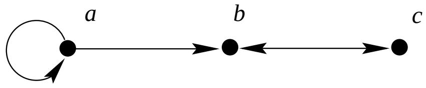
Overview

When studying a subject with a bewildering variety of specimens one hopes to discover simple structural patterns. This happened in the study of finite relational structures with the discovery of 0–1 laws in the mid 1970s.

A finite relational structure $\mathbf{S} = (S, R_1, \ldots, R_n)$ is a set S with a list of relations $R_1, \ldots, R_n$ on S. The most popular relations R are binary, that is, $R \subseteq S \times S$. But $R \subseteq S^k$, for any $k \geq 1$, is also permitted. For the purpose of this overview it suffices to consider relational structures $\mathbf{S} = (S, R)$ with a single binary relation. One can think of R as a directed edge relation on the set of vertices S and draw a picture, for example:



In this picture one sees that aRa, aRb, bRc, cRb. Such structures are *directed graphs.*

As specializations of directed graphs one has several well known classes. If the relation R is irreflexive (not aRa, for any $a \in S$) and symmetric ($aRb \Rightarrow bRa$) one has a *graph.* If it is reflexive, symmetric and transitive, then one has an *equivalence relation.* And if it is reflexive, antisymmetric and transitive, then one has a *poset.* Finite (directed) graphs exhibit incredible diversity and have provided a rich source of examples and problems, from Euler's analysis of the Seven Bridges of Königsberg problem to the modern study of complexity in computer science. Indeed, the interest in finite structures has blossomed in tandem with the growth of theoretical computer science.

One of the fundamental questions is 'What can be said about a randomly chosen structure?' Consider a property $\mathcal{P}$ and a class $\mathcal{K}$ of finite structures. One can ask 'What is the probability that a finite structure chosen randomly from $\mathcal{K}$ satisfies $\mathcal{P}$?' The simplest and most natural definition of this probability is to let p_n be the proportion of structures in $\mathcal{K}$ of size n that have the property $\mathcal{P}$, and then to let the probability that $\mathcal{P}$ holds for a randomly chosen structure from $\mathcal{K}$ be the limit of p_n as n goes to infinity (whenever this limit exists). This probability is called *the asymptotic density* of the collection of structures in $\mathcal{K}$ that satisfy φ.

There is a problem with this definition if there are infinitely many n such that $\mathcal{K}$ has no structures of size n, for then p_n is infinitely often undefined.

This is handled by considering only those p_n that are defined. With this understanding it turns out that the general theory is a minor variation of the theory where one assumes that the p_n are eventually well-defined. So, for the purpose of this overview, it will be assumed that the classes $\mathcal{K}$ have structures of size n for all n greater than some N.

Glebskij, Logan, Liogonkij and Talanov (1969), and independently Fagin (1976), considered properties defined by first-order sentences. For example, the sentence $\forall x \exists y (xRy)$ says, in the case of finite graphs, that there are no isolated vertices; and the sentence $\forall x \forall y \big[(xRy) \rightarrow \exists z (xRz \wedge zRy)\big]$ says one can interpolate a vertex between adjacent vertices. For $\mathcal{K}$ the class of finite directed graphs, or the class of finite graphs, they showed, for any first-order sentence φ, that the probability of φ holding is either 0 or 1. Hence [directed] graphs have a *first-order 0–1 law.* If the probability of a property holding in a class $\mathcal{K}$ is 1 then the property is *almost certainly true* in $\mathcal{K}$.

The method used to prove these results does not generalize readily. First the result is proved for *labeled structures*, using the set of vertices $\{1, \ldots, n\}$ for n-element structures. The following two directed graphs are identified when counting up to isomorphism, but are considered distinct when counting labeled structures:

When counting up to isomorphism one is said to be counting *unlabeled* structures.

To prove a 0–1 law for labeled structures, the original method is to understand the structures in $\mathcal{K}$ well enough to propose a basis Φ for the almost certainly true sentences. Then one proves that each member of Φ is indeed almost certainly true. Finally, to take a labeled 0–1 law back to the unlabeled case one needs to know that the property of being *rigid*[1] is almost certainly true in $\mathcal{K}$. Finding a basis Φ, and proving that rigidity is almost certainly true, are both serious obstacles to extending the applications of this method. Compton was able to carry this out for *posets*, but there are precious few other examples.

