1.1 max $f(x) = x_1 x_2^2 x_3^3$
s.t. $g(x) = x_1^2 + x_2^2 + x_3^2 - 1 = 0$

$$0 = \nabla f(x) + \lambda \nabla g(x) \Rightarrow \begin{bmatrix} x_2^2 x_3^3 \\ 2x_1 x_2 x_3^3 \\ 3x_1 x_2^2 x_3^2 \end{bmatrix} + \lambda \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$\lambda = 0 \Rightarrow x_2 x_3 = 0 \Rightarrow f(x) = 0$
but f can have positive and negative values for feasible x, so these can't be extrema.
Similarly, we may assume $x_1, x_2, x_3 \neq 0$.

$$\begin{aligned}
(1) &\Rightarrow x_2^2 x_3^3 + 2\lambda x_1 = 0 \Leftrightarrow x_1 x_2^2 x_3^3 + 2\lambda x_1^2 = 0 \quad (*) \quad \text{as } x_1 \neq 0 \\
(2) &\Rightarrow 2x_1 x_2 x_3^3 + 2\lambda x_2 = 0 \Leftrightarrow 2x_1 x_2^2 x_3^3 + 2\lambda x_2^2 = 0 \quad \text{as } x_2 \neq 0 \\
(3) &\Rightarrow 3x_1 x_2^2 x_3^2 + 2\lambda x_3 = 0 \Leftrightarrow 3x_1 x_2^2 x_3^3 + 2\lambda x_3^2 = 0 \quad \text{as } x_3 \neq 0
\end{aligned}$$

add these to get

$$6x_1 x_2^2 x_3^3 + 2\lambda \underbrace{(x_1^2 + x_2^2 + x_3^2)}_{=1 \text{ for feasible } (x_1, x_2, x_3)} = 0$$

$$\Rightarrow 6x_1 x_2^2 x_3^3 + 2\lambda = 0$$
$$\Rightarrow \lambda = -3x_1 x_2^2 x_3^3$$

$$\Rightarrow x_1 x_2^2 x_3^3 + 2(-3x_1 x_2^2 x_3^3) x_1^2 = 0 \quad \text{by } (*)$$
$$\Rightarrow \underbrace{x_1 x_2^2 x_3^3}_{\neq 0} (1 - 6x_1^2) = 0$$
$$\Rightarrow x_1 = \pm\sqrt{1/6}$$

Similarly, we get $x_2 = \pm\sqrt{1/3}$ and $x_3 = \pm\sqrt{1/2}$.

Continued...

1.1 Continued...

This yields 8 different solutions,

$(x_1, x_2, x_3) = (\pm\sqrt{1/6}, \pm\sqrt{1/3}, \pm\sqrt{1/2})$.

$|f(x)| = (\sqrt{1/6})(\sqrt{1/3})^2(\sqrt{1/2})^3 = \sqrt{1/432}$
for all these solutions.

Thus, we have that the solutions above give all the extreme values of $f(x)$ and whether they are minimizers or maximizers depends only on the number of "+" signs chosen.

Eg., $(\sqrt{1/6}, \sqrt{1/3}, \sqrt{1/2})$ is a maximizer while
$(-\sqrt{1/6}, \sqrt{1/3}, \sqrt{1/2})$ is a minimizer.

Let $f(x_1,x_2,x_3)=x_3$.

1.2] Consider the program

$$\text{Max } x_3$$
$$\text{s.t. } F(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2-x_1x_3-x_2x_3+2x_1+2x_2+2x_3-2=0.$$

We could also minimize, but we'll do both cases at the same time with Lagrange multipliers.

$0=\nabla f(x)+\lambda \nabla F(x)$

$$\Rightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}+\lambda\begin{bmatrix}2x_1-x_3+2\\2x_2-x_3+2\\2x_3-x_1-x_2+2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

$\lambda=0$ violates (3), so $\lambda\neq 0$.

