1.1. ~~Descending~~ Ascending Order

$$\begin{array}{r} 128.3 \\ +\ 24.47 \\ \hline 152.77 \end{array} \rightarrow \begin{array}{r} 152.7 \\ +\ 3.163 \\ \hline 155.863 \end{array} \rightarrow \begin{array}{r} 155.8 \\ +\ 0.4825 \\ \hline 156.2825 \end{array} \rightarrow 156.2$$

Ascending Order

$$\begin{array}{r} 0.4825 \\ +3.163 \\ \hline 3.6455 \end{array} \rightarrow \begin{array}{r} 3.645 \\ +24.47 \\ \hline 28.115 \end{array} \rightarrow \begin{array}{r} 28.11 \\ +128.3 \\ \hline 156.41 \end{array} \rightarrow 156.4$$

Infinite-Precision

$$\begin{array}{r} 128.3 \\ 24.47 \\ 3.163 \\ +\ 0.4825 \\ \hline 156.4155 \end{array}$$

1.2.] Using "format long" command in Matlab, we see the roots are

$$x_1 = 99.98999899979995l$$
$$x_2 = 0.01000100020005 0$$

(precision 17).

Using the quadratic formula and precision 5, we get

$b^2 = 10000$
$b^2 - 4ac = 10000 - 4 = 9996$
$\sqrt{b^2-4ac} = \sqrt{9996} \longrightarrow 99.979$ (precision 5)

$100 + 99.979 = 199.979 \longrightarrow 199.97$
$$\frac{199.97}{2} = 99.985$$

$100 - 99.979 = 0.021$
$$\frac{0.021}{2} = 0.0105$$

So, the roots are $x_1 = 99.985$, $x_2 = 0.0105$

The first is correct up to 4 digits, the second is correct up to 3 digits.

-Continued-

1.2) New algorithm:

Use the quadratic formula for finding the root largest in magnitude (i.e., absolute values) then use the formula $x_1 x_2 = c/a$ to compute the other root.

The accuracy here is "hurt" when subtracting two numbers of similar sizes (i.e., a difference which is close to 0.). The algorithm above helps to avoid this.

2. Summary

$p(x) = x^3 - 3x - 2$ $q(x) = x^2 - x - 2$

for p(x):

x_0	Newton	fzero	roots
5.95	2	2	
2.95	2	2	2, −1, −1
0	−1	2	
−2	−1	2	

didn't find the root "−1"

for q(x):

x_0	Newton	fzero	roots
5.95	2	2	
2.95	2	2	2, −1
0	−1	−1	
−2	−1	−1	

3.1.

(a) $x_n \Rightarrow 0$

$$\frac{|0-x_{n+1}|}{|0-x_n|^p} = \frac{e^{-(n+1)^2}}{e^{-n^2 p}} = e^{-(n+1)^2+n^2p} = e^{n^2(p-1)+n(-2)-1}$$

If $p=1$, limit is 0. If $p>1$, limit is ∞. <u>Superlinear</u>

(b) Not well-defined

(c) $x_n \Rightarrow 0$

$$\frac{|0-x_{n+1}|}{|0-x_n|^p} = \frac{\frac{1}{n+1}a^{n+1}}{(\frac{1}{n}a^n)^p} = \frac{n^p}{n+1}a^{n+1-np} = \frac{n^p}{n+1}a^{n(1-p)+1}$$

If $p=1$, limit is a. If $p>1$, let $p=1+\varepsilon$, where $\varepsilon>0$. Then

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \underbrace{\frac{n^{1+\varepsilon}}{n+1}}_{\text{blows up}}\underbrace{a^{-\varepsilon n+1}}_{\text{blows up}} = \infty$$

blows up (since $0<a<1$ means $a^{+\varepsilon}\in(0,1)\Rightarrow a^{-\varepsilon}>1$).

<u>Linear</u>

(d) $x_n \Rightarrow 0$

$$\frac{|0-x_{n+1}|}{|0-x_n|^p} = \frac{a^{\log(n+1)}}{a^{p\log n}} = a^{\log(n+1)-p\log n} = a^{\log(n+1)-\log n^p}$$

$$= a^{\log\left(\frac{n+1}{n^p}\right)}$$

Continued...

