ancient writers. Thus Seneca, one of Sir W. Hamilton's authorities, *Epistle* LXXXVIII., "The philosopher seeks and knows the *causes* of natural things, of which the mathematician searches out and computes the numbers and the measures." It may be remarked, in passing, that in whatever degree the belief has prevailed, that the business of philosophy is immediately with *causes*; in the same degree has every science whose object is the investigation of *laws*, been lightly esteemed. Thus the Epistle to which we have referred, bestows, by contrast with Philosophy, a separate condemnation on Music and Grammar, on Mathematics and Astronomy, although it is that of Mathematics only that Sir W. Hamilton has quoted.

Now we might take our stand upon the conviction of many thoughtful and reflective minds, that in the extent of the meaning above stated, Philosophy is impossible. The business of true Science, they conclude, is with laws and phenomena. The nature of Being, the mode of the operation of Cause, the *why*,

*The arguments are in general better than the authorities. Many writers quoted in condemnation of mathematics (Aristo, Seneca, Jerome, Augustine, Cornelius Agrippa, &c.) have borne a no less explicit testimony against other sciences, nor least of all, against that of logic. The treatise of the last named writer *De Vanitate Scientiarum*, must surely have been referred to by mistake.—*Vide* cap. CII.

they hold to be beyond the reach of our intelligence. But we do not require the vantage-ground of this position; nor is it doubted that whether the aim of Philosophy is attainable or not, the desire which impels us to the attempt is an instinct of our higher nature. Let it be granted that the problem which has baffled the efforts of ages, is not a hopeless one; that the "science of a real existence," and "the research of causes," "that kernel" for which "Philosophy is still militant," do not transcend the limits of the human intellect. **I am then compelled to assert, that according to this view of the nature of Philosophy, *Logic forms no part of it.* On the principle of a true classification, we ought no longer to associate Logic and Metaphysics, but Logic and Mathematics.**

Boole said that Logic is related to mathematics, not philosophy.

Should any one after what has been said, entertain a doubt upon this point, I must refer him to the evidence which will be afforded in the following Essay. He will there see Logic resting like Geometry upon axiomatic truths, and its theorems constructed upon that general doctrine of symbols, which constitutes the foundation of the recognised Analysis. In the Logic of Aristotle he will be led to view a collection of the formulæ of the science, expressed by another, but, (it is thought) less perfect scheme of symbols. I feel bound to contend for the absolute exactness of this parallel. It is no escape from the conclusion to which it points to assert, that Logic not only constructs a science, but also inquires into the origin and the nature of its own principles,—a distinction which is denied to Mathematics. "It is wholly beyond the domain of mathematicians," it is said, "to inquire into the origin and nature of their principles."—*Review*, page 415. But upon what ground can such a distinction be maintained? What definition of the term Science will be found sufficiently arbitrary to allow such differences?

The application of this conclusion to the question before us is clear and decisive. The mental discipline which is afforded by the study of Logic, *as an exact science*, is, in species, the same as that afforded by the study of Analysis.

Is it then contended that either Logic or Mathematics can supply a perfect discipline to the Intellect? The most careful and unprejudiced examination of this question leads me to doubt whether such a position can be maintained. The exclusive claims of either must, I believe, be abandoned, nor can any others, partaking of a like exclusive character, be admitted in their room. **It is an important observation, which has more than once been made, that it is one thing to arrive at correct premises, and another thing to deduce logical conclusions, and that the business of life depends more upon the former than upon the latter.** The study of the exact sciences may teach us the one, and it may give us some general preparation of knowledge and of practice for the attainment of the other, but it is to the union of thought with action, in the field of Practical Logic, the arena of Human Life, that we are to look for its fuller and more perfect accomplishment.

I desire here to express my conviction, that with the advance of our knowledge of all true science, an ever-increasing harmony will be found to prevail among its separate branches. The view which leads to the rejection of one, ought, if consistent, to lead to the rejection of others. And indeed many of the authorities which have been quoted against the study of Mathematics, are even more explicit in their condemnation of Logic. "Natural science," says the Chian Aristo, "is above us, Logical science does not concern us." When such conclusions are founded (as they often are) upon a deep conviction of the preeminent value and importance of the study of Morals, we admit the premises, but must demur to the inference. For it has been well said by an ancient writer, that it is the "characteristic of the liberal sciences, not that they conduct us to Virtue, but that they prepare us for Virtue;" and Melancthon's sentiment, "abeunt studia in mores," has passed into a proverb. Moreover, there is a common ground upon which all sincere votaries of truth may meet, exchanging with each other the language of Flamsteed's appeal to Newton, "The works of the Eternal Providence will be better understood through your labors and mine."

Let us employ the symbol 1, or unity, to represent the Universe, and let us understand it as comprehending every conceivable class of objects whether actually existing or not, it being premised that the same individual may be found in more than one class, inasmuch as it may possess more than one quality in common with other individuals. **Let us employ the letters X, Y, Z, to represent the individual members of classes,** X applying to every member of one class, as members of that particular class, and Y to every member of another class as members of such class, and so on, according to the received language of treatises on Logic.

In LT Boole used 1 to denote the *universe of discourse*—this could be either *the* universe of MAL or a *limited* universe (introduced by De Morgan in 1847 in his book "Formal Logic").

We would say 'let X, Y, Z denote classes'.
(In LT capital letters are used to denote terms.)

Further let us conceive a class of symbols x, y, z, possessed of the following character.

x, y, z denote elective operations.

The symbol x operating upon any subject comprehending individuals or classes, shall be supposed to select from that subject all the Xs which it contains. In like manner the symbol y, operating upon any subject, shall be supposed to select from it all individuals of the class Y which are comprised in it, and so on.

Boole did not permit a class to have just one element in MAL, but he did in LT.

When no subject is expressed, we shall suppose 1 (the Universe) to be the subject understood, so that we shall have

$$x = x \quad (1),$$

Clearly $X = x(1)$. Modern algebraists are happy to deal with algebras of operators without requiring that a subject be present.

the meaning of either term being the selection from the Universe of all the Xs which it contains, and the result of the operation

In MAL, did Boole regard a class as merely a notion from common language, and not to be considered as a mathematical object?

being in common language, the class X, *i. e.* the class of which each member is an X.

From these premises it will follow, that **the product xy** will represent, in succession, the selection of the class Y, and the selection from the class Y of such individuals of the class X as are contained in it, the result being the class whose members are both Xs and Ys. And in like manner the product xyz will represent a compound operation of which the successive elements are the selection of the class Z, the selection from it of such individuals of the class Y as are contained in it, and the selection from the result thus obtained of all the individuals of the class X which it contains, the final result being the class common to X, Y, and Z.

This does not follow from 'these premises'. For a modern algebra of elective operations the *product xy* would be defined as the composition $(xy)(\mathrm{Z}) = x(y(\mathrm{Z})) = \mathrm{X} \cap \mathrm{Y} \cap \mathrm{Z}$. Thus the product xy of two elective operations x, y is again an elective operation, the one determined by the class $\mathrm{X} \cap \mathrm{Y}$. But Boole seems to say that xy is $\mathrm{X} \cap \mathrm{Y}$.

From the nature of the operation which the symbols x, y, z, are conceived to represent, we shall designate them as **elective symbols.** An expression in which they are involved will be called an **elective function,** and an equation of which the members are elective functions, will be termed an **elective equation.**

Boole's definitions of elective symbols, functions and equations.

It will not be necessary that we should here enter into the analysis of that mental operation which we have represented by the elective symbol. It is not an act of Abstraction according to the common acceptation of that term, because we never lose sight of the concrete, but it may probably be referred to an exercise of the faculties of Comparison and Attention. Our present concern is rather with the laws of combination and of succession, by which its results are governed, and of these it will suffice to notice the following.

1st. The result of an act of election is independent of the grouping or classification of the subject.

The distributive law.

Thus it is indifferent whether from a group of objects considered as a whole, we select the class X, or whether we divide the group into two parts, select the Xs from them separately, and then connect the results in one aggregate conception.

Boole was verbally describing
$x(U \cup \mathrm{V}) = x(U) \cup x(\mathrm{V})$
where $U \cap \mathrm{V} = \emptyset$. He does not comment on whether this law applies when $U \cap \mathrm{V} \neq \emptyset$.

We may express this law mathematically by the equation

$$x(u+v) = xu + xv,$$

$u + v$ representing the undivided subject, and u and v the component parts of it.

2nd. It is indifferent in what order two successive acts of election are performed.

The commutative law for multiplication.

Whether from the class of animals we select sheep, and from the sheep those which are horned, or whether from the class of animals we select the horned, and from these such as are sheep, the result is unaffected. In either case we arrive at the class *horned sheep*.

The symbolical expression of this law is

$$xy = yx.$$

3rd. The result of a given act of election performed twice, or any number of times in succession, is the result of the same act performed once.

$x^2 = x$ is, in modern terminology, the *idempotent law*. Boole called $x^n = x$ the *index law* on the next page. This is really an infinite collection of laws, namely $x^2 = x$, $x^3 = x$, etc.

If from a group of objects we select the Xs, we obtain a class of which all the members are Xs. If we repeat the operation on this class no further change will ensue: in selecting the Xs we take the whole. Thus we have

$$xx = x,$$

or

$$x^2 = x;$$

The index law only applies to variables, but one can prove that there are many other elective functions that satisfy the index law, for example, $1 - x$ and $x + y - xy$.

and supposing the same operation to be n times performed, we have

$$x^n = x,$$

which is the mathematical expression of the law above stated.*

The laws we have established under the symbolical forms

$$x(u + v) = xu + xv \qquad (1)$$

$$xy = yx \qquad (2)$$

$$x^n = x \qquad (3)$$

$x(u + v)$ is the first instance where Boole has presented an elective function that is not totally defined (one needs $U \cap V = \emptyset$). The next occurrences of such elective functions are in his Elimination Theorem on p. 32. The justification of the use of partially defined functions to obtain results about totally defined functions is a major topic in LT; unfortunately Boole's justification is not correct.

An example where $+^n = +$ is due to Gregory in "On the real nature of symbolical algebra", Transactions of the Royal Society of Edinburgh, 14 (1840), 208-16 (p. 208).

*The office of the elective symbol x, is to select individuals comprehended in the class X. Let the class X be supposed to embrace the universe; then, whatever the class Y may be, we have

$$xy = y.$$

The office which x performs is now equivalent to the symbol $+$, in one at least of its interpretations, and the index law (3) gives

$$+^n = +,$$

which is the known property of that symbol.

are sufficient for the basis of a Calculus. From the first of these, it appears that elective symbols are *distributive*, from the second that they are *commutative*; properties which they possess in common with symbols of *quantity*, and in virtue of which, **all the processes of Common Algebra are applicable to the present system.** The one and sufficient **axiom** involved in this application is that **equivalent operations performed upon equivalent subjects produce equivalent results.***

The third law (3) we shall denominate **the index law.** It is peculiar to elective symbols, and will be found of great importance in enabling us to reduce our results to forms meet for interpretation.

From the circumstance that **the processes of algebra** may be applied to the present system, it is not to be inferred that the interpretation of an elective equation will be unaffected by such processes. The expression of a truth cannot be negatived by . . . . (pagebreak in MAL)

Boole incorrectly assumed that the distributive law and the commutative law, along with his single inference rule (called an axiom), were all one needed to justify using the algebra of numbers. [Cont'd below at (**).]

Better: "It only applies to elective symbols, and will be found of great importance in enabling us to reduce our results to forms suitable for interpretation".

It is safe to assume that the "processes of algebra" include using equations and equational arguments that are valid in the algebra of numbers.

(**) The *valid* equations and equational arguments of Boole's Algebra are precisely those that can be derived from those that hold in the integers $\mathbb{Z}$ along with the index law $x^n = x$ for variables.

*It is generally asserted by writers on Logic, that all reasoning ultimately depends on an application of the dictum of Aristotle, *de omni et nullo.* "Whatever is predicated universally of any class of things, may be predicated in like manner of any thing comprehended in that class." But it is agreed that this dictum is not immediately applicable in all cases, and that in a majority of instances, a certain previous process of reduction is necessary. What are the elements involved in that process of reduction? Clearly they are as much a part of general reasoning as the dictum itself.

Another mode of considering the subject resolves all reasoning into an application of one or other of the following canons, viz.

1. If two terms agree with one and the same third, they agree with each other.

2. If one term agrees, and another disagrees, with one and the same third, these two disagree with each other.

But the application of these canons depends on mental acts equivalent to those which are involved in the before-named process of reduction. We have to select individuals from classes, to convert propositions, &c., before we can avail ourselves of their guidance. Any account of the process of reasoning is insufficient, which does not represent, as well the laws of the operation which the mind performs in that process, as the primary truths which it recognises and applies.

It is presumed that the laws in question are adequately represented by the fundamental equations of the present Calculus. The proof of this will be found in its capability of expressing propositions, and of exhibiting in the results of its processes, every result that may be arrived at by ordinary reasoning.

a legitimate operation, but it may be limited. The equation $y = z$ implies that the classes Y and Z are equivalent, member for member. Multiply it by a factor x, and we have

$$xy = xz,$$

which expresses that the individuals which are common to the classes X and Y are also common to X and Z, and *vice versâ.* This is a perfectly legitimate inference, but the fact which it declares is a less general one than was asserted in the original proposition.

OF EXPRESSION AND INTERPRETATION.

A **Proposition** is a sentence which either **affirms or denies,** as, All men are mortal, No creature is independent.

A Proposition has necessarily two **terms,** as *men*, *mortal*; the former of which, or the one spoken of, is called **the subject;** the latter, or that which is affirmed or denied of the subject, **the predicate.** These are connected together by **the copula** *is*, or *is not*, or by some other modification of the substantive verb.

The substantive verb is the only verb recognised in Logic; all others are resolvable by means of the verb *to be* and a participle or adjective, *e. g.* "The Romans conquered"; the word conquered is both copula and predicate, being equivalent to "were (copula) victorious" (predicate).

A Proposition must either be **affirmative or negative,** and must be also either **universal or particular.** Thus we reckon in all, four kinds of **pure categorical Propositions.**

The four kinds of (pure) categorical propositions in Aristotelian logic. In LT the categorical propositions will be replaced by *primary propositions*, defined as propositions about classes.

1st. **Universal-affirmative,** usually represented by A,

Ex. All Xs are Ys.

2nd. **Universal-negative,** usually represented by E,

Ex. No Xs are Ys.

3rd. **Particular-affirmative,** usually represented by I,

Ex. Some Xs are Ys.

4th. **Particular-negative,** usually represented by O,*

Ex. Some Xs are not Ys.

1. To express the class, not-X, that is, the class including all individuals that are not Xs.

The class X and the class not-X together make the Universe. But the Universe is 1, and the class X is determined by the symbol x, therefore **the class not-X will be determined by the symbol $1 - x$.**

The minus sign ($-$) is introduced here; Boole said not-X is expressed by $1 - x$. The claim of "therefore" needs clarification. The general subtraction operation "$-$" does not appear until equation (15) on page 32, and not again until page 43. It is not defined, only used.

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*The above is taken, with little variation, from the Treatises of Aldrich and Whately.

Hence the office of the symbol $1-x$ attached to a given subject will be, to select from it all the not-Xs which it contains.

And in like manner, as the product xy expresses the entire class whose members are both Xs and Ys, the symbol $y(1-x)$ will represent the class whose members are Ys but not Xs, and the symbol $(1-x)(1-y)$ the entire class whose members are neither Xs nor Ys.

2. To express the Proposition, All Xs are Ys.

All X's are Y's is expressed by $xy = x$, or equivalently, by $x(1-y) = 0$. Boole changed this to $x = vy$ in LT, following a suggestion of Graves. See pp. 45, 82.

As all the Xs which exist are found in the class Y, it is obvious that to select out of the Universe all Ys, and from these to select all Xs, is the same as to select at once from the Universe all Xs.

Hence

$$xy = x,$$

or

$$x(1-y) = 0. \tag{4}$$

Item (4) is the first time Boole used "0", and it is undefined. "$= 0$" seems to function as a predicate, with $p(\vec{x}) = 0$ meaning that $p(\vec{x})$ "does not exist" if it is a term that denotes a class.

3. To express the Proposition, No Xs are Ys.

No X's are Y's is expressed by $xy = 0$. Boole changed this to $x = v(1-y)$ in LT. The word "terms" as used here means "elements".

To assert that no Xs are Ys, is the same as to assert that there are no terms common to the classes X and Y. Now all individuals common to those classes are represented by xy. Hence the Proposition that No Xs are Ys, is represented by the equation

$$xy = 0. \tag{5}$$

4. To express the Proposition, Some Xs are Ys.

Some X's are Y's is expressed by $v = xy$. (It is assumed that V denotes a non-empty class.) Boole changed this to $vx = vy$ in LT.

If some Xs are Ys, there are some terms common to the classes X and Y. Let those terms constitute a separate class V, to which there shall correspond a separate elective symbol v, then

$$v = xy. \tag{6}$$

And as v includes all terms common to the classes X and Y, we can indifferently interpret it, as Some Xs, or Some Ys.

TYPO: The phrase "as v includes" should be "as V includes".

5. To express the Proposition, Some Xs are not Ys.

In the last equation write $1 - y$ for y, and we have

$$v = x(1 - y), \tag{7}$$

the interpretation of v being indifferently Some Xs or Some not-Ys.

Some X's are not Y's is expressed by $v = x(1 - y)$. Boole changed this to $vx = v(1 - y)$ in LT.

The above equations involve the complete theory of categorical Propositions, and so far as respects the employment of analysis for the deduction of logical inferences, nothing more can be desired. **But it may be satisfactory to notice some particular forms deducible from the third and fourth equations, and susceptible of similar application.**

Equation (3) is on p. 17, and (4) is on p. 21. They are used to derive forms of (6) and (7) which will be used in analyzing syllogisms.