$\lambda\neq 0$ & (1) $\Rightarrow 2x_1-x_3+2=0 \Rightarrow x_3=-2x_1-2$

$\lambda\neq 0$ & (2) $\Rightarrow 2x_2-x_3+2=0 \Rightarrow x_3=-2x_2-2$

these imply $x_1=x_2$.

We require feasibility for minimizing and maximizing solutions, so sub in $x_2=x_1$ and $x_3=-2x_1-2$ into $F(x_1,x_2,x_3)=0$. This yields

$$2x_1^2+(-2x_1-2)^2-x_1(-2x_1-2)-x_1(-2x_1-2)+4x_1+2(-2x_1-2)-2=0$$
$$\Rightarrow 2x_1^2+4x_1^2+8x_1+4+2x_1^2+2x_1+2x_1^2+2x_1+4x_1-4x_1-4-2=0$$
$$\Rightarrow 10x_1^2+12x_1-2=0$$
$$\Rightarrow 5x_1^2+6x_1-1=0$$

Continued...

1.2] Continued...

$$\Rightarrow x_1 = \frac{-6 \pm \sqrt{36 - 4(5)(-1)}}{10} = \frac{-6 \pm \sqrt{56}}{10} = \frac{-6 \pm 2\sqrt{14}}{10}$$

$$= \frac{-3 \pm \sqrt{14}}{5}$$

For $x_1 = \frac{-3-\sqrt{14}}{5}$ we have $x_2 = \frac{-3-\sqrt{14}}{5}$

and $x_3 = -2\left(\frac{-3-\sqrt{14}}{5}\right) - 2$

$$= \frac{6+2\sqrt{14}-10}{5}$$

$$= \frac{-4+2\sqrt{14}}{5}$$

For $x_1 = \frac{-3+\sqrt{14}}{5}$ we have $x_3 = -2\left(\frac{-3+\sqrt{14}}{5}\right) - 2$

$$= \frac{6-2\sqrt{14}-10}{5}$$

$$= \frac{-4-2\sqrt{14}}{5}$$

These values satisfy the $\nabla f(x) + \lambda \nabla g(x) = 0$ condition for some λ so the above two values for x_3 are the two extrema.

Alternative Solution: Use implicit differentiation and set $\frac{\partial x_3}{\partial x_1} = \frac{\partial x_3}{\partial x_2} = 0$.

Will probably need 2nd order optimality conditions to prove a solution is a maximizer or minimizer.

1.3 Max $2x_1 + 3x_2$

$\sqrt{x_1} + \sqrt{x_2} = 5$

let $f(x) = 2x_1 + 3x_2$

& $g(x) = \sqrt{x_1} + \sqrt{x_2} - 5$

Lagrange multiplier rule:

$$0 = \nabla f(x) + \lambda \nabla g(x) \Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} \frac{1}{2\sqrt{x_1}} \\ \frac{1}{2\sqrt{x_2}} \end{bmatrix} = 0$$

$$\Rightarrow -\lambda = 4\sqrt{x_1}$$

$$\& \; -\lambda = 6\sqrt{x_2}$$

$$\Rightarrow 4\sqrt{x_1} = 6\sqrt{x_2}$$

$$\Rightarrow 2\sqrt{x_1} = 3\sqrt{x_2} \quad (*)$$

For (x_1, x_2) to be feasible we need

$$\sqrt{x_1} + \sqrt{x_2} = 5$$

$$\Rightarrow \sqrt{x_1} = 5 - \sqrt{x_2}$$

$$\Rightarrow 2\sqrt{x_1} = 10 - 2\sqrt{x_2}$$

$$\Rightarrow 3\sqrt{x_2} = 10 - 2\sqrt{x_2} \quad \text{by } (*)$$

$$\Rightarrow 5\sqrt{x_2} = 10$$

$$\Rightarrow \sqrt{x_2} = 2$$

$$\Rightarrow x_2 = 4$$

$$\Rightarrow x_1 = 9 \quad \text{by } (*)$$

So Lagrange multiplier rule suggests $x = (9, 4)$ as the optimal solution, but this is wrong.

By inspection, we see that $x^* = (0, 25)$ is the optimal solution, so what went wrong?