3.1] Continued...

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = a^{\log\left(\lim_{n\to\infty} \frac{n+1}{n^p}\right)} = \begin{cases} a^{\log 1} & \text{if } p=1 \\ \infty & \text{if } p>1 \end{cases}$$

If $p=1$, limit is $a^{\log 1} = a^0 = 1$

If $p>1$, limit is ∞, since $\lim_{n\to\infty} \frac{n+1}{n^p} = 0$

and $\log x \to -\infty$ as $x \to 0$ (from the right).

Also, we used "$a^{-\infty} = \infty$" since $0<a<1$.

Linear

(e) $x_n \to 0$

$$\frac{|0-x_{n+1}|}{|0-x_n|^p} = \frac{a^{(n+1)\log(n+1)}}{a^{np\log n}} = a^{(n+1)\log(n+1) - np\log n}$$

$$= a^{\log\left(\frac{(n+1)^{n+1}}{n^{np}}\right)}$$

$$= a^{\log\left[\left(\frac{n+1}{n^p}\right)^n (n+1)\right]}$$

$$= a^{\log\left(\left(\frac{1}{n^{p-1}}\right)^n \left(\frac{n+1}{n}\right)^n (n+1)\right)}$$

Continued...

3.1) Continued... for $p=1$,

$$\lim_{n\to\infty} \underbrace{\left(\frac{1}{n^{p-1}}\right)^n}_{=1} \underbrace{\left(\frac{n+1}{n}\right)^n}_{\substack{=e \\ \text{in limit}}} \underbrace{(n+1)}_{\to\infty} = \infty$$

using the fact $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

so $a^{\log\left[\lim_{n\to\infty}\left(\frac{1}{n^{p-1}}\right)^n\left(\frac{n+1}{n}\right)^n(n+1)\right]} = 0$ since $0<a<1$.

for $p>1$,

$$\lim_{n\to\infty}\left(\frac{1}{n^{p-1}}\right)^n\left(\frac{n+1}{n}\right)^n(n+1) = e\cdot\lim_{n\to\infty}\frac{n+1}{n^{n(p-1)}}$$

$$\xrightarrow{\text{L'Hôpital}} = e\cdot\lim_{n\to\infty}\frac{1}{(p-1)e^{n\ln n}(1+\ln n)} \quad (*)$$

$= e\cdot 0$ since $p-1\neq 0$ and the denominator grows large.

$=0$

so $a^{\log\left[\lim_{n\to\infty}\left(\frac{1}{n^{p-1}}\right)^n\left(\frac{n+1}{n}\right)^n(n+1)\right]} = \infty$ since $0<a<1$.

$$\left[\ (*)\ \text{Note: } \frac{d}{dx}x^x = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\left[\frac{x}{x}+\frac{\ln x}{1}\right] = e^{x\ln x}[1+\ln x]\ \right]$$

Superlinear convergence

Assume $f(x^*)=f'(x^*)=0$.

3.2) Want to show $\|x^*-x_{k+1}\| \le c\,\|x^*-x_k\|$, for some $c\in(0,1)$, and for all k sufficiently large.

$$\|x^*-x_{k+1}\| = \|x^*-[x_k - f(x_k)/f'(x_k)]\| = \|x^*-x_k + f(x_k)/f'(x_k)\|$$

Taylor Series
$$\begin{cases} f(x_k) = \cancelto{0}{f(x^*)} + \cancelto{0}{f'(x^*)}(x_k-x^*) + \dfrac{f''(t_1)(x_k-x^*)^2}{2} & \text{for some } t_1\in(x^*,x_k) \\ f'(x_k) = \cancelto{0}{f'(x^*)} + f''(t_2)(x_k-x^*) & \text{for some } t_2\in(x^*,x_k) \end{cases}$$

So
$$\begin{aligned} \|x^*-x_{k+1}\| &= \|x^*-x_k + f(x_k)/f'(x_k)\| \\ &= \left\| x^*-x_k + \frac{1}{2}\frac{f''(t_1)(x_k-x^*)^2}{f''(t_2)(x_k-x^*)} \right\| \\ &= \left\| x^*-x_k + \frac{1}{2}\frac{f''(t_1)}{f''(t_2)}(x_k-x^*) \right\| \\ &= \left| 1 - \frac{1}{2}\frac{f''(t_1)}{f''(t_2)} \right| \|x^*-x_k\| \end{aligned}$$

f is twice continuously differentiable, so $f''(t_1)\to f''(t_2)$ as $t_1\to t_2$.