If we multiply the equation (6) by x, we have

$$vx = x^2 y = xy \quad \text{by (3)}.$$

Comparing with (6), we find

$$v = vx,$$

or

$$v(1 - x) = 0. \tag{8}$$

And multiplying (6) by y, and reducing in a similar manner, we have

$$v = vy,$$

or

$$v(1 - y) = 0. \tag{9}$$

Comparing (8) and (9),

$$vx = vy = v. \tag{10}$$

(10) is equivalent to the single equation $v = vxy$, which in turn is implied by $v = xy$.

And further comparing (8) and (9) with (4), we have as the equivalent of this system of equations the Propositions

$$\left.\begin{array}{l}\text{All Vs are Xs}\\ \text{All Vs are Ys}\end{array}\right\}.$$

The system (10) might be used to replace (6), or the single equation

$$vx = vy, \tag{11}$$

might be used, assigning to vx the interpretation, Some Xs, and to vy the interpretation, Some Ys. But it will be observed that

$v = vxy$ implies $vx = vy$. Thus $v = xy \Rightarrow v = vxy \Rightarrow vx = vy$.

It now seems best to encode 'Some X is Y' by $v = vxy$.

this system does not express quite so much as the single equation (6), from which it is derived. Both, indeed, express the Proposition, Some Xs are Ys, but the system (10) does not imply that the class V includes *all* the terms that are common to X and Y.

In like manner, from the equation (7) which expresses the Proposition Some Xs are not Ys, we may deduce the system

$$vx = v(1-y) = v, \tag{12}$$

(12) is equivalent to the single equation $v = vx(1-y)$, which is implied by $v = x(1-y)$.

in which the interpretation of $v(1-y)$ is Some not-Ys. Since in this case $vy = 0$, we must of course be careful not to interpret vy as Some Ys.

If we multiply the first equation of the system (12), viz.

$$vx = v(1-y),$$

by y, we have

$$vxy = vy(1-y);$$

$$\therefore vxy = 0, \tag{13}$$

$v = vx(1-y)$ implies $vx = v(1-y)$. Thus

$$\begin{aligned} v = x(1-y) \quad &\Rightarrow \quad v = vx(1-y) \\ &\Rightarrow \quad vx = v(1-y). \end{aligned}$$

It now seems best to encode "Some X is not Y" by $v = vx(1-y)$.

which is a form that will occasionally present itself. It is not necessary to revert to the primitive equation in order to interpret this, for the condition that vx represents Some Xs, shews us by virtue of (5), that its import will be

Some Xs are not Ys,

the subject comprising *all* the Xs that are found in the class V.

Universally in these cases, difference of form implies a difference of interpretation with respect to the auxiliary symbol v, and each form is interpretable by itself.

Further, these differences do not introduce into the Calculus a needless perplexity. It will hereafter be seen that **they give a precision and a definiteness to its conclusions, which could not otherwise be secured.**

Boole needed the alternate forms in the Table on p. 25 to derive the valid syllogisms.

Finally, we may remark that all the equations by which particular truths are expressed, are deducible from any one general equation, expressing any one general Proposition, from which those particular Propositions are necessary deductions.

This says that if $\Phi \therefore \Phi'$ is valid then from any equation ε expressing Φ one can deduce an equation ε' whose interpretation is Φ'.

This has been partially shewn already, but it is much more fully exemplified in the following scheme.

To illustrate the previous remark Boole started with the proposition that X and Y are equivalent, expressed by $x = y$. Then he proceeded to show that each of the consequences All X is Y, All Y is X, Some X is Y and some Some Y is X is the interpretation of an equation derived from $x = y$.

The general equation

$$x = y,$$

implies that the classes X and Y are equivalent, member for member; that every individual belonging to the one, belongs to the other also. Multiply the equation by x, and we have

$$x^2 = xy;$$
$$\therefore x = xy,$$

which implies, by (4), that all Xs are Ys. Multiply the same equation by y, and we have in like manner

$$y = xy;$$

the import of which is, that all Ys are Xs. Take either of these equations, the latter for instance, and writing it under the form

$$(1 - x)y = 0,$$

we may regard it as an equation in which y, an unknown quantity, is sought to be expressed in terms of x. **Now it will be shewn when we come to treat of the Solution of Elective Equations (and the result may here be verified by substitution) that the most general solution of this equation is**

$$y = vx,$$

which implies that All Ys are Xs, and that Some Xs are Ys. Multiply by x, and we have

$$vy = vx,$$

which indifferently implies that some Ys are Xs and some Xs are Ys, being the particular form at which we before arrived.

Checking that $y = vx$ is a solution, by substitution into $(1 - x)y = 0$, does not guarantee that it is the most general solution.

TYPO: Multiply by v (not by x) This is Boole's algebraic proof that 'All Y is X' implies 'Some X is Y', known as Conversion by Limitation.

For convenience of reference the above and some other results have been classified in the annexed Table, the first column of which contains **propositions**, the second **equations**, and the third the **conditions of final interpretation**. It is to be observed, that the **auxiliary equations** which are given in this column are not independent: they are implied either in the equations of the second column, or in the condition for (pagebreak in MAL)

The auxiliary equations play a significant role in Boole's method of showing that certain premises do not belong to a valid syllogism, by deducing 0=0.

the interpretation of v. But it has been thought better to write them separately, for greater ease and convenience. And it is further to be borne in mind, that although three different forms are given for the expression of each of the *particular* propositions, everything is really included in the first form.

TABLE.

The class X	x	
The class not-X	$1-x$	
All Xs are Ys All Ys are Xs	$x=y$	
All Xs are Ys	$x(1-y)=0$	
No Xs are Ys	$xy=0$	
All Ys are Xs Some Xs are Ys	$y=vx$	$vx =$ Some Xs $v(1-x)=0$.
No Ys are Xs Some not-Xs are Ys	$y=v(1-x)$	$v(1-x) =$ some not-Xs $vx=0$.
Some Xs are Ys	$v=xy$ or $vx=vy$ or $vx(1-y)=0$	$v =$ some Xs or some Ys $vx =$ some Xs, $vy =$ some Ys $v(1-x)=0$, $v(1-y)=0$.
Some Xs are not Ys	$v=x(1-y)$ or $vx=v(1-y)$ or $vxy=0$	$v =$ some Xs, or some not-Ys $vx =$ some Xs, $v(1-y) =$ some not-Ys $v(1-x)=0$, $vy=0$.

One cannot derive $y = vx$ from $y = yx$ in equational logic. However in first-order logic one can easily derive $(\exists v)(y = vx)$ from $y = yx$. Then a standard step would be to say 'choose a v such that $y = vx$' to get back into equational logic.

$y = vx$ implies $y = yx$ in Boole's Algebra—just multiply both sides of $y = vx$ by $1 - x$. Thus from $y = vx$ one can conclude 'All Y is X'.

To get v to behave like 'some' one needs an additional assumption, namely $y \neq 0$. As before derive $(\exists v)(y = vx)$, and then choose a suitable v so that $y = vx$. Then one has $y = vx = vy$, all $\neq 0$, so one can conclude that 'Some X is Y'.

Boole wanted his algebra to apply to the traditional Aristotelian logic. To properly handle 'some' the simplest solution is to require that all class variables X, Y, etc., refer to non-empty classes. This means adding the assertions $x \neq 0$, $y \neq 0$, etc., to the assumptions about Boole's Algebra.

Assuming Boole wanted to treat contraries not-X, not-Y, etc., on an equal footing with simply named classes, require also that no class symbol refers to the universe 1, that is, $x \neq 1$, etc.

These negated equations complicate a rigorous version of Boole's Algebra. In modern logic they are dropped, and one can no longer derive particular propositions from universal propositions. Thus, for example, Conversion by Limitation is rejected in modern logic. Boole's Algebra is more elegant when the non-empty, non-universe restrictions on class symbols are dropped. But then it does not faithfully reflect Aristotelian logic.

OF THE CONVERSION OF PROPOSITIONS.

A Proposition is said to be **converted** when its terms are transposed; when nothing more is done, this is called **simple conversion;** *e. g.*

No virtuous man is a tyrant, *is converted into*

No tyrant is a virtuous man.

Logicians also recognise conversion *per accidens*, or by limitation, *e. g.*

All birds are animals, *is converted into*

Some animals are birds.

And **conversion by *contraposition* or *negation*,** as

Every poet is a man of genius, *converted into*

He who is not a man of genius is not a poet.

In one of these three ways every Proposition may be illatively converted, viz. E **and** I **simply,** A **and** O **by negation,** A **and** E **by limitation.**

The primary canonical forms already determined for the expression of Propositions, are

All Xs are Ys,	$x(1-y) = 0,$	.... A.
No Xs are Ys,	$xy = 0,$	.... E.
Some Xs are Ys,	$v = xy,$	.... I.
Some Xs are not Ys,	$v = x(1-y)$	....O.

On examining these, we perceive that E and I are symmetrical with respect to x and y, so that x being changed into y, and y into x, the equations remain unchanged. Hence E and I may be interpreted into

No Ys are Xs,

Some Ys are Xs,

respectively. Thus we have the known rule of the Logicians, that **particular affirmative and universal negative Propositions admit of simple conversion.**

The acceptance of conversion per accidens implies that a universal subject of a categorical proposition must be non-empty. This kind of conversion was rejected by C.S. Peirce in 1880, making the algebra of logic more elegant.

Converting a proposition Φ into Ψ *illatively* means that (a) Ψ is a converted form of Φ, and (b) Φ implies Ψ.

The four kinds of categorical propositions in Aristotelian logic were introduced on p. 20 and their primary equational forms were given in the next two pages. Here Boole applies the four kinds A,E,I,O to the equational expressions as well as to the propositions.

Some of the RULES that follow are labelled so as to match Boole's Laws of Transformation on p. 30. The other RULES can be derived from these Laws.

RULE 3 (Simple Conversion).

The equations A and O may be written in the forms

$$(1-y)\{1-(1-x)\}=0,$$
$$v=(1-y)\{1-(1-x)\}.$$

Now these are precisely the forms which we should have obtained if we had in those equations changed x into $1-y$, and y into $1-x$, which would have represented the changing in the original Propositions of the Xs into not-Ys, and the Ys into not-Xs, the resulting Propositions being

All not-Ys are not-Xs,

Some not-Ys are not not-Xs. (a)

Or we may, by simply inverting the order of the factors in the second member of O, and writing it in the form

$$v=(1-y)x,$$

interpret it by I into

Some not-Ys are Xs,

which is really another form of (a). Hence follows the rule, that **universal affirmative and particular negative Propositions admit of negative conversion, or, as it is also termed, conversion by contraposition.**

RULE for Conversion by Contraposition.

The equations A and E, written in the forms

$$(1-y)x=0,$$
$$yx=0,$$

give on solution the respective forms

$$x=vy,$$
$$x=v(1-y),$$

the correctness of which may be shewn by substituting these values of x in the equations to which they belong, and observing that those equations are satisfied quite independently of the nature of the symbol v. The first solution may be interpreted into

Substitution can be used to verify that one has a solution, but it does not show that one has the most general solution.

Some Ys are Xs,

and the second into

Some not-Ys are Xs.

From which it appears that **universal-affirmative, and universal-negative Propositions are convertible by limitation, or, as it has been termed, *per accidens*.**

RULE for Conversion by Limitation.

The above are the laws of Conversion recognized by Abp. Whately. **Writers differ however as to the admissibility of negative conversion. The question depends on whether we will consent to use such terms as not-X, not-Y.** Agreeing with those who think that such terms ought to be admitted, even although they change the *kind* of the Proposition, I am constrained to observe that **the present classification of them is faulty and defective.** Thus the conversion of No Xs are Ys, into All Ys are not-Xs, though perfectly legitimate, is not recognised in the above scheme. It may therefore be proper to examine the subject somewhat more fully.

In *Formal Logic* (1847) De Morgan argued strenuously for the right to include the contrary of terms, like "not-X", in logic. (De Morgan wrote x for the contrary of X.)

Should we endeavour, from the system of equations we have obtained, **to deduce the laws not only of the conversion, but also of the general transformation of propositions,** we should be led to recognise the following distinct elements, each connected with a distinct mathematical process.

Boole's transformation laws are on p. 30.

1st. The negation of a term, *i. e.* the changing of X into not-X, or not-X into X.

2nd. The translation of a Proposition from one *kind* to another, as if we should change

All Xs are Ys into Some Xs are Ys   A into I,

which would be lawful; or

All Xs are Ys into No Xs are Y.   A into E,

which would be unlawful.

3rd. The **simple conversion** of a Proposition.

Three minimal ways to transform a categorical proposition into another categorical proposition. Every transformation that Boole considered can be expressed as a composition of these three.

The conditions in obedience to which these processes may lawfully be performed, may be deduced from the equations by which Propositions are expressed.

We have

All Xs are Ys .... $x(1-y)=0$.   A,

No Xs are Ys .... $xy=0$.   E.

Write E in the form

$$x\{1-(1-y)\}=0,$$

and it is interpretable by A into

All Xs are not-Ys,

so that we may change

No Xs are Ys into All Xs are not-Ys.

In like manner A interpreted by E gives

No Xs are not-Ys,

so that we may change

All Xs are Ys into No Xs are not-Ys.

From these cases we have the following **Rule: A universal-affirmative Proposition is convertible into a universal-negative, and, *vice versâ*, by negation of the predicate.**

Again, we have

Some Xs are Ys .... $v=xy$,

Some Xs are not Ys .... $v=x(1-y)$.

These equations only differ from those last considered by the presence of the term v. The same reasoning therefore applies, and we have the **Rule**—

A particular-affirmative proposition is convertible into a particular-negative, and *vice versâ*, by negation of the predicate.

Assuming the universal Propositions

All Xs are Ys .... $x(1-y)=0$,

No Xs are Ys .... $xy=0$.

Multiplying by v, we find

$$vx(1-y)=0,$$
$$vxy=0,$$

which are interpretable into

Some Xs are Ys .... I,

Some Xs are not Ys .... O.

The next three RULES are due to Boole. They are *not* conversions, as indicated by the examples he provided, but *lawful transformations*, defined on the next page. It would be clearer if the phrase "is convertible into" in these rules is replaced by "can be lawfully transformed into".

In the following, $\widehat{\mathrm{X}}$ can be either X or not-X, and $\widehat{\mathrm{Y}}$ can be either Y or not-Y.

RULE 1a:

All $\widehat{\mathrm{X}}$ *is* $\widehat{\mathrm{Y}}$ $\Rightarrow$ *No* $\widehat{\mathrm{X}}$ *is not-*$\widehat{\mathrm{Y}}$.

No $\widehat{\mathrm{X}}$ *is* $\widehat{\mathrm{Y}}$ $\Rightarrow$ *All* $\widehat{\mathrm{X}}$ *is not-*$\widehat{\mathrm{Y}}$.

RULE: 1b

Some $\widehat{\mathrm{X}}$ *is* $\widehat{\mathrm{Y}}$

$\Rightarrow$ *Some* $\widehat{\mathrm{X}}$ *is not not-*$\widehat{\mathrm{Y}}$.

Some $\widehat{\mathrm{X}}$ *is not* $\widehat{\mathrm{Y}}$

$\Rightarrow$ *Some* $\widehat{\mathrm{X}}$ *is not-*$\widehat{\mathrm{Y}}$.

Hence **a universal-affirmative is convertible into a particular-affirmative, and a universal-negative into a particular-negative without negation of subject or predicate.**

RULE 2:
All $\widehat{\mathrm{X}}$ *is* $\widehat{\mathrm{Y}}$ $\Rightarrow$ *Some* $\widehat{\mathrm{X}}$ *is* $\widehat{\mathrm{Y}}$.
No $\widehat{\mathrm{X}}$ *is* $\widehat{\mathrm{Y}}$ $\Rightarrow$ *Some* $\widehat{\mathrm{X}}$ *is not* $\widehat{\mathrm{Y}}$.

Combining the above with the already proved rule of **simple conversion,** we arrive at the following **system of independent laws of transformation.**

Boole's Laws of Transformation:

1st. An affirmative Proposition may be changed into its corresponding negative (A into E, or I into O), and *vice versa*, by negation of the predicate.

1st is RULE 1a and RULE 1b.

2nd. A universal Proposition may be changed into its corresponding particular Proposition, (A into I, or E into O).

2nd is RULE 2.

3rd. In a particular-affirmative, or universal-negative Proposition, the terms may be mutually converted.

3rd is RULE 3.

Wherein **negation of a term** is the changing of X into not-X, and *vice versâ*, and is not to be understood as affecting the *kind* of the Proposition.

Negating a term does not affect the *kind* of a proposition.

Every lawful transformation is reducible to the above rules. Thus we have

All Xs are Ys,	
No Xs are not-Ys	by 1st rule,
No not-Ys are Xs	by 3rd rule,
All not-Ys are not-Xs	by 1st rule,

which is an example of *negative conversion.* Again,

No Xs are Ys,	
No Ys are Xs	3rd rule,
All Ys are not-Xs	1st rule,

which is the case already deduced.

Given Boole's modified collection of categorical propositions, where not-X and X are treated equally:

THEOREM: A proposition Φ implies a proposition Ψ iff one can obtain Ψ by applying a suitable sequence of these three laws of transformation to Φ.

This is left to the reader to check. Only the third law involves conversion.

Boole omits the Laws of Transformation in LT, perhaps to avoid deriving Some not-Y is not-X from All X is Y?

See lines 11, 12 on p. 28 where this case is stated, not deduced.

OF SYLLOGISMS.

