C&O 367, Winter 2008
Assignment 2, Due Feb. 4 (at start of class; late assign will not be accepted)

To Hand in:
1. Section 1: Problems 3,5;
2. Section 2: Problems 1;
3. Section 3: Problems 1,2,5;
4. Section 4: Problems 1.
5. Section 5: Problems 1. (This BONUS question is optional.)

1 Background/Theory

1. Suppose that $C[a,b]$ denotes the vector space of all real valued continuous functions on the interval $[a,b]$. Show that
$$\langle f, g\rangle = \int_a^b f(t)g(t)dt$$
defines an *inner product* on $C[a,b]$.

2. Find an (simple) example of a function that is continuous on $[-1,1]$ but not Lipschitz continuous. (No oscillating junk is permitted.)

3. (9 marks) Compute the gradient and Hessian of the following functions:
   (a) $f(x) = x^tAx$, where A is an $n \times n$ matrix (A is not assumed to be symmetric).
   (b) $f(x) = \frac{1}{2}x^tAx + b^tx$ (A symmetric).
   (c) $f(x) = g(x)^tg(x)$ where $g : \Re^n \to \Re^m$ and g is twice differentiable.

4. Let $x : [-1,+1] \to \Re^n$, $x \in C^2([-1,+1])$, and let $f \in C^2(\Re^n)$. Compute the gradient and Hessian of $f(x(t))$ at $t = 0$.

5. (5 marks) Consider the problem
$$\min_x x^2 - xy + 2y^2 - 2x + e^{x+y}.$$
Is the first-order necessary optimality condition also sufficient for optimality? Why?

6. (10 marks) Suppose $f : \Re \to \Re$ is convex and bounded above. Show that f is constant.

7. (10 marks)From the text: page 32, #7 b),e); page 33, #12 b),f);#16 a).

2 Optimality Conditions

1. (10 marks) Suppose that $q(x) = \frac{1}{2}x^TPx + b^Tx + \alpha$ is a quadratic function on $\mathbb{R}^n$ and $P = P^T \in \mathcal{S}^n$, the space of $n \times n$ symmetric matrices. Prove that the following three statements are equivalent.
   (a) $q(x)$ is bounded below.
   (b) $P \succeq 0$ (positive semidefinite) and $b \in \mathcal{R}(P)$, the range of P.
   (c) $-P^\dagger b \in \operatorname{argmin}_x\{q(x)\}$ ($P^\dagger$ denotes Moore-Penrose generalized inverse of P).

3 Convexity

1. (10 marks) A set C is *midpoint convex* if

$$a, b \in C \text{ implies } \frac{1}{2}(a+b) \in C.$$

Suppose that C is closed and midpoint convex. Show that C is convex.

2. (10 marks) Let $C \subset \mathbb{R}^n$ be the solution set of a quadratic inequality

$$C = \{x \in \mathbb{R}^n : x^T A x + b^T x + c \leq 0\}$$

where $A \in \mathcal{S}^n, b \in \mathbb{R}^n, c \in \mathbb{R}$.

   (a) Show that C is convex if $A \succeq 0$.

   (b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$, for some $\lambda \in \mathbb{R}$.

Are the converses of these statements true?

3. The conic hull of a set C, is the smallest convex cone that contains the set C. Consider the set of rank-k *outer products*, defined as $\{XX^T : X \in \mathbb{R}^{n \times k}, \text{rank } X = k\}$. Describe its conic hull in simple terms.

4. Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

5. (10 marks) Let $C \subset \mathbb{R}^n$. Define the distance to the farthest point of C

$$f(x) = \sup_{y \in C} \|x - y\|.$$

Show that f is a convex function.

4 MATLAB

1. (10 marks) Consider Rosenbrock's test function

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Find the minimum point for this function from the initial estimate $x^{(1)} = (-1 \ 1)^T$. Use the MATLAB package and use the following three methods:

   (a) Newton with approximate linesearch,

   (b) quasi-Newton with BFGS update with approximate linesearch,

   (c) steepest descent with exact line search.

You can experiment with different approximate linesearches, e.g. backtracking with initial steplength of 1. Or you can use MATLAB to do the exact line search for you. (Please include well-documented programs.)

HINT: Try $\boxed{\text{help bandem}}$ in MATLAB. Then look for the files related to this demo in the optimization toolbox. Alternatively, you can try $\boxed{\text{optimtool}}$ in MATLAB.

5 BONUS Question

1. Consider the problem of minimizing $f : \mathbb{R}^n \to \mathbb{R}$, where

$$f(x) = \frac{\|Ax - b\|_2^2}{c^T x + d}, \qquad \mathrm{dom}\, f = \{x : c^T x + d > 0\}$$

and $\mathrm{rank}\, A = n, b \notin \mathcal{R}(A)$, range of A. Show that the minimizer x^* of f is given by

$$x^* = x_1 + t x_2,$$

where $x_1 = (A^T A)^{-1} A^T b, x_2 = (A^T A)^{-1} c$, and $t \in \mathbb{R}$ can be found by solving a quadratic equation.