Continued...

1.3. continued...

$\nabla g(x^*)$ does not exist!

So we needed to check the cases where $\nabla g(x)$ doesn't exist. The only feasible points where $\nabla g(x)$ doesn't exist are $(0,25)$ and $(25,0)$.

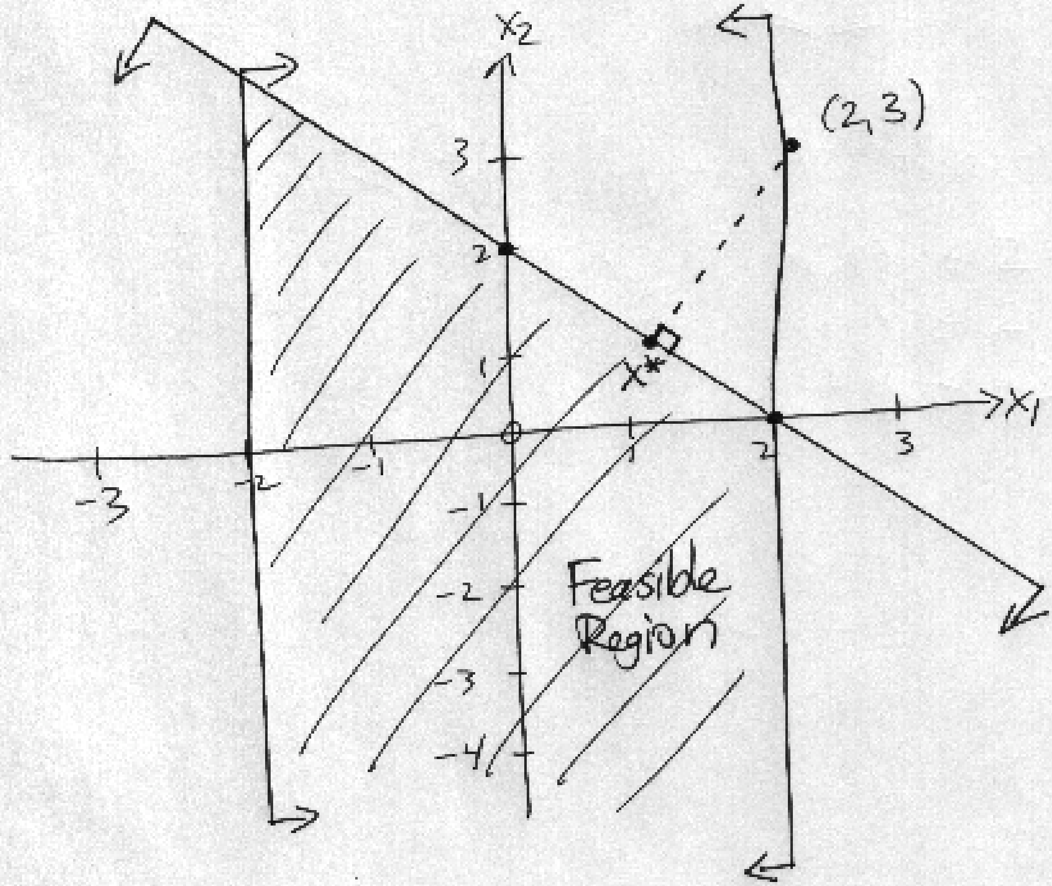
The function $f(x)$ evaluates to 50 at $(25,0)$ and 75 at $(0,25)$.

$$f(9,4) = 18 + 12 = 30$$

So f is minimized at $(9,4)$ and maximized at $(0,25)$.

2. The closest point to (2,3) satisfying the conditions is the optimal solution to the following convex program

$$\begin{aligned} \min\ & \|(x_1,x_2)-(2,3)\|^2 \\ \text{s.t.}\ & x_1+x_2 \le 2 \\ & |x_1| \le 2 \end{aligned} \quad \text{or equivalently} \quad \begin{aligned} \min\ & (x_1-2)^2+(x_2-3)^2 \\ \text{s.t.}\ & x_1+x_2 \le 2 \\ & x_1 \le 2 \\ & -x_1 \le 2 \end{aligned}$$



x^* is the closest point in the feasible region to (2,3).