A Syllogism consists of three Propositions, the last of which, called **the conclusion,** is a logical consequence of the two former, called **the premises;** *e. g.*

Premises, $\begin{cases}\text{All Ys are Xs.}\\ \text{All Zs are Ys.}\end{cases}$

Conclusion, All Zs are Xs.

Every syllogism has three and only three terms, whereof that which is the subject of the conclusion is called **the *minor* term,** the predicate of the conclusion, **the *major* term,** and the remaining term common to both premises, **the *middle* term.** Thus, in ths above formula, Z is the minor term, X the major term, Y the middle term.

The **figure of a syllogism** consists in the situation of the middle term with respect to the terms of the conclusion. The varieties of figure are exhibited in the annexed scheme.

1st Fig.	2nd Fig.	3rd Fig.	4th Fig.
YX	XY	YX	XY
ZY	ZY	YZ	YZ
ZX	ZX	ZX	ZX

When we designate the three propositions of a syllogism by their usual symbols (*A, E, I, O*), and in their actual order, we are said to determine **the mood of the syllogism.** Thus the syllogism given above, by way of illustration, belongs to the mood AAA in the first figure.

The moods of all syllogisms commonly received as valid, are represented by the vowels in the following mnemonic verses.

Fig. 1.—bArbArA, cElArEnt, dArII, fErIO que prioris.

Fig. 2.—cEsArE, cAmEstrEs, fEstIno, bArOkO, secundæ.

Fig. 3.—Tertia dArAptI, dIsAmIs, dAtIsI, fElAptOn,
bOkArdO, fErIsO, habet: quarta insuper addit.

Fig. 4.—brAmAntIp, cAmEnEs, dImArIs, fEsapO, frEsIsOn.

The equation by which we express any Proposition concerning the classes X and Y, is an equation between the symbols x and y, and the equation by which we express any (pagebreak in MAL)

The abstract, which fills most of this page, gives a summary of Aristotelian syllogisms.

If the conclusion follows from the premises, then one has a *valid* syllogism. Otherwise the syllogism is *invalid.*

TYPO: in the above

Boole failed to capitalize two instances of the relevant vowels in these verses, namely in fEstInO and fEsApO. In 1847 De Morgan ([15], p. 130) said of these mnemonic verses for syllogisms: "the magic words by which they have been denoted for many centuries, words which I take to be more full of meaning than any that ever were made".

Proposition concerning the classes Y and Z, is an equation between the symbols y and z. If from two such equations we eliminate y, the result, if it do not vanish, will be an equation between x and z, and will be interpretable into a Proposition concerning the classes X and Z. And it will then constitute the third member, or Conclusion, of a Syllogism, of which the two given Propositions are the premises.

Boole's Elimination Theorem uses elective functions that are usually only partially defined. He could have avoided this by using the forms
$ay = b,\ a'y = b'\ \therefore\ ab' = a'b.$

The result of the elimination of y from the equations

$$\begin{aligned} ay + b &= 0, \\ a'y + b' &= 0, \end{aligned} \tag{14}$$

is the equation

$$ab' - a'b = 0. \tag{15}$$

Boole borrowed this elimination theorem from the algebra of numbers. The best possible elimination theorem for his algebra of logic appears in LT—using this, (15) is replaced by (15′):
$(b^2 + b'^2)\big((a+b)^2 + (a'+b')^2\big) = 0.$
(15) is a consequence of (15′) in Boole's Algebra.

Now the equations of Propositions being of the first order with reference to each of the variables involved, all the cases of elimination which we shall have to consider, will be reducible to the above case, the constants a, b, a', b', being replaced by functions of x, z, and the auxiliary symbol v.

Below is a table of the equations (see p. 26) of the different categorical propositions $\Phi(\mathrm{X}, \mathrm{Y})$, along with the coefficients a, b when the equations are written in the form $ay + b = 0$:

	Equation	a	b
A	$x(1-y) = 0$	x	$-x$
A	$y(1-x) = 0$	$1-x$	0
E	$xy = 0$	x	0
I	$v = xy$	x	$-v$
O	$v = x(1-y)$	x	$v - x$
O	$v = y(1-x)$	$1-x$	$-v$

As to the choice of equations for the expression of our premises, the only restriction is, that the equations must not *both* be of the form $ay = 0$, for in such cases elimination would be impossible. When both equations are of this form, it is necessary to solve one of them, and it is indifferent which we choose for this purpose. If that which we select is of the form $xy = 0$, its solution is

$$y = v(1-x), \tag{16}$$

if of the form $(1-x)y = 0$, the solution will be

$$y = vx, \tag{17}$$

and these are the only cases which can arise. The reason of this exception will appear in the sequel. **For the sake of uniformity we shall, in the expression of particular propositions, confine ourselves to the forms**

By "the reason of" Boole means "the need for". See his Class 2nd, pp. 35–36.

In his analysis of syllogisms, Boole used secondary forms for particular propositions.

$vx = vy$, Some Xs are Ys,

$vx = v(1-y)$, Some Xs are not Ys,

These have a closer analogy with (16) and (17), than the other forms which might be used.

Between the forms about to be developed, and the Aristotelian canons, some points of difference will occasionally be observed, of which it may be proper to forewarn the reader.

To the right understanding of these it is proper to remark, that **the essential structure of a Syllogism is, in some measure, arbitrary.** Supposing the order of the premises to be fixed, and the distinction of the major and the minor term to be thereby determined, it is purely a matter of choice which of the two shall have precedence in the Conclusion. Logicians have settled this question in favour of the minor term, but it is clear, that this is a convention. Had it been agreed that the major term should have the first place in the conclusion, a logical scheme might have been constructed, less convenient in some cases than the existing one, but superior in others. What it lost in *barbara*, it would gain in *bramantip*. **Convenience is *perhaps* in favour of the adopted arrangement,*** **but it is to be remembered that it is *merely* an arrangement.**

Now the method we shall exhibit, not having reference to one scheme of arrangement more than to another, will always give the more general conclusion, regard being paid only to its abstract lawfulness, considered as a result of pure reasoning. And therefore we shall sometimes have presented to us the spectacle of conclusions, which a logician would pronounce informal, but never of such as a reasoning being would account false.

The Aristotelian canons, however, beside restricting the *order* of the terms of a conclusion, limit their nature also;—and this limitation is of more consequence than the former. We may, by a change of figure, replace the particular conclusion . . . . (pagebreak in MAL)

Regarding the limited nature of the terms in the conclusion, Boole is mainly concerned that one cannot have a contrary like 'not-Z' as the subject.

*The contrary view was maintained by Hobbes. The question is very fairly discussed in Hallam's *Introduction to the Literature of Europe*, vol. III. p. 309. In the rhetorical use of Syllogism, the advantage appears to rest with the rejected form.

of *bramantip* by the general conclusion of *barbara*; but we cannot thus reduce to rule such inferences, as

Some not-Xs are not Ys.

Yet there are cases in which such inferences may lawfully be drawn, and in unrestricted argument they are of frequent occurrence. **Now if an inference of this, or of any other kind, is lawful in itself, it will be exhibited in the results of our method.**

We may by restricting the canon of interpretation confine our expressed results within the limits of the scholastic logic; but this would only be to restrict ourselves to the use of a part of the conclusions to which our analysis entitles us.

The classification we shall adopt will be purely mathematical, and we shall afterwards consider the logical arrangement to which it corresponds. It will be sufficient, for reference, to name the premises and the Figure in which they are found.

Class 1st.—Forms in which v does not enter.

Those which admit of an inference are AA, EA, Fig. 1; AE, EA, Fig. 2; AA, AE, Fig. 4.

Ex. AA, Fig. 1, and, by mutation of premises (change of order), AA, Fig. 4.

$$\begin{array}{lll} \text{All Ys are Xs,} & y(1-x)=0, & \text{or } (1-x)y=0, \\ \text{All Zs are Ys,} & z(1-y)=0, & \text{or } \quad zy-z=0. \end{array}$$

Eliminating y by (13) we have

$$z(1-x)=0,$$
$$\therefore \text{ All Zs are Xs.}$$

A convenient mode of effecting the elimination, is to write the equation of the premises, so that y shall appear only as a factor of one member in the first equation, and only as a factor of the opposite member in the second equation, and then to multiply the equations, omitting the y. This method we shall adopt.

Boole said a convenient way to eliminate y in the equations for the premises $\Phi_1(\mathrm{X},\mathrm{Y})$ and $\Phi_2(\mathrm{Y},\mathrm{Z})$ would be to write the equations in the form $\begin{array}{ccc} ay & = & b \\ d & = & cy \end{array}$ and then multiply the corresponding sides, omitting y, to obtain $ad=bc$.

Ex. AE, Fig. 2, and, by mutation of premises, EA, Fig, 2.

$$\begin{array}{lrcl} \text{All Xs are Ys,} & x(1-y)=0, & \text{or} & x=xy \\ \text{No Zs are Ys,} & zy=0, & & \underline{zy=0} \\ & & & zx=0 \end{array}$$

$\therefore$ No Zs are Xs.

The only case in which there is no inference is AA, Fig. 2,

$$\begin{array}{lrl} \text{All Xs are Ys,} & x(1-y)=0, & x=xy \\ \text{All Zs are Ys,} & z(1-y)=0, & \underline{zy=z} \\ & & xz=xz \\ & & \therefore\ 0=0. \end{array}$$

If one makes the plausible assumption that one can replace given premises by equivalent premises, and use conclusions to the latter as valid conclusions to the original premises, then replacing the premises 'All X is Y', 'All Z is Y' by their contrapositives 'All not-Y is not-X ', 'All not-Y is not-Z' leads to the conclusion 'Some not-X is not-Z'. Why did Boole not allow this?

Class 2nd.—When v is introduced by the solution of an equation.

The lawful cases directly or indirectly* **determinable by the Aristotelian Rules** are AE, Fig. 1; AA, AE, EA, Fig. 3; EA, Fig. 4.

The lawful cases not so determinable, are EE, Fig. 1; EE, Fig. 2; EE, Fig. 3; EE, Fig. 4.

Ex. AE, Fig. 1, and, by mutation of premises, EA, Fig. 4.

$$\begin{array}{lrl} \text{All Ys are Xs,} & y(1-x)=0, & y=vx \quad (a) \\ \text{No Zs are Ys,} & zy=0, & \underline{0=zy} \\ & & 0=vzx \end{array}$$

$\therefore$ Some Xs are not Zs.

The reason why **we cannot interpret $vzx=0$ into Some Zs are not-Xs**, is that by the very terms of the first equation (a) the interpretation of vx is fixed, as Some Xs; **v is regarded as the representative of Some, only with reference to the class X**.

An important restriction as to when v can be read as 'some'.

*We say *directly* or *indirectly*, mutation or conversion of premises being in some instances required. Thus, AE (fig. 1) is resolvable by Fesapo (fig. 4), or by Ferio (fig. 1). **Aristotle and his followers rejected the fourth figure as only a modification of the first, but this being a mere question of form, either scheme may be termed Aristotelian.**

For the reason of our employing a solution of one of the primitive equations, see the remarks on (16) and (17). Had we solved the second equation instead of the first, we should have had

$$\begin{aligned} (1-x)y &= 0, \\ v(1-z) &= y, \quad (a) \\ v(1-z)(1-x) &= 0, \quad (b) \end{aligned}$$

$\therefore$ Some not-Zs are Xs.

Here it is to be observed, that the second equation (a) fixes the meaning of $v(1-z)$, as Some not-Zs. The full meaning of the result (b) is, that all the not-Zs which are found in the class Y are found in the class X, and it is evident that this could not have been expressed in any other way.

Ex. 2. AA, Fig. 3.

All Ys are Xs, $\quad y(1-x)=0, \quad y=vx$

All Ys are Zs, $\quad y(1-z)=0, \quad 0=y(1-z)$

$$\overline{\quad 0=vx(1-z) \quad}$$

$\therefore$ Some Xs are Zs.

Had we solved the second equation, we should have had as our result, Some Zs are Xs. The form of the final equation particularizes what Xs or what Zs are referred to, and this remark is general.

The following, EE, Fig. 1, and, by mutation, EE, Fig. 4, is an example of a lawful case not determinable by the Aristotelian Rules.

No valid EE cases can be handled by the Aristotelian Rules.

No Ys are Xs, $\quad xy=0, \quad 0=xy$

No Zs are Ys, $\quad zy=0, \quad y=v(1-z)$

$$\overline{\quad 0=v(1-z)x \quad}$$

$\therefore$ Some not-Zs are not Xs.

Class 3rd.—When v is met with in one of the equations, but not introduced by solution.

The lawful cases determinable *directly* or *indirectly* by the Aristotelian Rules, are AI, EI, Fig. 1; AO, EI, OA, IE, Fig. 2; AI, AO, EI, EO, IA, IE, OA, OE, Fig. 3; IA, IE, Fig. 4.

The two cases EO and OE Fig. 3 actually belong in the 'not so determinable' list.

Those not so determinable are OE, Fig. 1; EO, Fig. 4.

The cases in which no inference is possible, are AO, EO, IA, IE, OA, Fig. 1; AI, EO, IA, OE, Fig. 2; OA, OE, AI, EI, AO, Fig. 4.

EI Fig. 4 belongs in the 'directly determinable' list.

Ex. 1. AI, Fig. 1, and, by mutation, IA, Fig. 4.

All Ys are Xs, $\quad y(1-x)=0$

Some Zs are Ys, $\quad vz=vy$

$$vz(1-x)=0$$

$\therefore$ Some Zs are Xs.

Ex. 2. AO, Fig. 2, and, by mutation, OA, Fig. 2.

All Xs are Ys, $\quad x(1-y)=0, \quad x=xy$

Some Zs are not Ys, $\quad vz=v(1-y), \quad vy=v(1-z)$

$$vx=vx(1-z)$$

$$vxz=0$$

$\therefore$ Some Zs are not Xs.

The interpretation of vz as Some Zs, is implied, it will be observed, in the equation $vz=v(1-y)$ considered as representing the proposition Some Zs are not Ys.

The cases not determinable by the Aristotelian Rules are OE, Fig. 1, and, by mutation, EO, Fig. 4.

Some Ys are not Xs, $\quad vy=v(1-x)$

No Zs are Ys, $\quad 0=zy$

$$0=v(1-x)z$$

$\therefore$ Some not-Xs are not Zs.

The equation of the first premiss here permits us to interpret $v(1-x)$, but it does not enable us to interpret vz.

Of cases in which no inference is possible, we take as examples—

AO, Fig. 1, and, by mutation, OA, Fig. 4,

All Ys are Xs, $y(1-x)=0$,

Some Zs are not Ys, $vz = v(1-y)$ (a)

$$\begin{array}{r} y(1-x)=0 \\ v(1-z)=vy \\ \hline v(1-z)(1-x)=0 \quad (b) \\ 0=0 \end{array}$$

since the auxiliary equation in this case is $v(1-z)=0$.

Practically it is not necessary to perform this reduction, but it is satisfactory to do so. The equation (a), it is seen, defines vz as Some Zs, but it does not define $v(1-z)$, so that we might stop at the result of elimination (b), and content ourselves with saying, that it is not interpretable into a relation between the classes X and Z.

Take as a second example AI, Fig. 2, and, by mutation, IA, Fig. 2,

All Xs are Ys, $x(1-y)=0$,

Some Zs are Ys, $vz = vy$,

$$\begin{array}{r} x=xy \\ vy=vz \\ \hline vx=vxz \\ v(1-z)x=0 \\ 0=0, \end{array}$$

the auxiliary equation in this case being $v(1-z)=0$.

Indeed in every case in this class, in which no inference is possible, the result of elimination is reducible to the form $0=0$. Examples therefore need not be multiplied.

Class 4th.—When v enters into both equations.

No inference is possible in any case, but there exists a distinction among the unlawful cases which is peculiar to this class. The two divisions are,

1st. When the result of elimination is reducible by the auxiliary equations to the form $0=0$. The cases are II, OI, (pagebreak in MAL)

Fig. 1; II, OO, Fig. 2; II, IO, OI, OO, Fig. 3; II, IO, Fig. 4.

2nd. When the result of elimination is not reducible by the auxiliary equations to the form $0 = 0$.

The cases are IO, OO, Fig. 1; IO, OI, Fig. 2; OI, OO, Fig. 4.

Let us take as an example of the former case, II, Fig. 3.

Boole was careful to use distinct v for distinct premises in MAL. In LT this only happened in Chap. XV.

Some Xs are Ys,	$vx = vy,$	$vx = vy$
Some Zs are Ys,	$v'z = v'y,$	$v'y = v'z$
		$vv'x = vv'z$

Now the auxiliary equations $v(1-x) = 0$, $v'(1-z) = 0$, give

$$vx = v, \quad v'z = v'.$$

Substituting we have

$$vv' = vv',$$
$$\therefore 0 = 0.$$

As an example of the latter case, let us take IO, Fig. 1,

Some Ys are Xs,	$vy = vx,$	$vy = vx$
Some Zs are not Ys,	$v'z = v'(1-y),$	$v'(1-z) = v'y$
		$vv'(1-z) = vv'x$

Now the auxiliary equations being $v(1-x) = 0$, $v'(1-z) = 0$, the above reduces to $vv' = 0$. It is to this form that all similar cases are reducible. Its interpretation is, that the classes v and v' have no common member, as is indeed evident.

The above classification is purely founded on mathematical distinctions. We shall now inquire what is the logical division to which it corresponds.

The lawful cases of the first class comprehend all those in which, from two universal premises, a universal conclusion may be drawn. We see that they include the premises of *barbara* and *celarent* in the first figure, of *cesare* and *camestres* in the second, and of *bramantip* and *camenes* in the fourth. . . (pagebreak in MAL)

The premises of *bramantip* are included, because they admit of an universal conclusion, although not in the same figure.