Note: $t_1\to t_2$ as $k\to\infty$ since $t_1,t_2\in(x^*,x_k)$ and $x_k\to x^*$ as $k\to\infty$.

Choose $k_0\in\mathbb{N}$ such that $\dfrac{f''(t_1)}{f''(t_2)}\in\left(\frac{1}{2},\frac{3}{2}\right)$ $\forall k\geq k_0$.

Then $\|x^*-x_{k+1}\| \le \left|1-\frac{1}{2}\left(\frac{1}{2}\right)\right| \|x^*-x_k\| = 3/4\,\|x^*-x_k\|$

as desired.

4.1] $f(x) = 1/x_1 + e^{x_2} - \log x_1 - \log x_2 - \log x_3$

$$\nabla f(x) = \begin{bmatrix} -x_1^{-2} - \frac{1}{x_1} \\ e^{x_2} - \frac{1}{x_2} \\ -1/x_3 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 2x_1^{-3} + x_1^{-2} & 0 & 0 \\ 0 & e^{x_2} + x_2^{-2} & 0 \\ 0 & 0 & x_3^{-2} \end{bmatrix}$$

$\nabla^2 f(x) \succ 0$ for $x_1, x_2, x_3 > 0$

So f is strictly convex.

4.2] $A = LL^T$ where

note the transpose

$$L = \begin{bmatrix} 4.5826 & -1.9640 & 1.3093 & 0.8729 \\ 0 & 1.4639 & 0.3904 & -0.1952 \\ 0 & 0 & 1.4606 & -0.7303 \\ 0 & 0 & 0 & 1.6330 \end{bmatrix}^T$$

$$\left.\begin{array}{l} \Delta_1 = 21 > 0 \\ \Delta_2 = 45 > 0 \\ \Delta_3 = 96 > 0 \\ \Delta_4 = 256 > 0 \end{array}\right\} \text{leading principle minors of } A \text{ are positive.}$$

$\therefore A$ is positive definite.

$\mathcal{U}^T = \mathcal{U}^{-1}$.

<u>Fact:</u> $\|x\| = \|\mathcal{U}x\|$ for all orthogonal matrices $\mathcal{U}$.

<u>5.1</u> $A = A^T \Rightarrow A = \mathcal{U}D\mathcal{U}^T$ where $\mathcal{U}^T = \mathcal{U}^{-1}$ (i.e., orthogonal) and D is the diagonal matrix of eigenvalues of A.

$$\min_{\|x\|=1} x^T A x = \min_{\|x\|=1} x^T(\mathcal{U}D\mathcal{U}^T)x$$

$$= \min_{\|x\|=1} (\mathcal{U}^T x)^T D (\mathcal{U}^T x)$$

$$= \min_{\|x\|^2=1} (\mathcal{U}^T x)^T D (\mathcal{U}^T x)$$

$$= \min_{\|\mathcal{U}^T x\|^2=1} (\mathcal{U}^T x)^T D (\mathcal{U}^T x) \quad \text{since } \mathcal{U}^T \text{ is orthogonal}$$

$$= \min_{\|y\|^2=1} y^T D y \qquad \text{letting } y = \mathcal{U}^T x$$

$$= \min \ \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

subject to $y_1^2 + \cdots + y_n^2 = 1$

$= \text{smallest } \lambda_i$

$\max_{\|x\|=1} x^T A x = \text{largest } \lambda_i$ follows similarly.