In the 1980s an alternate approach to proving 0–1 laws was developed by Compton. By considering *adequate* classes $\mathcal{K}$ of finite structures, that is, classes closed under disjoint union and components, he was able to show that the single condition $a(n-1)/a(n) \rightarrow 1$ is sufficient for a first-order 0–1 law. Here $a(n)$ counts the total number of structures of size n in $\mathcal{K}$ (counting up to isomorphism). This method does not apply to the original examples of graphs and directed graphs as these classes grow so rapidly that one has $a(n-1)/a(n) \rightarrow 0$. However, it is a beautiful technique that does apply to a truly wide range of slowly growing classes such as equivalence relations, permutations, linear forests, etc.

[1] A structure is rigid if the only automorphism is the identity map.

Furthermore, Compton's technique yields a 0–1 law for *monadic second-order logic*. This logic extends the well known first-order logic by allowing quantification over subsets—it is able to express far more properties than first-order logic. For example, in first-order logic one cannot say that a graph is *connected*, but in monadic second-order logic this is expressible by saying that 'if the domain is partitioned into two sets then it is always possible to find two vertices, one from each partition set, with an edge between them':

$$\forall U \forall V \Big[\Big(\exists x (Ux) \ \wedge \ \exists y (Vy) \ \wedge \ \forall x \big(Ux \leftrightarrow \neg\, Vx \big) \Big) \\ \rightarrow \ \exists x \exists y \, \big(Ux \ \wedge \ Vy \ \wedge \ xRy \big) \Big].$$

A considerable portion of the work in Compton's treatment is devoted to studying *Dirichlet density*. (This density is the limiting value of a quotient of two generating series.) He also uses the *fundamental identity*

$$\sum_{n \geq 0} a(n) x^n \ = \ \prod_{n \geq 1} \big(1 - x^n\big)^{-p(n)}$$

that relates $a(n)$ to $p(n)$, where $p(n)$ is the number of *components* of size n (counting up to isomorphism). On the surface, Dirichlet density appears to have little to do with the probability that a randomly chosen structure has a property, but under a natural hypothesis one can show that if the probability exists then so does the Dirichlet density, and they are equal. Thus Dirichlet density extends probability (defined as asymptotic density), and the main goal is to find theorems for a converse result that guarantees that if the Dirichlet density exists then it is the asymptotic density. Such converses are called Tauberian Theorems. For the development of the very large part of Compton's theory just described in this paragraph, one does not need the details of the relations involved in the individual structures, but rather just the additive number system associated with $\mathcal{K}$.

Consider the example of the class $\mathcal{K}$ of finite graphs. If one defines the sum of two finite graphs to be their disjoint union then, by identifying isomorphic graphs, one ends up with an additive number system $\mathcal{A}$. The graphs are the 'numbers' in this abstract system. The graph on the empty set gives the zero element of $\mathcal{A}$. The only information that is needed for the asymptotic work with additive number systems is the addition table for the elements and the 'size' (or *norm*) of a graph (defined as the number of vertices). The indecomposable elements, the nonzero elements that cannot be written as the sum of two nonzero elements, are precisely the connected graphs. Clearly every nonzero element of $\mathcal{A}$ can be expressed uniquely as a sum of indecomposable elements since every finite graph with a nonempty set of vertices is uniquely expressible as a disjoint union of connected graphs. The essential features of an additive number system are just the addition operation and the size function. The adjective 'additive' is derived from the fact that the size function $\|\mathsf{a}\|$ is additive, that is, $\|\mathsf{a} + \mathsf{b}\| = \|\mathsf{a}\| + \|\mathsf{b}\|$.