The lawful cases of the second class are those in which a particular conclusion only is deducible from two universal premises.

The lawful cases of the third class are those in which a conclusion is deducible from two premises, one of which is universal and the other particular.

The fourth class has no lawful cases.

Among the cases in which no inference of any kind is possible, we find six in the fourth class distinguishable from the others by the circumstance, that the result of elimination does not assume the form $0 = 0$. The cases are

$$\left\{\begin{array}{l}\text{Some Ys are Xs,}\\ \text{Some Zs are not Ys,}\end{array}\right\} \quad \left\{\begin{array}{l}\text{Some Ys are not Xs,}\\ \text{Some Zs are not Ys,}\end{array}\right\} \quad \left\{\begin{array}{l}\text{Some Xs are Ys,}\\ \text{Some Zs are not Ys,}\end{array}\right\}$$

and the three others which are obtained by mutation of premises.

It might be presumed that some logical peculiarity would be found to answer to the mathematical peculiarity which we have noticed, and in fact there exists a very remarkable one. If we examine each pair of premises in the above scheme, we shall find that there *is virtually* no middle term, i. e. *no medium of comparison*, in any of them. Thus, in the first example, the individuals spoken of in the first premiss are asserted to belong to the class Y, but those spoken of in the second premiss are *virtually* asserted to belong to the class not-Y: nor can we by any lawful transformation or conversion alter this state of things. The comparison will still be made with the class Y in one premiss, and with the class not-Y in the other.

Now in every case beside the above six, there will be found a middle term, either expressed or implied. I select two of the most difficult cases.

In AO, Fig. 1, viz.

All Ys are Xs,
Some Zs are not Ys,

we have, by *negative conversion* of the first premiss,

All not-Xs are not-Ys,
Some Zs are not Ys,

and the middle term is now seen to be not-Y.

Again, in EO, Fig. 1,

No Ys are Xs,
Some Zs are not Ys,

a proved conversion of the first premiss (see *Conversion of Propositions*), gives

All Xs are not-Ys,
Some Zs are not-Ys,

and the middle term, the true medium of comparison, is plainly not-Y, although as the not-Ys in the one premiss *may be* different from those in the other, no conclusion can be drawn.

The mathematical condition in question, therefore,—the irreducibility of the final equation to the form $0 = 0$,—adequately represents the logical condition of there being no middle term, or common medium of comparison, in the given premises.

I am not aware that the distinction occasioned by the presence or absence of a middle term, in the strict sense here understood, has been noticed by logicians before. The distinction, though real and deserving attention, is indeed by no means an obvious one, and it would have been unnoticed in the present instance but for the peculiarity of its mathematical expression.

See the Postscript, page 82, where he makes a correction, attributing this observation to De Morgan.

What appears to be novel in the above case is the proof of the existence of combinations of premises in which there (pagebreak in MAL) is absolutely no medium of comparison. When such a medium of comparison, or true middle term, does exist, the condition that its quantification in both premises together shall exceed its quantification as a single whole, has been ably and clearly shewn by Professor De Morgan to be necessary to lawful inference (*Cambridge Memoirs*, Vol. VIII. Part 3). And this is undoubtedly the true principle of the Syllogism, viewed from the standing-point of Arithmetic.

I have said that it would be possible to impose conditions of interpretation which should **restrict the results of this calculus to the Aristotelian forms.** Those conditions would be,

1st. That we should agree not to interpret the forms $v(1-x)$, $v(1-z)$.

2ndly. That we should agree to reject every interpretation in which the order of the terms should violate the Aristotelian rule.

Or, instead of the second condition, it might be agreed that, the conclusion being determined, the order of the premises should, if necessary, be changed, so as to make the syllogism formal.

From the *general* character of the system it is indeed plain, that it may be made to represent any conceivable scheme of logic, by imposing the conditions proper to the case contemplated.

We have found it, in a certain class of cases, to be necessary to replace the two equations expressive of universal Propositions, by their solutions; and it may be proper to remark, that it would have been allowable in all instances to have done this,* so that **every case of the Syllogism, without ex- . .** (pagebreak in MAL)

Noting that in a few cases of algebraically demonstrating a syllogism he needed to replace an equation by its solution, Boole claimed that one could always use the solution form of the equations to express universal propositions.

*It may be satisfactory to illustrate this statement by an example. In *Barbara*, we should have

All Ys are Xs, $\quad y = vx$

All Zs are Ys, $\quad z = v'y$

$$z = vv'x$$

$\therefore$ All Zs are Xs.

(This footnote continues till p. 45.)

ception, might have been treated by equations comprised in the general forms

$$
\begin{array}{llrl}
y = vx, & \text{or} & y - vx = 0 & \text{A,} \\
y = v(1-x), & \text{or} & y + vx - v = 0 & \text{E,} \\
vy = vx, & & vy - vx = 0 & \text{I,} \\
vy = v(1-x), & & vy + vx - v = 0 & \text{O.}
\end{array}
$$

In the Postscript, p. 82, he said that these equations for A, E, I, O are preferable to his original choices in the chapter Of Expression and Interpretation. The first column gives the equations used in LT.

(footnote from p. 42 continues)

Or, we may multiply the resulting equation by $1 - x$, which gives

$$z(1-x) = 0,$$

whence the same conclusion, All Zs are Xs.

Some additional **examples of the application of the system of equations in the text to the demonstration of general theorems**, may not be inappropriate.

Let y be the term to be eliminated, and let x stand indifferently for either of the other symbols, then each of the equations of the premises of any given syllogism may be put in the form

$$ay + bx = 0, \qquad (\alpha)$$

if the premiss is affirmative, and in the form

$$ay + b(1-x) = 0, \qquad (\beta)$$

if it is negative, a and b being either constant, or of the form $\pm v$. To prove this in detail, let us examine each kind of proposition, making y successively subject and predicate.

A, All Ys are Xs,	$y - vx = 0,$	(γ)
All Xs are Ys,	$x - vy = 0,$	(δ)
E, No Ys are Xs,	$xy = 0,$	
No Xs are Ys,	$y - v(1-x) = 0,$	(ε)
I, Some Xs are Ys,		
Some Ys are Xs,	$vx - vy = 0,$	(ζ)
O, Some Ys are not Xs,	$vy - v(1-x) = 0,$	(η)
Some Xs are not Ys,	$vx = v(1-y),$	
	$\therefore vy - v(1-x) = 0.$	(θ)

The affirmative equations (γ), (δ) and (ζ), belong to (α), and the negative equations (ε), (η) and (θ), to (β). It is seen that the two last negative equations are alike, but there is a difference of interpretation. In the former

$$v(1-x) = \text{Some not-Xs},$$

in the latter,

$$v(1-x) = 0.$$

The utility of the two general forms of reference, (α) and (β), will appear from the following application.

1st. *A conclusion drawn from two affirmative propositions* is itself affirmative.

By (α) we have for the given propositions,

$$
\begin{aligned}
ay + bx &= 0, \\
a'y + b'z &= 0,
\end{aligned}
$$

Perhaps the system we have actually employed is better, as distinguishing the cases in which v only *may* be employed,

---

and eliminating

$$ab'z - a'bx = 0,$$

which is of the form (α). Hence, if there is a conclusion, it is affirmative.

(footnote from pp. 42,43 continues)

2nd. ***A conclusion drawn from an affirmative and a negative proposition is negative.***

By (α) and (β), we have for the given propositions

$$ay + bx = 0,$$
$$a'y + b'(1-z) = 0,$$
$$\therefore\ a'bx - ab'(1-z) = 0,$$

which is of the form (β). Hence the conclusion, if there is one, is negative.

3rd. ***A conclusion drawn from two negative premises will involve a negation, (not-X, not-Z) in both subject and predicate, and will therefore be inadmissible in the Aristotelian system, though just in itself.***

For the premises being

$$ay + b(1-x) = 0,$$
$$a'y + b'(1-z) = 0,$$

the conclusion will be

$$ab'(1-z) - a'b(1-x) = 0,$$

which is only interpretable into a proposition that has a negation in each term.

4th. ***Taking into account those syllogisms only, in which the conclusion is the most general, that can be deduced from the premises,—if, in an Aristotelian syllogism, the minor premises be changed in quality (from affirmative to negative or from negative to affirmative), whether it be changed in quantity or not, no conclusion will be deducible in the same figure.***

An Aristotelian proposition does not admit a term of the form not-Z in the subject, —Now on changing the quantity of the minor proposition of a syllogism, we transfer it from the general form

TYPO: changing the quality

$$ay + bz = 0,$$

to the general form

$$a'y + b'(1-z) = 0,$$

see (α) *and* (β), or *vice versâ*. And therefore, in the equation of the conclusion, there will be a change from z to $1-z$, or *vice versâ*. But this is equivalent to the change of Z into not-Z, or not-Z into Z. Now the subject of the original conclusion must have involved a Z and not a not-Z, therefore the subject of the new conclusion will involve a not-Z, and the conclusion will not be admissible in the Aristotelian forms, except by conversion, which would render necessary a change of Figure.

Now **the conclusions of this calculus are always the most general that can be drawn,** and therefore the above demonstration must not be supposed to extend to a syllogism, in which a particular conclusion is deduced, when a universal one is possible. This is the case with *bramantip* only, among the Aristotelian forms, and therefore the transformation of *bramantip* into *camenes*, and *vice versâ*, is the case of restriction contemplated in the preliminary statement of the theorem.

An unsubstantiated claim

from those in which it *must*. But for the demonstration of certain general properties of the Syllogism, the above system is, from its simplicity, and from the mutual analogy of its forms, very convenient. We shall apply it to the following theorem.*

Given the three propositions of a Syllogism, prove that **there is but one order in which they can be legitimately arranged**, and determine that order.

Graves Theorem for AC-syllogisms, followed by its "proof", based on the assumption that a weak form of the elimination theorem gives the most general conclusion.

All the forms above given for the expression of propositions, are particular cases of the general form,

$$a + bx + cy = 0.$$

5th. ***If for the minor premiss of an Aristotelian syllogism, we substitute its contradictory, no conclusion is deducible in the same figure.***

(footnote that started on p. 42 finishes)

It is here only necessary to examine the case of *bramantip*, all the others being determined by the last proposition.

On changing the minor of *bramantip* to its contradictory, we have AO, Fig. 4, and this admits of no legitimate inference.

Hence the theorem is true without exception. Many other general theorems may in like manner be proved.

---

*This elegant theorem was communicated by the **Rev. Charles Graves**, Fellow and Professor of Mathematics in Trinity College, Dublin, **to whom the Author desires further to record his grateful acknowledgments** for a very judicious examination of the former portion of this work, and for some new applications of the method. The following example of Reduction *ad impossibile* is among the number:

I am not aware of any surviving documents detailing Graves' comments on Boole's work.

Reducend Mood,	All Xs are Ys,	$1-y=v'(1-x)$
Baroko	Some Zs are not Ys	$vz=v(1-y)$
	Some Zs are not Xs	$vz=vv'(1-x)$
Reduct Mood,	All Xs are Ys	$1-y=v'(1-x)$
Barbara	All Zs are Xs	$z(1-x)=0$
	All Zs are Ys	$z(1-y)=0.$

The conclusion of the reduct mood is seen to be the contradictory of the suppressed minor premiss. Whence, &c. It may just be remarked that **the mathematical test of contradictory propositions is, that on eliminating one elective symbol between their equations, the other elective symbol vanishes.** The *ostensive* reduction of *Baroko* and *Bokardo* involves no difficulty.

Professor Graves suggests the employment of the equation $x = vy$ for the primary expression of the Proposition All Xs are Ys, and remarks, that on multiplying both members by $1-y$, we obtain $x(1-y) = 0$, the equation from which we set out in the text, and of which the previous one is a solution.

Assume then for the premises of the given syllogism, the equations

$$a + bx + cy = 0, \tag{18}$$

$$a' + b'z + c'y = 0, \tag{19}$$

then, eliminating y, we shall have for the conclusion

$$ac' - a'c + bc'x - b'cz = 0. \tag{20}$$

Now taking this as one of our premises, and either of the original equations, suppose (18), as the other, if by elimination of a common term x, between them, we can obtain a result equivalent to the remaining premiss (19), it will appear that there are more than one order in which the Propositions may be lawfully written; but if otherwise, one arrangement only is lawful.

If one multiplies (18) by c', and then subtracts (20) from it, one has the elimination result $c(a'+b'z+c'y) = 0$ which is more general than (21).

Effecting then the elimination, we have

$$bc(a' + b'z + c'y) = 0, \tag{21}$$

which is equivalent to (19) multiplied by a factor bc. Now on examining the value of this factor in the equations A, E, I, O, we find it in each case to be v or $-v$. But it is evident, that if an equation expressing a given Proposition be multiplied by an extraneous factor, derived from another equation, its interpretation will either be limited or rendered impossible. Thus there will either be no result at all, or the result will be a *limitation* of the remaining Proposition.

The first appearance of "$-$" as a unary operation.

If, however, one of the original equations were

$$x = y, \quad \text{or} \quad x - y = 0,$$

the factor bc would be -1, and would *not* limit the interpretation of the other premiss. Hence if the first member of a syllogism should be understood to represent the double proposition All Xs are Ys, and All Ys are Xs, it would be indifferent in what order the remaining Propositions were written.

The second appearance of "$-$" as a unary operation.

A more general form of the above investigation would be, to express the premises by the equations

$$a + bx + cy + dxy = 0, \qquad (22)$$
$$a' + b'z + c'y + d'zy = 0. \qquad (23)$$

After the double elimination of y and x we should find

$$(bc - ad)(a' + b'z + c'y + d'zy) = 0;$$

and it would be seen that the factor $bc - ad$ must in every case either vanish or express a limitation of meaning.

The determination of the order of the Propositions is sufficiently obvious.

OF HYPOTHETICALS.

A hypothetical Proposition is defined to be *two or more categoricals united by a copula* (or conjunction), and **the different kinds of hypothetical Propositions** are named from their respective conjunctions, viz. conditional (if), disjunctive (either, or), &c.

In conditionals, that categorical Proposition from which the other results is called **the *antecedent*,** that which results from it **the *consequent*.**

Of **the conditional syllogism** there are two, and only two formulæ.

1st. The constructive,

If A is B, then C is D,
But A is B, therefore C is D.

2nd. The Destructive,

If A is B, then C is D,
But C is not D, therefore A is not B.

A dilemma is a complex conditional syllogism, with several antecedents in the major, and a disjunctive minor.

The first paragraph is essentially the same as one finds in Whately's *Elements of Logic*. It is easy, as Boole noted (on p. 57), to make additions to the standard hypothetical syllogisms (which are stated on pp. 56–57). Hypothetical propositions will be replaced in LT by *secondary propositions*, defined simply as propositions about propositions.

If we examine either of the forms of conditional syllogism above given, we shall see that the validity of the argument does not depend upon any considerations which have reference to the terms A, B, C, D, considered as the representatives of individuals or of classes. We may, in fact, represent the Propositions A is B, C is D, by the arbitrary symbols X and Y respectively, and express our syllogisms in such forms as the following:

If X is true, then Y is true,
But X is true, therefore Y is true.

Thus, **what we have to consider is** not objects and classes of objects, but **the truths of Propositions,** namely, of those elementary Propositions which are embodied in the terms of our hypothetical premises.

To the symbols X, Y, Z, representative of Propositions, we may appropriate the elective symbols x, y, z, in the following sense.

The hypothetical Universe, 1, shall comprehend all conceivable cases and conjunctures of circumstances.

The elective symbol x attached to any subject expressive of such cases **shall select those cases in which the Proposition X is true**, and similarly for Y and Z.

If we confine ourselves to the contemplation of a given proposition X, and hold in abeyance every other consideration, **then two cases only are conceivable, viz. first that the given Proposition is true, and secondly that it is false.*** As these cases together make up the **Universe of the Proposition**, and as the former is determined by the elective symbol x, the latter is determined by the symbol $1 - x$.

But if other considerations are admitted, each of these cases will be resolvable into others, individually less extensive, the

A clearer formulation is, given a collection $\mathfrak{X}$ of propositional variables, to define a *case* to be a mapping $\mu : \mathfrak{X} \to \{T, F\}$, that is, an assignment of truth-values to the propositional variables. Then the hypothetical universe is $\{T, F\}^{\mathfrak{X}}$, and for $\mathrm{X} \in \mathfrak{X}$, the elective operation x selects the cases μ for which $\mu(\mathrm{X}) = T$, and $1 - x$ selects the cases μ for which $\mu(\mathrm{X}) = F$.

Boole does not make a clear connection between the Universe of a Proposition and the Hypothetical Universe.

By "other considerations" he meant other propositional variables, like Y, Z, etc.

*It was upon the obvious principle that a Proposition is either true or false, that the Stoics, applying it to assertions respecting future events, endeavoured to establish the doctrine of Fate. It has been replied to their argument, that it involves "an abuse of the word *true*, the precise meaning of which is id quod *res est*. An assertion respecting the future is neither true nor false."—*Copleston on Necessity and Predestination*, p. 36. Were the Stoic axiom, however, presented under the form, It is either certain that a given event will take place, or certain that it will not; the above reply would fail to meet the difficulty. The proper answer would be, that no merely verbal definition can settle the question, what is the actual course and constitution of Nature. When we affirm that it is either certain that an event will take place, or certain that it will not take place, we tacitly assume that the order of events is necessary, that the Future is but an evolution of the Present; so that the state of things which is, completely determines that which shall be. But this (at least as respects the conduct of moral agents) is the very question at issue. Exhibited under its proper form, the Stoic reasoning does not involve an abuse of terms, but a *petitio principii*.