2. $(x-y)^T(x^*-y) \geq \|x^*-y\|^2 \quad \forall x \in C$

$\Leftrightarrow (x-y)^T(x^*-y) - (x^*-y)^T(x^*-y) \geq 0 \quad \forall x \in C$

$\Leftrightarrow [(x-y)-(x^*-y)]^T(x^*-y) \geq 0 \quad \forall x \in C$

$\Leftrightarrow (x-x^*)^T(x^*-y) \geq 0 \quad \forall x \in C$

$\Leftrightarrow (x-x^*)^T(y-x^*) \leq 0 \quad \forall x \in C$

$\Leftrightarrow$ $x^* \in C$ is closest to y by Thm 5.1.1.

3. $C_1^\circ = \text{int}(C_1)$

$A - B := \{a-b : a \in A, b \in B\}$

<u>Claim:</u> $C_1^\circ - C_2$ is convex

let $x_1 - y_1, x_2 - y_2 \in C_1^\circ - C_2$, where $x_1, x_2 \in C_1^\circ$ and $y_1, y_2 \in C_2$, and $\lambda \in [0,1]$.

$$\lambda(x_1 - y_1) + (1-\lambda)(x_2 - y_2) = \underbrace{\left[\lambda(x_1) + (1-\lambda)x_2\right]}_{\substack{\in C_1^\circ \\ \text{since } C_1^\circ \text{ is convex} \\ \text{(see Thm 5.1.8)}}} - \underbrace{\left[\lambda y_1 + (1-\lambda) y_2\right]}_{\substack{\in C_2 \\ \text{since } C_2 \text{ is} \\ \text{convex.}}} \in C_1^\circ - C_2$$

<u>Claim:</u> $C_1^\circ - C_2$ has interior points.

If $C_2 = \emptyset$ then $C_1^\circ - C_2 = C_1^\circ \neq \emptyset$ so $\text{int}(C_1^\circ - C_2) = \text{int}(C_1^\circ) = C_1^\circ \neq \emptyset$

by hypothesis

If $C_2 \neq \emptyset$ then let $x - y \in C_1^\circ - C_2$, $x \in C_1^\circ$, $C_2 \ni y$.

$x \in C_1^\circ \Rightarrow \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq C_1^\circ$

$\Rightarrow B(x-y, \varepsilon) \subseteq C_1^\circ - \{y\} \subseteq C_1^\circ - C_2$

$\Rightarrow x - y \in \text{int}(C_1^\circ - C_2)$

<u>Case:</u> $0 \in \text{cl}(C_1^\circ - C_2)$

$C_1^\circ \cap C_2 = \emptyset \Rightarrow 0 \notin C_1^\circ - C_2 \Rightarrow 0 \notin \text{int}(C_1^\circ - C_2)$

but $0 \in \text{cl}(C_1^\circ - C_2) \Rightarrow 0 \in \partial(C_1^\circ - C_2)$

boundary

By the Support Thm (5.1.9), $\exists a \in \mathbb{R}^n$, $a \neq 0$ such that $a \cdot \tilde{x} \leq a \cdot 0 \quad \forall \tilde{x} \in C_1^\circ - C_2$

$\Rightarrow a \cdot (x - y) \leq 0 \quad \forall x \in C_1^\circ, y \in C_2$

continued...

3. continued...

$$\Rightarrow \quad a\cdot x \leq a\cdot y \quad \forall x\in C_1^\circ, y\in C_2$$

$$\Rightarrow \quad a\cdot x \leq a\cdot y \quad \forall x\in C_1, y\in C_2$$

[Indeed, if $\exists\, x\in C_1$ and $y\in C_2$ s.t. $a\cdot x > a\cdot y$, then $\exists\, \tilde{x}\in C_1^\circ$ such that $a\cdot\tilde{x} > a\cdot y$, contradicting the second last implication.]