To simplify the presentation, additive number systems are interpolated between adequate classes of structures and the fundamental identities:

Adequate Classes of Structures	Additive Number Systems	Fundamental Identities

This interpolation allows one to separate the logical aspects from the number-theoretic aspects. Every adequate class gives rise to a (unique) additive number system, and every additive number system can be derived from (many) adequate classes; and each additive number system gives rise to a (unique) fundamental identity that essentially defines the number system.

Given an additive number system $\mathcal{A}$, subsets called *partition sets* play a crucial role in this work. A partition set has the form $\gamma_1\mathsf{P}_1 + \cdots + \gamma_k\mathsf{P}_k$, where $\mathsf{P}_1, \ldots, \mathsf{P}_k$ is a partition of the set P of indecomposable elements of $\mathcal{A}$, and where each of the γ_i is in one of the three forms $(\geq m_i)$, m_i, $(\leq m_i)$. The elements of such a partition set are the members of $\mathcal{A}$ which can be written as a sum of γ_1 members of P_1 plus $\ldots$ plus γ_k members of P_k. (Repeats are allowed when counting elements.) Thus $1\mathsf{P}$ is just P, and $(\geq 0)\mathsf{P}$ is the entire set of 'numbers' in $\mathcal{A}$. And if $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3$ is a partition of the set P of indecomposable elements of $\mathcal{A}$ then the partition set $(\leq 5)\mathsf{P}_1 + (\geq 4)\mathsf{P}_2 + 2\mathsf{P}_3$ is the set of elements which can be expressed as the sum of at most 5 elements from P_1 plus the sum of at least 4 elements from P_2 plus the sum of exactly 2 elements from P_3.

Now, if $\mathcal{K}$ is an adequate class of structures and $\mathcal{A}$ the corresponding additive number system, then the members of $\mathcal{K}$ that satisfy a given monadic second-order sentence can be described, when viewed in $\mathcal{A}$, as a disjoint union of finitely many partition sets. Thus, if one can show that every partition set of $\mathcal{A}$ has density 0 or 1 then $\mathcal{K}$ has a monadic second-order 0–1 law.

The method of Compton, to establish a 0–1 law for $\mathcal{K}$, is indeed to show that all partition sets of $\mathcal{A}$ have asymptotic density 0 or 1. The necessary and sufficient condition for this to hold is that $a(n-1)/a(n) \to 1$. To obtain interesting conditions that guarantee $a(n-1)/a(n) \to 1$, one turns to the fundamental identity. This ties in with the established work in asymptotic additive number theory, a subject often described by referring to the famous results of Hardy and Ramanujan on the number of partitions of an integer. The applications presented in Chapter 4 are based on an analysis of Bateman and Erdös of additive number systems which have at most one indecomposable of each size.

In a subsequent paper Compton turned to more general limit laws for logic, where one only requires that each sentence φ have a probability of holding (the probability need not be 0 or 1). Again, additive number systems are used to shift to the study of conditions that guarantee that every partition set has an asymptotic density. A rather delicate analysis is needed to establish the main result of Compton. One of the striking corollaries is that if $a(n)$ is asymptotic to $C\beta^n$ then $\mathcal{K}$ has a limit law. Further applications come from the asymptotics of Knopfmacher, Knopfmacher, and Warlimont. By applying the Cauchy integral formula

$$a(n) \;=\; \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\mathbf{A}(z)}{z^{n+1}}dz$$

to $\mathbf{A}(z) = \sum a(k)z^k$, they are able to find asymptotics for $a(n)$ when $p(n) = C\beta^n + \mathrm{o}(\eta^n)$, where $0 < \eta < \beta$. This gives numerous applications of Compton's theorem, for example, to two-colored linear forests.

In the late 1980s Compton was considering the application of his ideas to the study of randomly chosen members of classes $\mathcal{K}$ of finite algebraic structures—the class of finite Abelian groups is an excellent example. Here the key operation is *direct product*, not disjoint union. The analog of a component (of a relational structure) is, in this setting, a (*directly*) *indecomposable* structure—that is, a structure which has at least two elements and is not isomorphic to a direct product of smaller structures. The indecomposable finite Abelian groups are precisely the $\mathbf{Z}_{p^n}$, where $\mathbf{Z}_{p^n}$ is the group of integers modulo p^n, for p a prime number. As is well known, every nontrivial finite Abelian group can be uniquely expressed as a product of indecomposable Abelian groups. This is the *unique factorization property* for finite Abelian groups. However, unique factorization does not hold for many important classes of algebraic structures, for example, semigroups.