It should be added, that enlightened advocates of the doctrine of Necessity in the present day, viewing the end as appointed only in and through the means, justly repudiate those practical ill consequences which are the reproach of Fatalism.

number of which will depend upon the number of foreign considerations admitted. Thus if we associate the Propositions X and Y, the total number of conceivable cases will be found as exhibited in the following scheme.

	Cases.	Elective expressions.	
1st	X true, Y true	xy	
2nd	X true, Y false	$x(1-y)$	
3rd	X false, Y true	$(1-x)y$	
4th	X false, Y false	$(1-x)(1-y)$	(24)

The proposition "X and Y" is expressed by xy, the proposition "X and not-Y" by $x(1-y)$, etc.

Given n propositional variables $X_1, \dots, X_n$ there are 2^n 'constituent' propositions $\widehat{X}_1$ *and* … *and* $\widehat{X}_n$, where each $\widehat{X}_i$ is either X_i or not-X_i. Their expressions give the constituents of $x_1, \dots, x_n$.

If we add the elective expressions for the two first of the above cases the sum is x, which is the elective symbol appropriate to the more general case of X being true independently of any consideration of Y; and if we add the elective expressions in the two last cases together, the result is $1-x$, which is the elective expression appropriate to the more general case of X being false.

Thus the extent of the hypothetical Universe does not at all depend upon the number of circumstances which are taken into account. And it is to be noted that however few or many those circumstances may be, the sum of the elective expressions representing every conceivable case will be unity. Thus let us consider the three Propositions, X, It rains, Y, It hails, Z, It freezes. The possible cases are the following:

	Cases.	Elective expressions.
1st	It rains, hails, and freezes,	xyz
2nd	It rains and hails, but does not freeze	$xy(1-z)$
3rd	It rains and freezes, but does not hail	$xz(1-y)$
4th	It freezes and hails, but does not rain	$yz(1-x)$
5th	It rains, but neither hails nor freezes	$x(1-y)(1-z)$
6th	It hails, but neither rains nor freezes	$y(1-x)(1-z)$
7th	It freezes, but neither hails nor rains	$z(1-x)(1-y)$
8th	It neither rains, hails, nor freezes	$(1-x)(1-y)(1-z)$
		$1 = \text{sum}$

Expression of Hypothetical Propositions.

To express that a given Proposition X is true.

The symbol $1 - x$ selects those cases in which the Proposition X is false. But if the Proposition is true, there are no such cases in its hypothetical Universe, therefore

$$1 - x = 0,$$

or

$$x = 1. \tag{25}$$

To express that a given Proposition X is false.

The elective symbol x selects all those cases in which the Proposition is true, and therefore if the Proposition is false,

$$x = 0. \tag{26}$$

And in every case, having determined the elective expression appropriate to a given Proposition, we assert the truth of that Proposition by equating the elective expression to unity, and its falsehood by equating the same expression to 0.

To express that two Propositions, X and Y, are simultaneously true.

The elective symbol appropriate to this case is xy, therefore the equation sought is

$$xy = 1. \tag{27}$$

To express that two Propositions, X and Y, are simultaneously false.

The condition will obviously be

$$(1 - x)(1 - y) = 1,$$

or

$$x + y - xy = 0. \tag{28}$$

To express that either the Proposition X is true, or the Proposition Y is true.

To assert that either one or the other of two Propositions is true, is to assert that it is not true, that they are both false. Now the elective expression appropriate to their both being false is $(1 - x)(1 - y)$, therefore the equation required is

$$(1 - x)(1 - y) = 0,$$

or

$$x + y - xy = 1. \tag{29}$$

Translating modern propositional logic into Boole's system. First translate modern propositional formulas $\Phi(\vec{\mathrm{X}})$ into terms $p_\Phi(\vec{x})$ as follows: $p_{\mathrm{X}} := x$ for a variable X, and given p_Φ and p_Ψ let

$p_{\neg\Phi} := 1 - p_\Phi$

$p_{\Phi\wedge\Psi} := p_\Phi \cdot p_\Psi$

$p_{\Phi\vee\Psi} := p_\Phi + p_\Psi - p_\Phi \cdot p_\Psi$

$p_{\Phi\triangle\Psi} := p_\Phi + p_\Psi - 2p_\Phi \cdot p_\Psi$

$p_{\Phi\rightarrow\Psi} := 1 - p_\Phi + p_\Phi \cdot p_\Psi$

$p_{\Phi\leftrightarrow\Psi} := 1 - p_\Phi - p_\Psi + 2p_\Phi \cdot p_\Psi$

Then a propositional formula $\Phi(\vec{\mathrm{X}})$ is a tautology iff $p_\Phi(\vec{x}) = 1$ is a law of Boole's Algebra, and it is a contradiction iff $p_\Phi(\vec{x}) = 0$ is a law of Boole's Algebra. Furthermore a propositional formula argument $\Phi_1, \ldots, \Phi_k \therefore \Phi$ is valid iff the equational argument $p_{\Phi_1} = 1, \ldots, p_{\Phi_k} = 1 \therefore p_\Phi = 1$ is valid in Boole's Algebra.

And, by indirect considerations of this kind, may every disjunctive Proposition, however numerous its members, be expressed. But **the following general Rule** will usually be preferable.

Rule. ***Consider what are those distinct and mutually exclusive cases of which it is implied in the statement of the given Proposition, that some one of them is true, and equate the sum of their elective expressions to unity. This will give the equation of the given Proposition.***

For the sum of the elective expressions for all distinct conceivable cases will be unity. Now all these cases being mutually exclusive, and it being asserted in the given Proposition that some one case out of a given set of them is true, it follows that all which are not included in that set are false, and that their elective expressions are severally equal to 0. Hence the sum of the elective expressions for the remaining cases, viz. those included in the given set, will be unity. Some one of those cases will therefore be true, and as they are mutually exclusive, it is impossible that more than one should be true. Whence the Rule in question.

And in the application of this Rule it is to be observed, that if the cases contemplated in the given disjunctive Proposition are not mutually exclusive, they must be resolved into an equivalent series of cases which are mutually exclusive.

Thus, if we take the Proposition of the preceding example, viz. Either X is true, or Y is true, and assume that the two members of this Proposition are not exclusive, insomuch that in the enumeration of possible cases, we must reckon that of the Propositions X and Y being both true, then **the mutually exclusive cases which fill up the Universe of the Proposition**, with their elective expressions, are

1st,	X true and Y false,	$x(1-y)$,
2nd,	Y true and X false,	$y(1-x)$,
3rd,	X true and Y true,	xy,

Boole does not speak of a propositional formula being true or false except when it is a propositional variable. Evidently he views a propositional formula $\Phi(\mathrm{X}_1, \ldots, \mathrm{X}_m)$ as asserting a disjunction of implied truth assignments of the variables. Then the equational expression for $\Phi(\mathrm{X}_1, \ldots, \mathrm{X}_m)$ is obtained by equating to 1 the sum of the expressions associated with the implied truth-value assignments to the variables.

It seems Boole was only considering simple propositions, like those one finds in a traditional book on logic. He does not describe a method to find the implied truth-value assignments; in particular, he does not have the truth-table method of determining which assignments make a proposition true.

and the sum of these elective expressions equated to unity gives

$$x + y - xy = 1. \tag{30}$$

as before. But if we suppose the members of the disjunctive Proposition to be exclusive, then the only cases to be considered are

1st, X true, Y false, $x(1-y)$,
2nd, Y true, X false, $y(1-x)$,

and the sum of these elective expressions equated to 0, gives

TYPO: equated to 1.

$$x - 2xy + y = 1. \tag{31}$$

(31) is the first time terms with explicit numerical coefficients other than $0, \pm 1$ appear in MAL.

The subjoined examples will further illustrate this method.

To express the Proposition, Either X is not true, or Y is not true, the members being exclusive.

The mutually exclusive cases are

1st, X not true, Y true, $y(1-x)$,
2nd, Y not true, X true, $x(1-y)$,

and the sum of these equated to unity gives

$$x - 2xy + y = 1, \tag{32}$$

which is the same as (31), and in fact the Propositions which they represent are equivalent.

To express the Proposition, Either X is not true, or Y is not true, the members not being exclusive.

To the cases contemplated in the last Example, we must add the following, viz.

X not true, Y not true, $(1-x)(1-y)$.

The sum of the elective expressions gives

$$x(1-y) + y(1-x) + (1-x)(1-y) = 1,$$

or

$$xy = 0. \tag{33}$$

To express the disjunctive Proposition, Either X is true, or Y is true, or Z is true, the members being exclusive.

Here the mutually exclusive cases are

1st,	X true, Y false, Z false,	$x(1-y)(1-z)$,
2nd,	Y true, Z false, X false,	$y(1-z)(1-x)$,
3rd,	Z true, X false, Y false,	$z(1-x)(1-y)$,

and the sum of the elective expressions equated to 1, gives, upon reduction,

$$x + y + z - 2(xy + yz + zx) + 3xyz = 1. \tag{34}$$

The expression of the same Proposition, when the members are in no sense exclusive, will be

$$(1-x)(1-y)(1-z) = 0. \tag{35}$$

And it is easy to see that our method will apply to the expression of any similar Proposition, whose members are subject to any specified amount and character of exclusion.

To express the conditional Proposition, If X is true, Y is true.

Here it is implied that all the cases of X being true, are cases of Y being true. The former cases being determined by the elective symbol x, and the latter by y, we have, in virtue of (4), Item (4) is on p. 21.

$$x(1-y) = 0. \tag{36}$$

To express the conditional Proposition, If X be true, Y is not true.

The equation is obviously

$$xy = 0; \tag{37}$$

this is equivalent to (33), and in fact the disjunctive Proposition, Either X is not true, or Y is not true, and the conditional Proposition, If X is true, Y is not true, are equivalent.

To express that If X is not true, Y is not true.

In (36) write $1-x$ for x, and $1-y$ for y, we have

$$(1-x)y = 0.$$

The results which we have obtained admit of verification in many different ways. Let it suffice to take for more particular examination the equation

$$x - 2xy + y = 1, \tag{38}$$

which expresses the conditional Proposition, Either X is true, or Y is true, the members being in this case exclusive.

First, let the Proposition X be true, then $x = 1$, and substituting, we have

$$1 - 2y + y = 1, \qquad \therefore -y = 0, \quad \text{or} \quad y = 0,$$

The third appearance of "$-$" as a unary operation.

which implies that Y is not true.

Secondly, let X be not true, then $x = 0$, and the equation gives

$$y = 1, \tag{39}$$

which implies that Y is true. In like manner we may proceed with the assumptions that Y is true, or that Y is false.

Again, in virtue of the property $x^2 = x$, $y^2 = y$, we may write the equation in the form

$$x^2 - 2xy + y^2 = 1,$$

This does not qualify as an equational deduction since it has the form

$$\varepsilon \ \therefore\ \varepsilon_1 \text{ or } \varepsilon_2,$$

the conclusion being a disjunction of two equations.

and extracting the square root, we have

$$x - y = \pm 1, \tag{40}$$

and this represents the actual case; for, as when X is true or false, Y is respectively false or true, we have

$$\begin{aligned} x &= 1 \quad \text{or} \quad 0, \\ y &= 0 \quad \text{or} \quad 1, \\ \therefore x - y &= 1 \quad \text{or} \quad -1. \end{aligned}$$

There will be no difficulty in the analysis of other cases.

Examples of Hypothetical Syllogism.

The treatment of every form of hypothetical Syllogism will consist in **forming the equations of the premises, and eliminating the symbol or symbols which are found in more than one of them.** The result will express the conclusion.

Better: "eliminating *some* of the symbols which are found ...". In the 4th hypothetical syllogism, p. 56, each of the three symbols appears in two of the premises.

1st. Disjunctive Syllogism.

Either X is true, or Y is true (exclusive),	$x + y - 2xy = 1$
But X is true,	$x = 1$
Therefore Y is not true,	$\therefore y = 0$
Either X is true, or Y is true (not exclusive),	$x + y - xy = 1$
But X is not true,	$x = 0$
Therefore Y is true,	$\therefore y = 1$

2nd. Constructive Conditional Syllogism.

If X is true, Y is true,	$x(1 - y) = 0$
But X is true,	$x = 1$
Therefore Y is true,	$\therefore 1 - y = 0$ or $y = 1$.

3rd. Destructive Conditional Syllogism.

If X is true, Y is true,	$x(1 - y) = 0$
But Y is not true,	$y = 0$
Therefore X is not true,	$\therefore x = 0$

4th. Simple Constructive Dilemma, the minor premiss exclusive.

If X is true, Y is true,	$x(1 - y) = 0,$	(41)
If Z is true, Y is true,	$z(1 - y) = 0,$	(42)
But Either X is true, or Z is true,	$x + z - 2xz = 1.$	(43)

From the equations (41), (42), (43), we have to eliminate x and z. **In whatever way we effect this,** the result is

$$y = 1;$$

whence it appears that the Proposition Y is true.

In MAL it seems that one needs 2 equations to eliminate 1 variable, 3 equations to eliminate 2 variables, and perhaps $n + 1$ equations to eliminate n variables. The improved reduction and elimination theorems in LT allow one to eliminate any number of variables from any number of equations.

5th. Complex Constructive Dilemma, the minor premiss not exclusive.

If X is true, Y is true,	$x(1 - y) = 0,$
If W is true, Z is true,	$w(1 - z) = 0,$
Either X is true, or W is true,	$x + w - xw = 1.$

From these equations, eliminating x, we have

$$y + z - yz = 1,$$

which expresses the Conclusion, Either Y is true, or Z is true, the members being non-exclusive.

6th. Complex Destructive Dilemma, the minor premiss exclusive.

If X is true, Y is true,	$x(1-y)=0$
If W is true, Z is true,	$w(1-z)=0$
Either Y is not true, or Z is not true,	$y+z-2yz=1.$

From these equations we must eliminate y and z. The result is

$$xw=0,$$

which expresses the Conclusion, Either X is not true, or Y is not true, the members *not being exclusive*.

TYPO:
change "or Y" to "or W".

7th. Complex Destructive Dilemma, the minor premiss not exclusive.

If X is true, Y is true,	$x(1-y)=0$
If W is true, Z is true,	$w(1-z)=0$
Either Y is not true, or Z is not true,	$yz=0.$

On elimination of y and z, we have

$$xw=0,$$

which indicates the same Conclusion as the previous example.

It appears from these and similar cases, that whether the members of the minor premiss of a Dilemma are exclusive or not, the members of the (disjunctive) Conclusion are never exclusive. This fact has perhaps escaped the notice of logicians.

The above are the principal forms of hypothetical Syllogism which logicians have recognised. **It would be easy, however, to extend the list,** especially by the blending of the disjunctive and the conditional character in the same Proposition, of which the following is an example.

Boole regarded the study of hypothetical syllogisms as incomplete. However, without the general notion of a propositional formula, he does not realize the full extent to which the accepted study is incomplete.

If X is true, then either Y is true, or Z is true,

$$x(1-y-z+yz)=0$$

But Y is not true,	$y=0$
Therefore If X is true, Z is true,	$\therefore\ x(1-z)=0.$

That which logicians term a *Causal* Proposition is properly a conditional Syllogism, the major premiss of which is suppressed.

The assertion that the Proposition X is true, *because* the Proposition Y is true, is equivalent to the assertion,

The Proposition Y is true,
Therefore the Proposition X is true;

and these are the minor premiss and conclusion of the conditional Syllogism,

If Y is true, X is true,
But Y is true,
Therefore X is true.

And thus causal Propositions are seen to be included in the applications of our general method.

Note, that there is a family of disjunctive and conditional Propositions, which do not, of right, belong to the class considered in this Chapter. Such are those in which the force of the disjunctive or conditional particle is expended upon the predicate of the Proposition, as if, speaking of the inhabitants of a particular island, we should say, that they are all *either Europeans or Asiatics*; meaning, that it is true of each individual, that he is either a European or an Asiatic. If we appropriate the elective symbol x to the inhabitants, y to Europeans, and z to Asiatics, then the equation of the above Proposition is

$$x = xy + xz, \quad \text{or} \quad x(1 - y - z) = 0; \quad (a)$$

to which we might add the condition $yz = 0$, since no Europeans are Asiatics. The nature of the symbols x, y, z, indicates that the Proposition belongs to those which we have before designated as *Categorical*. Very different from the above is the Proposition, Either all the inhabitants are Europeans, or they are all Asiatics. Here the disjunctive particle separates Propositions. The case is that contemplated in (31) of the present Chapter; and the symbols by which it is expressed, ... (pagebreak in MAL)

Item (31) is on p. 53.

although subject to the same laws as those of (a), have a totally different interpretation.*

The distinction is real and important. **Every Proposition which language can express may be represented by elective symbols, and the laws of combination of those symbols are in all cases the same;** but in one class of instances the symbols have reference to collections of objects, in the other, to the truths of constituent Propositions.

Boole had a much narrower view of what qualified as a proposition than logicians today. Furthermore, justification for his algebra of logic was sorely lacking.

*Some writers, among whom is Dr. Latham (*First Outlines*), regard it as the exclusive office of a conjunction to connect *Propositions*, not *words*. In this view I am not able to agree. The Proposition, Every animal is *either* rational *or* irrational, cannot be resolved into, *Either* every animal is rational, *or* every animal is irrational. The former belongs to pure categoricals, the latter to hypotheticals. In *singular* Propositions, such conversions would seem to be allowable. This animal is *either* rational *or* irrational, is equivalent to, *Either* this animal is rational, *or* it is irrational. This peculiarity of *singular* Propositions would almost justify our ranking them, though truly universals, in a separate class, as Ramus and his followers did.

PROPERTIES OF ELECTIVE FUNCTIONS.