Now define α to be such that

$$\max_{x\in C_1} a\cdot x \leq \alpha \leq \min_{y\in C_2} a\cdot y,$$

this gives the result.

Case 2: $0\notin cl(C_1^\circ - C_2)$

Then since $C_1^\circ - C_2$ is convex, so is $cl(C_1^\circ - C_2)$ by Thm 5.1.7.

So $cl(C_1^\circ - C_2)$ is a closed convex set, with $0\notin cl(C_1^\circ - C_2)$.

Basic Separation Thm (5.1.5) says that $\exists\, a\in\mathbb{R}^n$, $a\neq 0$, and an $\alpha\in\mathbb{R}$ such that

$$a\cdot\tilde{x} \leq \alpha < a\cdot 0 \quad \forall \tilde{x}\in Cl(C_1^\circ - C_2)$$

$$\Rightarrow a\cdot(\tilde{x}) \leq \alpha < 0 \quad \forall \tilde{x}\in C_1^\circ - C_2 \quad \text{as } C_1^\circ - C_2 \subseteq Cl(C_1^\circ - C_2)$$

$$\Rightarrow a\cdot(x-y) < 0 \quad \forall x\in C_1^\circ, y\in C_2$$

$$\Rightarrow a\cdot x < a\cdot y \quad \forall x\in C_1^\circ, y\in C_2$$

$$\Rightarrow a\cdot x \leq a\cdot y \quad \forall x\in C_1, y\in C_2$$

We can choose α as above yielding the desired result.

6. (a) $\min\ t_1^{-1} t_2^{-1}$

$\text{s.t.}\ \frac{1}{2}t_1 + \frac{1}{2}t_2 \le 1$

$t_1 > 0,\ t_2 > 0$

has the same solution as

$\min\ \log(t_1^{-1} t_2^{-1})$

$\text{s.t.}\ t_1 + t_2 \le 2$

$t_1 > 0,\ t_2 > 0$

since log is defined on the feasible region and is a strictly increasing function.

$$\begin{array}{llll} \min & \log(t_1^{-1}t_2^{-1}) = \min & -\log t_1 - \log t_2 = \min & -x_1 - x_2 \\ \text{s.t.} & t_1 + t_2 \le 2 \quad\quad \text{s.t.} & t_1 + t_2 \le 2 \quad\quad \text{s.t.} & e^{x_1} + e^{x_2} \le 2 \\ & t_1 > 0,\ t_2 > 0 & t_1 > 0,\ t_2 > 0 & x_1, x_2 \text{ free} \end{array}$$

Notice we made a change of variables $x_1 = \log t_1$
$x_2 = \log t_2$

$t_1 + t_2 \le 2$, $t_1 > 0, t_2 > 0$ is feasible $\Longleftrightarrow$ $e^{x_1} + e^{x_2} \le 2$, $x_1, x_2 \in \mathbb{R}$ is feasible

The resulting problem is now convex.

continued...

6.(b) Setting $x_1 x_2$ very small satisfies the constraint with strict inequality so the convex program is superconsistent (i.e., strictly feasible, satisfies Slater's condition).

So we can apply KKT Theorem.

feasible (x_1, x_2) is optimal $\Leftrightarrow \exists \lambda$ such that

(1) $\lambda \geq 0$

(2) $\lambda(e^{x_1} + e^{x_2} - 2) = 0$

(3) $\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} e^{x_1} \\ e^{x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

<u>Case $\lambda = 0$:</u>

Then (3) $\Rightarrow \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a contradiction.

<u>Case $\lambda > 0$:</u>

Then (3) $\Rightarrow$ $\left.\begin{array}{l} -1 + \lambda e^{x_1} = 0 \Rightarrow \lambda = 1/e^{x_1} \\ -1 + \lambda e^{x_2} = 0 \Rightarrow \lambda = 1/e^{x_2} \end{array}\right\} \Rightarrow x_1 = x_2$

$\lambda > 0$ and (2) $\Rightarrow e^{x_1} + e^{x_2} - 2 = 0$

$\Rightarrow 2e^{x_1} - 2 = 0$

$\Rightarrow e^{x_1} = 1$

$\Rightarrow x_1 = 0$

$\Rightarrow x_2 = 0$

Check: $(x_1, x_2) = (0,0)$ feasible? $e^0 + e^0 - 2 = 0$ ✓ So $(0,0)$ is feasible.