In the context of finite algebraic structures let us say that $\mathcal{K}$ is *adequate* if it is closed under direct product, direct factors, and it has the unique factorization property. Let $a(n)$ count the number of structures in $\mathcal{K}$ of size n, and let $p(n)$ count the number of indecomposables in $\mathcal{K}$ of size n. (All counting is done up to isomorphism.) From the fact that $\mathcal{K}$ is adequate one has the fundamental identity

$$\sum_{n\geq 1} a(n)n^{-s} \;=\; \prod_{n\geq 2}\left(1-n^{-s}\right)^{-p(n)}.$$

What is the probability that a randomly chosen structure from $\mathcal{K}$ satisfies a given property $\mathcal{P}$? If one defines p_n as before, namely the proportion of structures in $\mathcal{K}$ of size n that have the property $\mathcal{P}$, then very little can be said. However, if one uses the *global* count, that is, let P_n be the proportion of structures in $\mathcal{K}$ of size *at most* n that have the property $\mathcal{P}$, and if one lets the probability be the limit of P_n as n goes to infinity (provided this exists), then the results for relational structures have remarkable parallels in the algebraic setting.

In Part 2 multiplicative number systems are interpolated between adequate classes and fundamental identities, giving an exact analog of the use of additive number systems in Part 1. This is used, as in Part 1, to give a separation of the number theoretic and logical aspects.

Adequate Classes of Structures	—	Multiplicative Number Systems	—	Fundamental Identities

In the case of Abelian groups one obtains the associated number system by letting the 'numbers' be the finite groups $\mathbf{G}$, and letting multiplication be direct product. These number systems are said to be multiplicative because the size function is multiplicative, that is, $\|\mathsf{a} \cdot \mathsf{b}\| = \|\mathsf{a}\| \cdot \|\mathsf{b}\|$.

The multiplicative number system analog of Compton's theorem (on general limit laws for relational structures) has, as a corollary, the fact that if $A(x)$ is asymptotic to cx^α, where $A(x) = \sum_{n \le x} a(n)$, then $\mathcal{K}$ has a first-order limit law. This applies to the example of Abelian groups, and one can show, for example, that the probability of a finite Abelian group having an element of order 2 is $1 - \prod_{n \ge 1} \left(1 - 2^{-n}\right) \approx 0.71$.

Many of the interesting examples, like Abelian groups, have limit laws, but not 0–1 laws. Further examples are again found by turning to complex analysis, using Perron's integral formula, to generalize Oppenheim's asymptotics for the number of ways to factor the numbers less than or equal to n. From this analysis one concludes that if $p(n) = Cn^\alpha + O(n^\beta)$ with $C > 0$, $\alpha \ge 0$, and $\beta < \alpha$, then $\mathcal{K}$ has a first-order limit law. Thus the class of finite lattices that decompose into a product of chains has a first-order limit law.

Chapter 6, the last chapter of Part 1, covers the logic results for adequate classes (with respect to disjoint union) of purely relational structures; and Chapter 12, the last chapter of Part 2, covers the logic results for adequate classes (with respect to direct product) of structures.

These two chapters can be briefly summarized as follows. Given an adequate class $\mathcal{K}$ and a sentence φ, let $\mathcal{K}_\varphi$ be the class of members of $\mathcal{K}$ that satisfy φ. The goal is to prove that $\mathcal{K}_\varphi$ is a disjoint union of finitely many partition classes. Then the number theoretic results on asymptotic density of partition sets can be used.

Three tools are needed to prove these logic results: the Ehrenfeucht-Fraissé games in the additive case; and the Feferman-Vaught Theorem, with Skolem's analysis of sentences about finite Boolean algebras, in the multiplicative case. These tools are fully developed in Chapters 6 and 12.