SINCE elective symbols combine according to the laws of quantity, we may, by Maclaurin's theorem, expand a given function $\phi(x)$, in ascending powers of x, known cases of failure excepted. Thus we have

$$\phi(x) = \phi(0) + \phi'(0)x + \frac{\phi''(0)}{1 \cdot 2}x^2 + \&c. \qquad (44)$$

Now $x^2 = x$, $x^3 = x$, &c., whence

$$\phi(x) = \phi(0) + x\bigl\{\phi'(0) + \frac{\phi''(0)}{1 \cdot 2} + \&c.\bigr\}. \qquad (45)$$

Now if in (44) we make $x = 1$, we have

$$\phi(1) = \phi(0) + \phi'(0) + \frac{\phi''(0)}{1 \cdot 2} + \&c.,$$

whence

$$\phi'(0) + \frac{\phi''(0)}{1 \cdot 2} + \frac{\phi'''(0)}{1 \cdot 2 \cdot 3} + \&c. = \phi(1) - \phi(0).$$

Substitute this value for the coefficient of x in the second member of (45), and we have[*]

$$\phi(x) = \phi(0) + \bigl\{\phi(1) - \phi(0)\bigr\}x, \qquad (46)$$

In Boole's time the algebra of numbers included parts of what are now considered analysis, such as working with power series. From the modern point of view, Boole formally applied a power series expansion to a logical function $\phi(x)$ and derived the expansion theorem. In LT this approach is relegated to a footnote, as an alternate approach.

This proof of (46) and (47) is correct for $\phi(x)$ a (finite) term. For a term $\phi(x)$ one has $\phi(x) = ax + b$. Then $\phi(1) = a + b$, $\phi(0) = b$, so $\phi(x) = \bigl(\phi(1) - \phi(0)\bigr)x + \phi(0)$.

Boole may have been uncertain about how much of the algebra of numbers would be needed/useful in the algebra of logic, and thus opted to include the most general form of functions for which he thought he could prove the expansion theorem.

In the footnote on this page he acknowledged that not all functions in algebra are capable of a power series expansion, yet he still claimed that all satisfy the expansion theorem in his algebra of logic.

In Prop. 5 on p. 67 he claimed any valid equational inference

$$f = 0 \therefore g = 0$$

in his algebra is expressible by a function ψ in the sense that $\psi(f) = \psi(0)$ is a consequence of $f = 0$ and is equivalent to $g = 0$. Evidently he was not sure that ψ could be chosen to be a term.

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[*]Although this and the following theorems have only been proved for those forms of functions which are expansible by Maclaurin's theorem, they may be regarded as true for all forms whatever; this will appear from the applications. The reason seems to be that, as **it is only through the one form of expansion that elective functions become interpretable,** no conflicting interpretation is possible.

The development of $\phi(x)$ may also be determined thus. By the known formula for expansion in factorials,

$$\phi(x) = \phi(0) + \Delta\phi(0)x + \frac{\Delta^2\phi(0)}{1 \cdot 2}x(x-1) + \&c.$$

which we shall also employ under the form

$$\phi(x) = \phi(1)x + \phi(0)(1 - x). \tag{47}$$

This is the standard form of the expansion theorem in LT.

Every function of x, in which integer powers of that symbol are alone involved, is by this theorem reducible to the first order. The quantities $\phi(0)$, $\phi(1)$, we shall call **the moduli of the function $\phi(x)$.** They are of great importance in the theory of elective functions, as will appear from the succeeding Propositions.

The moduli of $\phi(x)$. The word 'moduli' does not appear in LT; instead $\phi(0)$, $\phi(1)$ are simply coefficients.

Prop. 1. Any two functions $\phi(x)$, $\psi(x)$, are equivalent, whose corresponding moduli are equal.

Prop. 1, and its converse below, say that $\phi(x) = \psi(x)$ is a law of Boole's algebra iff the corresponding moduli are equal, that is $\phi(0) = \psi(0)$ and $\phi(1) = \psi(1)$. This can be expanded to several variables, as stated on the next page, giving R01 for equations, namely $\phi(\vec{x}) = \psi(\vec{x})$ is a law of Boole's algebra iff $\mathbb{Z} \models_{01} \phi(\vec{x}) = \psi(\vec{x})$.

This is a plain consequence of the last Proposition. For since

$$\phi(x) = \phi(0) + \big\{\phi(1) - \phi(0)\big\}x,$$
$$\psi(x) = \psi(0) + \big\{\psi(1) - \psi(0)\big\}x,$$

it is evident that if $\phi(0) = \psi(0)$, $\phi(1) = \psi(1)$, the two expansions will be equivalent, and therefore the functions which they represent will be equivalent also.

The converse of this Proposition is equally true, viz.

If two functions are equivalent, their corresponding moduli are equal.

Among the most important applications of the above theorem, we may notice the following.

Suppose it required to determine for what forms of the function $\phi(x)$, the following equation is satisfied, viz.

$$\big\{\phi(x)\big\}^n = \phi(x).$$

Now x being an elective symbol, $x(x-1) = 0$, so that all the terms after the second, vanish. Also $\Delta\phi(0) = \phi(1) - \phi(0)$, whence

$$\phi\{x = \phi(0)\} + \{\phi(1) - \phi(0)\}x.$$

Correction: whence

$$\phi(x) = \phi(0) + \big\{\phi(1) - \phi(0)\big\}x$$

The mathematician may be interested in the remark, that this is not the only case in which an expansion stops at the second term. The expansions of the compound operative functions $\phi\left(\dfrac{d}{dx} + x^{-1}\right)$ and $\phi\left\{x + \left(\dfrac{d}{dx}\right)^{-1}\right\}$ are, respectively,

$$\phi\left(\frac{d}{dx}\right) + \phi'\left(\frac{d}{dx}\right)x^{-1},$$

and

$$\phi(x) + \phi'(x)\left(\frac{d}{dx}\right)^{-1}.$$

See *Cambridge Mathematical Journal*, Vol. IV. p. 219.

Here we at once obtain for the expression of the conditions in question,

$$\{\phi(0)\}^n = \phi(0). \quad \{\phi(1)\}^n = \phi(1). \tag{48}$$

Again, suppose it required to determine the conditions under which the following equation is satisfied, viz.

$$\phi(x)\psi(x) = \chi(x),$$

The general theorem at once gives

$$\phi(0)\psi(0) = \chi(0). \quad \phi(1)\psi(1) = \chi(1). \tag{49}$$

This result may also be proved by substituting for $\phi(x)$, $\psi(x)$, $\chi(x)$, their expanded forms, and equating the coefficients of the resulting equation properly reduced.

All the above theorems may be extended to functions of more than one symbol. For, as different elective symbols combine with each other according to the same laws as symbols of quantity, we can first expand a given function with reference to any particular symbol which it contains, and then expand the result with reference to any other symbol, and so on in succession, the order of the expansions being quite indifferent.

Thus the given function being $\phi(xy)$ we have

$$\phi(xy) = \phi(x0) + \{\phi(x1) - \phi(x0)\}y,$$

and expanding the coefficients with reference to x, and reducing

$$\begin{aligned}\phi(xy) = \phi(00) &+ \{\phi(10) - \phi(00)\}x + \{\phi(01) - \phi(00)\}y \\ &+ \{\phi(11) - \phi(10) - \phi(01) + \phi(00)\}xy,\end{aligned} \tag{50}$$

Expansion of $\phi(x, y)$ as a polynomial

to which we may give the elegant symmetrical form

$$\begin{aligned}\phi(xy) = \phi(00)(1-x)(1-y) &+ \phi(01)y(1-x) \\ &+ \phi(10)x(1-y) + \phi(11)xy,\end{aligned} \tag{51}$$

Expansion of $\phi(x, y)$ in form used in LT

wherein we shall, in accordance with the language already employed, designate $\phi(00)$, $\phi(01)$, $\phi(10)$, $\phi(11)$, as **the moduli of the function $\phi(xy)$.**

By inspection of the above general form, it will appear that any functions of two variables are equivalent, whose corresponding moduli are all equal.

Thus the conditions upon which depends the satisfaction of the equation,

$$\{\phi(xy)\}^n = \phi(xy)$$

are seen to be

$$\begin{aligned} \{\phi(00)\}^n &= \phi(00), & \{\phi(01)\}^n &= \phi(01), \\ \{\phi(10)\}^n &= \phi(10), & \{\phi(11)\}^n &= \phi(11). \end{aligned} \tag{52}$$

And the conditions upon which depends the satisfaction of the equation

$$\phi(xy)\psi(xy) = \chi(xy),$$

are

$$\begin{aligned} \phi(00)\psi(00) &= \chi(00), & \phi(01)\psi(01) &= \chi(01), \\ \phi(10)\psi(10) &= \chi(10), & \phi(11)\psi(11) &= \chi(11). \end{aligned} \tag{53}$$

It is very easy to assign by induction from (47) and (51), the general form of an expanded elective function. It is evident that if the number of elective symbols is m, the number of the moduli will be 2^m, and that their separate values will be obtained by interchanging in every possible way the values 1 and 0 in the places of the elective symbols of the given function. The several terms of the expansion of which the moduli serve as coefficients, will then be formed by writing for each 1 that recurs under the functional sign, the elective symbol x, &c., which it represents, and for each 0 the corresponding $1 - x$, &c., and regarding these as factors, the product of which, multiplied by the modulus from which they are obtained, constitutes a term of the expansion.

Thus, if we represent the moduli of any elective function $\phi(xy\dots)$ by $a_1, a_2, \dots, a_r$, the function itself, when expanded and arranged with reference to the moduli, will assume the form

$$\phi(xy) = a_1t_1 + a_2t_2 \cdots + a_rt_r, \tag{54}$$

in which $t_1t_2\dots t_r$ are functions of $x, y, \dots$, resolved into factors of the forms $x, y, \dots\ 1 - x, 1 - y, \dots$ &c. **These functions satisfy individually the index relations**

$$t_1^n = t_1, \quad t_2^n = t_2, \quad \&\text{c}. \tag{55}$$

and the further relations,

$$t_1t_2 = 0 \dots t_1t_2 = 0, \ \&\text{c}. \tag{56}$$

In general, $\phi(\vec{x}) = \sum_\sigma \phi(\sigma)\mathrm{C}_\sigma(\vec{x})$, where σ runs over all lists of 0s and 1s of length m, where $\vec{x}$ is $x_1, \dots, x_m$, and the constituent $\mathrm{C}_\sigma(\vec{x})$ is $\prod_i \mathrm{C}_{\sigma_i}(x_i)$ where $\mathrm{C}_1(x_i) = x_i$ and $\mathrm{C}_0(x_i) = 1 - x_i$.

The constituents $\mathrm{C}_\sigma(\vec{x})$ depend only on σ and $\vec{x}$, and not on $\phi(\vec{x})$. If *all* the variables occurring in $\phi(\vec{x})$ appear in the list $\vec{x}$, then the $\phi(\sigma)$ are integers.

Some Facts: In Boole's Algebra

$\mathrm{C}_\sigma(\vec{x})^2 = \mathrm{C}_\sigma(\vec{x})$,
$\mathrm{C}_\sigma(\vec{x}) \cdot \mathrm{C}_\tau(\vec{x}) = 0$ if $\sigma \neq \tau$,
$1 = \sum_\sigma \mathrm{C}_\sigma(\vec{x})$ are valid, and
$\mathrm{C}_\sigma(\tau) = 1$ if $\sigma = \tau$; $= 0$ otherwise.

$\phi(\vec{x}) = \psi(\vec{x})$ is valid iff $\phi(\sigma) = \psi(\sigma)$ for all σ.

$\phi(\vec{x})^n = \phi(\vec{x})$ is valid iff $\phi(\sigma)^n = \phi(\sigma)$ for all σ.

$\phi(\vec{x}) \cdot \psi(\vec{x}) = \chi(\vec{x})$ is valid iff $\phi(\sigma) \cdot \psi(\sigma) = \chi(\sigma)$ for all σ.

Change $\phi(xy)$ to $\phi(xy\dots)$ in (54). The t_i are called the *constituent* functions of ϕ on the next page. It would be better to call them the constituent functions of $xy\dots$.

The second t_1t_2 in (56) should probably be t_1t_r. (56) is clearer if written $t_it_j = 0$ for $1 \le i < j \le r$.

the product of any two of them vanishing. This will at once be inferred from inspection of the particular forms (47) and (51). Thus in the latter we have for the values of t_1, t_2, &c., the forms

$$xy, \quad x(1-y), \quad (1-x)y, \quad (1-x)(1-y);$$

and it is evident that these satisfy the index relation, and that their products all vanish. We shall designate $t_1 t_2 \ldots$ as the **constituent functions** of $\phi(xy)$, and we shall define the peculiarity of the vanishing of the binary products, by saying that **those functions are *exclusive*.** And indeed the classes which they represent are mutually exclusive.

The sum of all the constituents of an expanded function is unity. An elegant proof of this Proposition will be obtained by expanding 1 as a function of any proposed elective symbols. Thus if in (51) we assume $\phi(xy) = 1$, we have $\phi(11) = 1$, $\phi(10) = 1$, $\phi(01) = 1$, $\phi(00) = 1$, and (51) gives

$$1 = xy + x(1-y) + (1-x)y + (1-x)(1-y). \tag{57}$$

It is obvious indeed, that however numerous the symbols involved, all the moduli of unity are unity, whence the sum of the constituents is unity.

We are now prepared to enter upon the question of **the general interpretation of elective equations.** For this purpose we shall find the following Propositions of the greatest service.

The first member of an equation is the left-side of the equation.

$\phi(xy\ldots) = 0$ is equivalent to the collection of $t_i = 0$ whose modulus a_i is not 0. (It is also equivalent to $\sum\{t_i : a_i \neq 0\} = 0$.)

Change $at = 0$ to $t = 0$.

Note that each $t = 0$ is interpretable.

Prop. 2. If the first member of the general equation $\phi(xy\ldots) = 0$, be expanded in a series of terms, each of which is of the form at, a being a modulus of the given function, then **for every numerical modulus a which does not vanish, we shall have the equation**

$$at = 0,$$

and the combined interpretations of these several equations will express the full significance of the original equation.

For, representing the equation under the form

$$a_1 t_1 + a_2 t_2 \cdots + a_r t_r = 0. \tag{58}$$

Multiplying by t_1, we have, by (56),

$$a_1 t_1 = 0, \tag{59}$$

whence if a_1 is a numerical constant which does not vanish,

$$t_1 = 0,$$

and similarly for all the moduli which do not vanish. And inasmuch as from these **constituent equations** we can form the given equation, their interpretations will together express its entire significance.

It is not clear if Boole obtained $t_1 = 0$ from (59) by simply using a rule of inference, or by first multiplying both sides of (59) by $1/a_1$.

Thus if the given equation were

$$x - y = 0, \quad \text{Xs and Ys are identical,} \tag{60}$$

we should have $\phi(11) = 0$, $\phi(10) = 1$, $\phi(01) = -1$, $\phi(00) = 0$, so that the expansion (51) would assume the form

$$x(1-y) - y(1-x) = 0,$$

whence, by the above theorem,

$$\begin{aligned} x(1-y) &= 0, \qquad \text{All Xs are Ys,} \\ y(1-x) &= 0, \qquad \text{All Ys are Xs,} \end{aligned}$$

results which are together equivalent to (60).

It may happen that the simultaneous satisfaction of equations thus deduced, may require that one or more of the elective symbols should vanish. **This would only imply the nonexistence of a class: it may even happen that it may lead to a final result of the form**

$$1 = 0,$$

which would indicate the nonexistence of the logical Universe. Such cases will only arise when we attempt to unite contradictory Propositions in a single equation. The manner in which the difficulty seems to be evaded in the result is characteristic.

Boole did not treat 0 as an elective operation defined by a class. He said that if one derives $x = 0$, then the class X does not exist; and if one derives $1 = 0$ then the universe does not exist. Boole said such derivations need contradictory propositions. In LT Boole would let 0 denote the class with no members.

It appears from this Proposition, that **the differences in the interpretation of elective functions depend solely upon the number and position of the vanishing moduli.** No change in the value of a modulus, but one which causes it to vanish, produces any change in the interpretation of the equation in which it is found. If among the infinite number of different values which we are thus permitted to give to the moduli which do not vanish in a proposed equation, any one value should be . . . . (pagebreak in MAL)

Boole said that among the equivalent forms of an elective equation $\phi = 0$, the preferred choice is

$$\sum\{t_i : a_i \neq 0\} \ = \ 0,$$

where the complete expansion of ϕ is $\sum a_i t_i$.

preferred, it is unity, for when the moduli of a function are all either 0 or 1, the function itself satisfies the condition

$$\{\phi(xy\dots)\}^n = \phi(xy\dots),$$

and this at once introduces symmetry into our Calculus, and provides us with fixed standards for reference.

Prop. 3. If $w = \phi(xy\dots)$, w, x, y, $\dots$ being elective symbols, and if the second member be completely expanded and arranged in a series of terms of the form at, we shall be permitted to equate separately to 0 every term in which the modulus a does not satisfy the condition

$$a^n = a,$$

and to leave for the value of w the sum of the remaining terms.

Prop. 3 says that any equation $w = \phi(xy\dots)$ is equivalent to $w = \sum\{t_i : a_i = 1\}$ along with $t_i = 0$ for $a_i \notin \{0,1\}$, where $\sum a_i t_i$ is the complete expansion of $\phi(xy\dots)$.

This is the only time Boole refers to a *complete* expansion. The hallmark of a complete expansion is that all the coefficients of the constituents are numerical, and called modulii.