$\therefore$ $(0,0)$ is the optimal solution to the convex program.

$x_1 = 0 = \log t_1 \Rightarrow t_1 = 1$

$x_2 = 0 = \log t_2 \Rightarrow t_2 = 1$

So $(t_1, t_2) = (1,1)$ is optimal for the original problem.

13. Suppose C is a closed convex cone in $\mathbb{R}^n$.

<u>Claim:</u> C^* is a convex cone

For $t \geq 0$ and $x \in C^*$, want $tx \in C^*$.

$x \in C^* \Rightarrow \langle x, c \rangle \geq 0 \ \forall c \in C$

$\Rightarrow \langle tx, c \rangle \geq 0 \ \forall c \in C, t \geq 0$

$\Rightarrow tx \in C^*$

For $x, y \in C^*$ and $\lambda \in [0,1]$, want $\lambda x + (1-\lambda) y \in C^*$.

$x, y \in C^* \Rightarrow \langle x, c \rangle \geq 0$ and $\langle y, c \rangle \geq 0 \ \forall c \in C$

$\Rightarrow \langle \lambda x, c \rangle \geq 0$ and $\langle (1-\lambda) y, c \rangle \geq 0 \ \forall c \in C$

$\Rightarrow \langle \lambda x + (1-\lambda) y, c \rangle \geq 0 \ \forall c \in C$

$\Rightarrow \lambda x + (1-\lambda) y \in C^*$

This shows that C^* is a convex cone.

Note: we did not use any convexity assumptions on C, nor the fact it's a cone.

Now we must show $(C^*)^* = C$.

Let $c \in C$. Then $\langle c, y \rangle \geq 0 \ \forall y \in C^*$.

So $c \in (C^*)^*$ which implies $C \subseteq (C^*)^*$.

To show $(C^*)^* \subseteq C$, suppose not for a contradiction.

Let $y \in (C^*)^* \setminus C$. Since $y \notin C$ and C is closed and convex, we can use the Basic Separation Thm (5.1.5).

Continued...

13. Continued...

$\Rightarrow \exists a \in \mathbb{R}^n$, $a \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$a^T x \leq \alpha < a^T y \quad \forall x \in C. \qquad (**)$$

$$\Rightarrow \quad -a^T x \geq -\alpha > -a^T y \quad \forall x \in C \qquad (*)$$

Can $-a^T x < 0$ for some $x \in C$?

If so, $x \in C \Rightarrow tx \in C \;\; \forall t \geq 0$ as C is a convex cone

So $-a^T(tx) \to -\infty$ as $t \to \infty$

so for some $\bar{t} \geq 0$ we contradict $(*)$.

So we must have $-a^T x \geq 0 \;\; \forall x \in C$

$\Rightarrow -a \in C^*$

Since $y \in (C^*)^*$ we have that $-a^T y \geq 0$

$\Rightarrow a^T y \leq 0$

$\Rightarrow \alpha < 0$ from $(**)$

$\Rightarrow a^T x < 0$ from $(**)$ $\forall x \in C$

but, we can choose $t > 0$ small enough so that $a^T(tx) > \alpha$, violating $(**)$, a contradiction.

This completes the proof.

13. Continued...

$$((C^*)^* = C$$

$$\Rightarrow C = \bigcap_{y \in C^*} \{x : \langle x, y \rangle \geq 0\}$$ since this is another way of writing $(C^*)^*$

$\Rightarrow$ C is closed since it's an intersection of closed sets.

From earlier, we showed $\tilde{C}^*$ is a convex cone for <u>any</u> set $\tilde{C}$. Thus C is also a convex cone as $C = (C^*)^*$.

Thus, C is a closed convex cone.