The general proof is quite simple. From $w = \sum a_i t_i$ follows $w = \sum a_i^2 t_i$. Subtracting gives $\sum(a_i^2 - a_i)t_i = 0$. If a_j is not idempotent, that is $a_j \notin \{0,1\}$, then multiply the last sum by t_j to obtain $(a_j^2 - a_j)t_j = 0$. As the numerical coefficient $a_j^2 \neq a_j$, it follows that $t_j = 0$.

As the nature of the demonstration of this Proposition is quite unaffected by the number of the terms in the second member, we will for simplicity confine ourselves to the supposition of there being four, and suppose that the moduli of the two first only, satisfy the index law.

We have then

$$w = a_1t_1 + a_2t_2 + a_3t_3 + a_4t_4, \tag{61}$$

with the relations

$$a_1^n = a_1, \quad a_2^n = a_2,$$

in addition to the two sets of relations connecting t_1, t_2, t_3, t_4, in accordance with (55) and (56).

Squaring (61), we have

$$w = a_1t_1 + a_2t_2 + a_3^2t_3 + a_4^2t_4,$$

and subtracting (61) from this,

$$(a_3^2 - a_3)t_3 + (a_4^2 - a_4)t_4 = 0;$$

and it being an hypothesis, that the coefficients of these terms do not vanish, we have, by Prop. 2,

$$t_3 = 0, \quad t_4 = 0, \tag{62}$$

whence (61) becomes

$$w = a_1t_1 + a_2t_2.$$

The utility of this Proposition will hereafter appear.

Prop. 4. The functions $t_1 t_2 \dots t_r$ being mutually exclusive, we shall always have

$$\psi(a_1 t_1 + a_2 t_2 \cdots + a_r t_r) = \psi(a_1) t_1 + \psi(a_2) t_2 \cdots + \psi(a_r) t_r, \quad (63)$$

whatever may be the values of $a_1 a_2 \dots a_r$ or the form of ψ.

Let the function $a_1 t_1 + a_2 t_2 \cdots + a_r t_r$ be represented by $\phi(xy\dots)$, then the moduli $a_1 a_2 \dots a_r$ will be given by the expressions

$$\phi(11\dots), \quad \phi(10\dots), \quad (\dots) \quad \phi(00\dots).$$

Also

$$\begin{aligned}
&\psi(a_1 t_1 + a_2 t_2 \cdots + a_r t_r) = \psi\{\phi(xy\dots)\} \\
&= \psi\{\phi(11\dots)\}xy \cdots + \psi\{\phi(10\dots)\}x(1-y)\dots \\
&\quad + \psi\{\phi(00\dots)\}(1-x)(1-y)\dots \\
&= \psi(a_1)xy \cdots + \psi(a_2)x(1-y) \cdots + \psi(a_r)(1-x)(1-y)\dots \\
&= \psi(a_1)t_1 + \psi(a_2)t_2 \cdots + \psi(a_r)t_r. \qquad (64)
\end{aligned}$$

The proof assumes that $t_1, \dots, t_r$ is a list of all the constituents for some list of variables $\vec{x}$. Let $\psi(w) = p \cdot w + q$ where p and q are terms not involving w. Then for any *terms* a_i one has $\psi(\sum_i a_i t_i) = p \cdot \left(\sum_i a_i t_i\right) + q = \sum_i (p \cdot a_i + q) t_i = \sum_i \psi(a_i) t_i$.

It would not be difficult to extend the list of interesting properties, of which the above are examples. But those which we have noticed are sufficient for our present requirements. The following Proposition may serve as an illustration of their utility.

Prop. 5. Whatever process of reasoning we apply to a single given Proposition, the result will either be the same Proposition or a limitation of it.

Boole's proof of Prop. 5 shows, in particular, that if $q(\vec{x}) = 0$ is a consequence of $p(\vec{x}) = 0$ then $p(\sigma) = 0$ implies $q(\sigma) = 0$, for all σ. Thus the collection of constituents with non-zero coefficient in the expansion of $q = 0$ is a subset of those in the expansion of $p = 0$; hence $q = 0$ is either equivalent to $p = 0$ or a limitation of it. This can readily be generalized to show that $\{p_i(\vec{x}) = 0\} \therefore p(\vec{x}) = 0$ is valid iff $\{p_i(\sigma) = 0\} \therefore p(\sigma) = 0$ holds for all σ, a result which is the foundation principle in LT, and has been called the *Rule of 0 and 1.*

Let us represent the equation of the given Proposition under its most general form,

$$a_1 t_1 + a_2 t_2 \cdots + a_r t_r = 0, \quad (65)$$

resolvable into as many equations of the form $t = 0$ as there are moduli which do not vanish.

Now the most general transformation of this equation is

$$\psi(a_1 t_1 + a_2 t_2 \cdots + a_r t_r) = \psi(0), \quad (66)$$

provided that we attribute to ψ a perfectly arbitrary character, allowing it even to involve new elective symbols, having *any proposed relation* to the original ones.

The development of (66) gives, by the last Proposition,

$$\psi(a_1)t_1 + \psi(a_2)t_2 \cdots + \psi(a_r)t_r = \psi(0).$$

To reduce this to the general form of reference, it is only necessary to observe that since

$$t_1 + t_2 \cdots + t_r = 1,$$

we may write for $\psi(0)$,

$$\psi(0)(t_1 + t_2 \cdots + t_r),$$

whence, on substitution and transposition,

$$\{\psi(a_1) - \psi(0)\}t_1 + \{\psi(a_2) - \psi(0)\}t_2 \cdots + \{\psi(a_r) - \psi(0)\}t_r = 0.$$

From which it appears, that if a be any modulus of the original equation, the corresponding modulus of the transformed equation will be

$$\psi(a) - \psi(0).$$

If $a = 0$, then $\psi(a) - \psi(0) = \psi(0) - \psi(0) = 0$, whence there are no *new terms* in the transformed equation, and therefore there are no *new Propositions* given by equating its constituent members to 0.

Again, since $\psi(a) - \psi(0)$ may vanish without a vanishing, terms may be wanting in the transformed equation which existed in the primitive. Thus some of the **constituent truths** of the original Proposition may entirely disappear from the interpretation of the final result.

Lastly, if $\psi(a) - \psi(0)$ do not vanish, it must either be a numerical constant, or it must involve new elective symbols. In the former case, the term in which it is found will give

$$t = 0,$$

which is one of the constituents of the original equation: in the latter case we shall have

$$\{\psi(a - \psi(0)\}t = 0,$$ TYPO: $\psi(a) - \psi(0)$

in which t has a limiting factor. The interpretation of this equation, therefore, is a limitation of the interpretation of (65).

The purport of the last investigation will be more apparent to the mathematician than to the logician. As from any mathematical equation an infinite number of others may be deduced, it seemed to be necessary to shew that when the original equation expresses a logical Proposition, every member of the derived series, even when obtained by expansion under a functional sign, admits of exact and consistent interpretation.

OF THE SOLUTION OF ELECTIVE EQUATIONS.

In whatever way an elective symbol, considered as unknown, may be involved in a proposed equation, it is possible to assign its complete value in terms of the remaining elective symbols considered as known. It is to be observed of such equations, that from the very nature of elective symbols, they are necessarily linear, and that **their solutions have a very close analogy with those of linear differential equations,** arbitrary elective symbols in the one, occupying the place of arbitrary constants in the other. The method of solution we shall in the first place illustrate by particular examples, and, afterwards, apply to the investigation of general theorems.

Boole's first method of solving an equation $\phi(\vec{x}, w) = 0$ for w in terms of $\vec{x}$ is to **assume** the solution is a linear combination $\sum v_i t_i$ of the constituents t_i of the variables $\vec{x}$. This value of w is substituted into $\phi(\vec{x}, w) = 0$, perhaps with the left side expanded, to determine which v_i must be 0.

Given $(1-x)y = 0$, (All Ys are Xs), to determine y in terms of x.

Example: Solve $(1-x)y = 0$ for y.

As y is a function of x, we may assume $y = vx + v'(1-x)$, (such being the expression of an arbitrary function of x), the moduli v and v' remaining to be determined. We have then

$$(1-x)\{vx + v'(1-x)\} = 0,$$

or, on actual multiplication,

$$v'(1-x) = 0:$$

that this may be generally true, without imposing any restriction upon x, we must assume $v' = 0$, and there being no condition to limit v, we have

$$y = vx. \tag{67}$$

This is the complete solution of the equation. The condition that y is an elective symbol requires that v should be an elective symbol also (since it must satisfy the index law), its interpretation in other respects being arbitrary.

Similarly the solution of the equation, $xy = 0$, is

$$y = v(1 - x). \tag{68}$$

Given $(1 - x)zy = 0$, (All Ys which are Zs are Xs), to determine y.

Example: Solve $(1 - x)zy = 0$ for y.

As y is a function of x and z, we may assume

$$y = v(1 - x)(1 - z) + v'(1 - x)z + v''x(1 - z) + v'''zx.$$

And substituting, we get

$$v'(1 - x)z = 0,$$

whence $v' = 0$. The complete solution is therefore

$$y = v(1 - x)(1 - z) + v''x(1 - z) + v'''xz, \tag{69}$$

v', v'', v''', being arbitrary elective symbols, and the rigorous interpretation of this result is, that Every Y is *either* a not-X and not-Z, or an X and not-Z, or an X and Z.

It is deserving of note that the above equation may, in consequence of its linear form, be solved by adding the two particular solutions with reference to x and z; and replacing the arbitrary constants which each involves by an arbitrary function of the other symbol, the result is

$$y = x\phi(z) + (1 - z)\psi(x). \tag{70}$$

To shew that this solution is equivalent to the other, it is only necessary to substitute for the arbitrary functions $\phi(z)$, $\psi(x)$, their equivalents

$$wz + w'(1 - z) \quad \text{and} \quad w''x + w'''(1 - x),$$

we get

$$y = wxz + (w' + w'')x(1 - z) + w'''(1 - x)(1 - z).$$

In consequence of the perfectly arbitrary character of w' and w'', we may replace their sum by a single symbol w', whence

$$y = wxz + w'x(1 - z) + w'''(1 - x)(1 - z),$$

which agrees with (69).

The solution of the equation $wx(1-y)z = 0$, expressed by arbitrary functions, is

$$z = (1-w)\phi(xy) + (1-x)\psi(wy) + y\chi(wx). \tag{71}$$

These instances may serve to shew the analogy which exists between the solutions of elective equations and those of the corresponding order of linear differential equations. Thus the expression of the integral of a partial differential equation, either by arbitrary functions or by a series with arbitrary coefficients, is in strict analogy with the case presented in the two last examples. To pursue this comparison further would minister to curiosity rather than to utility. We shall prefer to contemplate the problem of the solution of elective equations under its most general aspect, which is the object of the succeeding investigations.

To solve the general equation $\phi(xy) = 0$, with reference to y.

The above examples did not illustrate that, when solving an equation for one variable, constraints on the other variables may be required. For example, to have a solution to $xy = z$ for y one needs $(1-x)z = 0$.

If we expand the given equation with reference to x and y, we have

$$\phi(00)(1-x)(1-y) + \phi(01)(1-x)y + \phi(10)x(1-y) + \phi(11)xy = 0, \tag{72}$$

the coefficients $\phi(00)$ &c. **being numerical constants.**

Now the general expression of y, as a function of x, is

$$y = vx + v'(1-x),$$

v and v' being unknown symbols to be determined. Substituting this value in (72), we obtain a result which may be written in the following form,

$$\big[\phi(10) + \big\{\phi(11) - \phi(10)\big\}v\big]x + \big[\phi(00) + \big\{\phi(00) - \phi(00)\big\}v'\big](1-x) = 0;$$

TYPO: Change $\phi(00) - \phi(00)$ to $\phi(01) - \phi(00)$.

and in order that this equation may be satisfied without any way restricting the generality of x, we must have

$$\phi(10) + \big\{\phi(11) - \phi(10)\big\}v = 0,$$
$$\phi(00) + \big\{\phi(01) - \phi(00)\big\}v' = 0,$$

from which we deduce

$$v = \frac{\phi(10)}{\phi(10) - \phi(11)}\,, \qquad v' = \frac{\phi(00)}{\phi(01) - \phi(00)}\,,$$

TYPO: Change $\phi(01) - \phi(00)$ to $\phi(00) - \phi(01)$.

On the next page Boole shows how to interpret these fractions when they are not 0 or 1.

wherefore

$$y = \frac{\phi(10)}{\phi(10) - \phi(11)}\,x + \frac{\phi(00)}{\phi(00) - \phi(01)}\,(1 - x). \qquad (73)$$

Had we expanded the original equation with respect to y only, we should have had

$$\phi(x0) + \big\{\phi(x1) - \phi(x0)\big\}y = 0;$$

but **it might have startled those who are unaccustomed to the processes of Symbolical Algebra, had we from this equation deduced**

$$y = \frac{\phi(x0)}{\phi(x0) - \phi(x1)}\,,$$

because of the apparently meaningless character of the second member. Such a result would however have been perfectly lawful, and the expansion of the second member would have given us the solution above obtained. I shall in the following example employ this method, and shall only remark that those to whom it may appear doubtful, may verify its conclusions by the previous method.

Boole tried to work with rational expressions $p(\vec{x})/q(\vec{x})$ just as in numerical algebra, but he had only limited success, namely when solving an equation of the form $q(\vec{x})w = p(\vec{x})$. His method of solution, starting with $w = p(\vec{x})/q(\vec{x})$, formally expanding the right side and interpreting it, does not work to solve $q(\vec{x})(w_1 + w_2) = p(\vec{x})$ for $w_1 + w_2$. It seems best to view Boole's use of expansions of rational expressions as a clever mnemonic device for solving $q(\vec{x})w = p(\vec{x})$ for w, or more generally, for solving $q(\vec{x})f(\vec{w}) = p(\vec{x})$ for $f(\vec{w})$ provided $f(\vec{w})$ is idempotent.

To solve the general equation $\phi(xyz) = 0$, or in other words to determine the value of z as a function of x and y.

Expanding the given equation with reference to z, we have

$$\phi(xy0) + \big\{\phi(xy1) - \phi(xy0)\big\} \cdot z = 0;$$

$$\therefore z = \frac{\phi(xy0)}{\phi(xy0) - \phi(xy1)}\,, \qquad (74)$$

and expanding the second member as a function of x and y by aid of the general theorem, we have

$$z = \frac{\phi(110)}{\phi(110) - \phi(111)}\,xy + \frac{\phi(100)}{\phi(100) - \phi(101)}\,x(1 - y)$$
$$+ \frac{\phi(010)}{\phi(010) - \phi(011)}\,(1 - x)y + \frac{\phi(000)}{\phi(000) - \phi(001)}\,(1 - x)(1 - y), \qquad (75)$$

and this is the complete solution required. By the same method we may resolve an equation involving any proposed number of elective symbols.

In the interpretation of any general solution of this nature, the following cases may present themselves.

The values of the moduli $\phi(00)$, $\phi(01)$, &c. being constant, one or more of the coefficients of the solution may assume the form $\frac{0}{0}$ or $\frac{1}{0}$. **In the former case, the indefinite symbol $\frac{0}{0}$ must be replaced by an arbitrary elective symbol v. In the latter case, the term, which is multiplied by a factor $\frac{1}{0}$ (or by any numerical constant except 1), must be separately equated to 0, and will indicate the existence of a subsidiary Proposition.** This is evident from (62).

Regarding the fractional moduli, in the cases where the coefficients of the solution are not 0 or 1 they can be any of m/n where m, n are integers satisfying $0 \neq m \neq n$ or $m = n = 0$. If ϕ is idempotent, then the only possible coefficients besides 0 and 1 are $0/0$ and $1/0$.

The arbitrary v is not necessarily the v used for "some".

Correction: any numerical constant except 0 or 1. This includes the "numerical" constants $m/0$, $m \neq 0$.

$\boxed{\text{Ex.}}$ Given $x(1-y) = 0$, All Xs are Ys, to determine y as a function of x.

Let $\phi(xy) = x(1-y)$, then $\phi(10) = 1$, $\phi(11) = 0$, $\phi(01) = 0$, $\phi(00) = 0$; whence, by (73),

$$\begin{aligned} y &= \frac{1}{1-0}\,x + \frac{0}{0-0}\,(1-x) \\ &= x + \tfrac{0}{0}(1-x) \\ &= x + v(1-x), \qquad (76) \end{aligned}$$

v being an arbitrary elective symbol. The interpretation of this result is that the class Y consists of the entire class X with an indefinite remainder of not-Xs. **This remainder is indefinite in the highest sense,** *i. e.* it may vary from 0 up to the entire class of not-Xs.

This does not imply that v is permitted to take on the value 0.

$\boxed{\text{Ex.}}$ Given $x(1-z)+z = y$, (the class Y consists of the entire class Z, with such not-Zs as are Xs), to find Z.

Here $\phi(xyz) = x(1-z) - y + z$, whence we have the following set of values for the moduli,

$$\begin{array}{llll} \phi(110) = 0, & \phi(111) = 0, & \phi(100) = 1, & \phi(101) = 1, \\ \phi(010) = -1, & \phi(011) = 0, & \phi(000) = 0, & \phi(001) = 1, \end{array}$$

and substituting these in the general formula (75), we have

$$z = \tfrac{0}{0}xy + \tfrac{1}{0}x(1-y) + (1-x)y, \qquad (77)$$

the infinite coefficient of the second term indicates the equation

$$x(1-y) = 0, \quad \text{All Xs are Ys;}$$

and the indeterminate coefficient of the first term being replaced by v, an arbitrary elective symbol, we have

$$z = (1-x)y + vxy,$$

the interpretation of which is, that the class Z consists of all the Ys which are not Xs, and an *indefinite* remainder of Ys which are Xs. **Of course this indefinite remainder may vanish.** The two results we have obtained are logical inferences (not very obvious ones) from the original Propositions, and they give us all the information which it contains respecting the class Z, and its constituent elements.

$\boxed{\text{Ex.}}$ Given $x = y(1-z) + z(1-y)$. The class X consists of all Ys which are not-Zs, and all Zs which are not-Ys: required the class Z.

We have

$$\begin{aligned}
&\phi(xyz) = x - y(1-z) - z(1-y),\\
&\phi(110) = 0, \quad \phi(111) = 1, \quad \phi(100) = 1, \quad \phi(101) = 0,\\
&\phi(010) = -1, \quad \phi(011) = 0, \quad \phi(000) = 0, \quad \phi(001) = -1;
\end{aligned}$$

whence, by substituting in (75),

$$z = x(1-y) + y(1-x), \tag{78}$$

the interpretation of which is, the class Z consists of all Xs which are not Ys, and of all Ys which are not Xs; an inference strictly logical.

$\boxed{\text{Ex.}}$ Given $y\{1 - z(1-x)\} = 0$, All Ys are Zs and not-Xs.

Proceeding as before to form the moduli, we have, on substitution in the general formulæ,

$$z = \tfrac{1}{0}xy + \tfrac{0}{0}x(1-y) + y(1-x) + \tfrac{0}{0}(1-x)(1-y),$$

or

$$\begin{aligned}
z &= y(1-x) + vx(1-y) + v'(1-x)(1-y)\\
&= y(1-x) + (1-y)\phi(x),
\end{aligned} \tag{79}$$

Also: $z = (1-x)y + v''(1-y)$.

with the relation

$$xy = 0:$$

from these it appears that No Ys are Xs, and that the class Z consists of all Ys which are not Xs, and of an indefinite remainder of not-Ys.

This method, in combination with **Lagrange's method of indeterminate multipliers**, may be very elegantly applied to the treatment of simultaneous equations. Our limits only permit us to offer a single example, but the subject is well deserving of further investigation.

Given the equations $x(1-z) = 0$, $z(1-y) = 0$, All Xs are Zs, All Zs are Ys, to determine the complete value of z with any subsidiary relations connecting x and y.

Adding the second equation multiplied by an indeterminate constant λ, to the first, we have

$$x(1-z) + \lambda z(1-y) = 0,$$

whence determining the moduli, and substituting in (75),

$$z = xy + \frac{1}{1-\lambda}\, x(1-y) + \tfrac{0}{0}(1-x)y, \tag{80}$$

from which we derive

$$z = xy + v(1-x)y,$$

with the subsidiary relation

$$x(1-y) = 0:$$

the former of these expresses that the class Z consists of all Xs that are Ys, with an indefinite remainder of not-Xs that are Ys; the latter, that All Xs are Ys, being in fact the conclusion of the syllogism of which the two given Propositions are the premises.

By assigning an appropriate meaning to our symbols, all the equations we have discussed would admit of interpretation in hypotheticals, but it may suffice to have considered them as examples of categoricals.

That peculiarity of elective symbols, in virtue of which every elective equation is reducible to a system of equations $t_1 = 0$, $t_2 = 0$, &c., so constituted, that all the binary products $t_1 t_2$, $t_1 t_3$, &c., vanish, represents a general doctrine in Logic with reference to the ultimate analysis of Propositions, of which it may be desirable to offer some illustration.

Boole first gave a simple example of the use of indeterminate multipliers to solve a system of two logical equations in three variables for one of its variables in terms of the other two variables. Essentially he was using the fact that $\phi(\vec{x}) = \psi(\vec{x}) = 0$ is equivalent to $(\forall\lambda)\Big(\phi(\vec{x}) + \lambda\psi(\vec{x}) = 0\Big)$, where λ ranges over $\{0,1\}$. Then he looked at the general theory of three equations in three variables on p. 78, saying that this exhibits all the ingredients needed to solve any number of equations in any number of variables. However his constraint equations are not strong enough to guarantee that a solution exists—see the margin comments on p. 80.

"That peculiarity" is the index law.

Any of these constituents t_1, t_2, &c. consists only of factors of the forms x, y, ... $1-w$, $1-z$, &c. In categoricals it therefore represents a compound class, *i. e.* a class defined by the presence of certain qualities, and by the absence of certain other qualities.

Each constituent equation $t_1 = 0$, &c. expresses a denial of the existence of some class so defined, and the different classes are mutually exclusive.

Thus all categorical Propositions are resolvable into a denial of the existence of certain compound classes, no member of one such class being a member of another.

This was already covered in Prop. 2 on p. 64.

The Proposition, All Xs are Ys, expressed by the equation $x(1-y) = 0$, is resolved into a denial of the existence of a class whose members are Xs and not-Ys.

The Proposition Some Xs are Ys, expressed by $v = xy$, is resolvable as follows. On expansion,

$$v - xy = vx(1-y) + vy(1-x) + v(1-x)(1-y) - xy(1-v);$$
$$\therefore vx(1-y) = 0, \quad vy(1-x) = 0, \quad v(1-x)(1-y) = 0, \quad (1-v)xy = 0.$$

The three first imply that there is no class whose members belong to a certain unknown Some, and are 1st, Xs and not Ys; 2nd, Ys and not Xs; 3rd, not-Xs and not-Ys. The fourth implies that there is no class whose members are Xs and Ys without belonging to this unknown Some.

From the same analysis it appears that ***all hypothetical Propositions may be resolved into denials of the coexistence of the truth or falsity of certain assertions.***

Thus the Proposition, If X is true, Y is true, is resolvable by its equation $x(1-y) = 0$, into a denial that the truth of X and the falsity of Y coexist.

And the Proposition Either X is true, or Y is true, members exclusive, is resolvable into a denial, first, that X and Y are both true; secondly, that X and Y are both false.

But it may be asked, is not something more than a system of negations necessary to the constitution of an affirmative Proposition? is not a positive element required? Undoubtedly there is need of one; and this positive element is supplied in categoricals by the assumption (which may be regarded as a prerequisite of reasoning in such cases) that **there *is* a Universe of conceptions,** and that each individual it contains either belongs to a proposed class or does not belong to it; in hypotheticals, by the assumption (equally prerequisite) that **there is a Universe of conceivable cases,** and that any given Proposition is either true or false. Indeed the question of the existence of conceptions (εἶ ἔστι) is preliminary to any statement of their qualities or relations (τί ἔστι).—*Aristotle, Anal. Post.* lib. II. cap. 2.

This means $1 \neq 0$.

It would appear from the above, that Propositions may be regarded as resting at once upon a positive and upon a negative foundation. Nor is such a view either foreign to the spirit of Deductive Reasoning or inappropriate to its Method; the latter ever proceeding by limitations, while the former contemplates the particular as derived from the general.

Demonstration of the Method of Indeterminate Multipliers, as applied to Simultaneous Elective Equations.

To avoid needless complexity, **it will be sufficient to consider the case of three equations involving three elective symbols,** those equations being the most general of the kind. It will be seen that the case is marked by every feature affecting the character of the demonstration, which would present itself in the discussion of the more general problem in which the number of equations and the number of variables are both unlimited.

Let the given equations be

$$\phi(xyz) = 0, \quad \psi(xyz) = 0, \quad \chi(xyz) = 0. \qquad (1)$$

Multiplying the second and third of these by the arbitrary constants h and k, and adding to the first, we have

$$\phi(xyz) + h\psi(xyz) + k\chi(xyz) = 0; \qquad (2)$$

Boole used indeterminate multipliers to reduce several equations to a single equation. Clearly

$$\phi = \psi = \chi = 0$$

is equivalent to

$$(\forall h)(\forall k)(\phi + h\psi + k\chi = 0).$$

Indeed it suffices to let h, k range over $\{0, 1\}$.

In LT this method will be mentioned, but the preferred method will use sums of squares; e.g., in this case the reduction would be

$$\phi^2 + \psi^2 + \chi^2 = 0.$$

and we are to shew, that in solving this equation with reference to any variable z by the general theorem (75), we shall obtain not only the general value of z independent of h and k, but also any subsidiary relations which may exist between x and y independently of z.

If we represent the general equation (2) under the form $F(xyz) = 0$, its solution may by (75) be written in the form

This is rather dubious since, for example, $F(110)$ might be 0.

$$z = \frac{xy}{1 - \dfrac{F(111)}{F(110)}} + \frac{x(1-y)}{1 - \dfrac{F(101)}{F(100)}} + \frac{y(1-x)}{1 - \dfrac{F(011)}{F(010)}} + \frac{(1-x)(1-y)}{1 - \dfrac{F(001)}{F(000)}};$$

and we have seen, that any one of these four terms is to be equated to 0, whose modulus, which we may represent by M, does not satisfy the condition $M^n = M$, or, which is here the same thing, whose modulus has any other value than 0 or 1.

Correction: …any other value than 0 or 1 or 0/0.

Consider the modulus (suppose M_1) of the first term, viz. $\dfrac{1}{1 - \dfrac{F(111)}{F(110)}}$, and giving to the symbol F its full meaning, we have

$$M_1 = \frac{1}{1 - \dfrac{\phi(111) + h\psi(111) + k\chi(111)}{\phi(110) + h\psi(110) + k\chi(110)}}.$$

It is evident that the condition $M_1^n = M_1$ cannot be satisfied unless the right-hand member be independent of h and k; and in order that this may be the case, **we must have the function $\dfrac{\phi(111) + h\psi(111) + k\chi(111)}{\phi(110) + h\psi(110) + k\chi(110)}$ independent of h and k.**

This argument is dubious, but the equations in (3) at the top of the next page are correct constraints (the ψ in the third equation should be ϕ); however they can be significantly strengthened as stated there in the margin.

Assume then

$$\frac{\phi(111) + h\psi(111) + k\chi(111)}{\phi(110) + h\psi(110) + k\chi(110)} = c,$$

c being independent of h and k; we have, on clearing of fractions and equating coefficients,

The requirement that $M_1^n = M_1$ for all h, k is equivalent to requiring that the three quotients at the bottom of the page do not have both $\dfrac{\neq 0}{0}$ and $\dfrac{0}{\neq 0}$ occurring. But one could have both $\dfrac{0}{0}$ and $\dfrac{\neq 0}{0}$ occuring, or one could have both $\dfrac{0}{0}$ and $\dfrac{0}{\neq 0}$ occuring.

$$\phi(111) = c\phi(110), \quad \psi(111) = c\psi(110), \quad \chi(111) = c\chi(110);$$

whence, eliminating c,

$$\frac{\phi(111)}{\phi(110)} = \frac{\psi(111)}{\psi(110)} = \frac{\chi(111)}{\chi(110)},$$

being equivalent to the triple system

$$\left.\begin{array}{l}\phi(111)\psi(110)-\phi(110)\psi(111)=0\\ \psi(111)\chi(110)-\psi(110)\chi(111)=0\\ \chi(111)\phi(110)\;-\chi(110)\psi(111)=0\end{array}\right\}\quad(3)$$

and it appears that if any one of these equations is not satisfied, the modulus M_1 will not satisfy the condition $M_1^n = M_1$, whence the first term of the value of z must be equated to 0, and we shall have

$$xy = 0,$$

a relation between x and y independent of z.

Now if we expand in terms of z each pair of the primitive equations (1), we shall have

$$\begin{aligned}\phi(xy0)+\{\phi(xy1)-\phi(xy0)\}z&=0,\\ \psi(xy0)+\{\psi(xy1)-\psi(xy0)\}z&=0,\\ \chi(xy0)+\{\chi(xy1)-\chi(xy0)\}z&=0,\end{aligned}$$

and successively eliminating z between each pair of these equations, we have

$$\begin{aligned}\phi(xy1)\psi(xy0)-\phi(xy0)\psi(xy1)&=0,\\ \psi(xy1)\chi(xy0)-\psi(xy0)\chi(xy1)&=0,\\ \chi(xy1)\phi(xy0)-\chi(xy0)\phi(xy1)&=0,\end{aligned}$$

which express all the relations between x and y that are formed by the elimination of z. Expanding these, and writing in full the first term, we have

$$\begin{aligned}&\{\phi(111)\psi(110)-\phi(110)\psi(111)\}xy+\&c.=0,\\ &\{\psi(111)\chi(110)-\psi(110)\chi(111)\}xy+\&c.=0,\\ &\{\chi(111)\phi(110)-\chi(110)\phi(111)\}xy+\&c.=0:\end{aligned}$$

and it appears from Prop. 2. that if the coefficient of xy in any of these equations does not vanish, we shall have the equation

$$xy = 0;$$

but the coefficients in question are the same as the first members of the system (3), and the two sets of conditions exactly agree. Thus, as respects the first term of the expansion, the method of indeterminate coefficients leads to the same result as ordinary elimination; and it is obvious that from their similarity of form, the same reasoning will apply to all the other terms.

One can improve (3) as follows:

$$\begin{aligned}&\phi(1,1,0)\psi(1,1,1)=\phi(1,1,1)\psi(1,1,0)=0\\ &\psi(1,1,0)\chi(1,1,1)=\psi(1,1,1)\chi(1,1,0)=0\\ &\chi(1,1,0)\phi(1,1,1)=\chi(1,1,1)\phi(1,1,0)=0,\end{aligned}$$

and add

$$\phi(1,1,0)\phi(1,1,1)=\psi(1,1,0)\psi(1,1,1)=\chi(1,1,0)\chi(1,1,1)=0.$$

Using the reduction and elimination theorems of LT, a system of equations

$$p_1(\vec{x},z)=\cdots=p_k(\vec{x},z)=0$$

can be solved for z iff for every i,j:

$$p_i(\vec{x},0)p_j(\vec{x},1)=0.$$

Correction: "…each of the primitive…"

This does not express all the relations between x and y after eliminating z. See the above discussion of systems of equations.

Suppose, in the second place, that the conditions (3) are satisfied so that M_1 is independent of h and k. It will then indifferently assume the equivalent forms

$$M_1 = \frac{1}{1 - \dfrac{\phi(111)}{\phi(110)}} = \frac{1}{1 - \dfrac{\psi(111)}{\psi(110)}} = \frac{1}{1 - \dfrac{\chi(111)}{\chi(110)}}$$

These forms need not be equivalent for M_1 to be idempotent. M_1 is idempotent iff one of (a), (b) and (c) holds:

(a) all three are $0/0$, or

(b) some are 1 and all others are $0/0$, or

(c) some are 0 and all others are $0/0$.

Then

$M_1 = 0/0$ in case (a);

$M_1 = 1$ in case (b);

$M_1 = 0$ in case (c).

These are the exact forms of the first modulus in the expanded values of z, deduced from the solution of the three primitive equations singly. If this common value of M_1 is 1 or $\frac{0}{0} = v$, the term will be retained in z; if any other constant value (except 0), we have a relation $xy = 0$, not given by elimination, but deducible from the primitive equations singly, and similarly for all the other terms. Thus in every case the expression of the subsidiary relations is a necessary accompaniment of the process of solution.

It is evident, upon consideration, that **a similar proof will apply to the discussion of a system indefinite as to the number both of its symbols and of its equations.**

POSTSCRIPT.

Some additional explanations and references which have occurred to me during the printing of this work are subjoined.

The remarks on the connexion between Logic and Language, p. 5, are scarcely sufficiently explicit. Both the one and the other I hold to depend very materially upon our ability to form general notions by the faculty of abstraction. **Language is an instrument of Logic, but not an indispensable instrument.**

To the remarks on Cause, p. 12, I desire to add the following: Considering Cause as an invariable antecedent in Nature, (which is Brown's view), whether associated or not with the idea of Power, as suggested by Sir John Herschel, the knowledge of its existence is a knowledge which is properly expressed by the word *that* (τὸ ὅτὶ), not by *why* (τὸ διότι). It is very remarkable that the two greatest authorities in Logic, modern and ancient, agreeing in the latter interpretation, differ most widely in its application to Mathematics. Sir W. Hamilton says that Mathematics . . . . (pagebreak in MAL) exhibit only the *that* (τὸ ὅτι): Aristotle says, The *why* belongs to mathematicians, for they have the demonstrations of Causes. *Anal. Post.* lib. I., cap. XIV. It must be added that Aristotle's view is consistent with the sense (albeit an erroneous one) which in various parts of his writings he virtually assigns to the word Cause, viz. an antecedent in Logic, a sense according to which the premises might be said to be the cause of the conclusion. This view appears to me to give even to his physical inquiries much of their peculiar character.

Upon reconsideration, I think that the view on p. 41, as to the presence or absence of a medium of comparison, would readily follow from Professor De Morgan's doctrine, and I therefore relinquish all claim to a discovery. The mode in which it appears in this treatise is, however, remarkable.

I have seen reason to change the opinion expressed in pp. 42, 43. The system of equations there given for the expression of Propositions in Syllogism is *always* preferable to the one before employed—first, in generality—secondly, in facility of interpretation.

Boole has decided that the secondary translations of propositions, which involve a parameter v, is always preferable to the primary translations. This will carry over to LT.

In virtue of the principle, that a Proposition is either true or false, every elective symbol employed in the expression of hypotheticals admits only of the values 0 and 1, which are the only quantitative forms of an elective symbol. It is in fact possible, setting out from the theory of Probabilities (which is purely quantitative), to arrive at a system of methods and processes for the treatment of hypotheticals exactly similar to those which have been given. The two systems of elective symbols and of quantity osculate, if I may use the expression, in the points 0 and 1. It seems to me to be implied by this, that unconditional truth (categoricals) and probable truth meet together in the constitution of contingent truth; (hypotheticals). **The general doctrine of elective symbols and all the more characteristic applications are quite independent of any quantitative origin.**

THE END.

MY INDEX FOR BOOLE'S BOOK MAL

The index I have compiled for Boole's *Mathematical Analysis of Logic* starts on the next page.

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