5.2]

If $A \succ 0 \Rightarrow Q_A(x) = x^T A x$ where A has positive eigenvalues

$$\Rightarrow Q_A(x) = x^T U D U^T x$$
$$= (U^T x)^T D (U^T x)$$

$$\lim_{\|x\| \to \infty} Q_A(x) = \lim_{\|x\| \to \infty} (U^T x)^T D (U^T x)$$
$$= \lim_{\|U^T x\| \to \infty} (U^T x)^T D (U^T x)$$
$$= \lim_{\|y\| \to \infty} y^T D y$$
$$= \lim_{\|y\| \to \infty} \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \quad \text{where } \lambda_i > 0, \forall i.$$
$$= \infty \quad \text{since at least one } y_i^2 \to \infty \text{ as } \|y\| \to \infty.$$

So $Q_A(x)$ is coercive.

If $A \not\succ 0 \Rightarrow \exists v$ s.t. $Av = \lambda v$ where $\lambda \leq 0$ (and $v \neq 0$)
$$\Rightarrow v^T A v = \lambda v^T v = \lambda \|v\|^2$$

So $Q_A(v) = \lambda \|v\|^2 \not\to +\infty$ as $\|v\| \to \infty$ since $\lambda \leq 0$

Thus, $Q_A(x)$ is not coercive.

6. $f(x) = 100(x_2 - x_1^2)^2 + (1-x_1)^2$

$$\nabla f(x) = \begin{bmatrix} 200(x_2 - x_1^2)(-2x_1) + 2(1-x_1)(-1) \\ 200(x_2 - x_1^2) \end{bmatrix} = \begin{bmatrix} -400x_1(x_2 - x_1^2) + 2x_1 - 2 \\ 200(x_2 - x_1^2) \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} -400[x_1(-2x_1) + (x_2 - x_1^2)] + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

$$= \begin{bmatrix} 800x_1^2 - 400(x_2 - x_1^2) + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

$$\begin{aligned} \det \nabla^2 f(x) &= (800x_1^2 - 400(x_2 - x_1^2) + 2)(200) - (-400x_1)^2 \\ &= 160000x_1^2 - 80000(x_2 - x_1^2) + 400 - 160000x_1^2 \\ &= -80000(x_2 - x_1^2) + 400 \end{aligned}$$

$$\begin{aligned} \det \nabla^2 f(x) = 0 &\Leftrightarrow -80000(x_2 - x_1^2) + 400 = 0 \\ &\Leftrightarrow -200(x_2 - x_1^2) = -1 \\ &\Leftrightarrow x_2 - x_1^2 = 1/200 = 0.005 \end{aligned}$$

$\therefore$ $\nabla^2 f(x)$ is singular $\Leftrightarrow \det \nabla^2 f(x) = 0 \Leftrightarrow x_2 - x_1^2 = 0.005$

Continued

6. continued

Let x be such that $f(x) < 0.0025$

$$f(x) = 100(x_2 - x_1^2)^2 + (1-x_1)^2 \quad \text{for all } x \in \mathbb{R}^2$$

$$\Rightarrow \quad 100(x_2 - x_1^2)^2 < 0.0025 \quad \text{since } (1-x_1)^2 \geq 0 \ \forall x_1$$

$$\Rightarrow \quad (x_2 - x_1^2)^2 < 0.0000025$$

$$\Rightarrow \quad x_2 - x_1^2 < 0.005 \qquad (*)$$

To show $\nabla^2 f(x)$ is positive definite, we can show the leading principle minors are positive.

$$\begin{aligned} \Delta_1 &= 800x_1^2 - 400(x_2 - x_1^2) + 2 \\ &> 800x_1^2 - 400(0.005) + 2 \quad \text{by } (*) \\ &= 800x_1^2 - 2 + 2 \\ &= 800x_1^2 \\ &\geq 0 \end{aligned}$$

$$\begin{aligned} \Delta_2 = \det \nabla^2 f(x) &= -80000(x_2 - x_1^2) + 400 \\ &> -80000(0.005) + 400 \\ &= 0 \end{aligned}$$

$\therefore \nabla^2 f(x) > 0$ for all x such that $f(x) < 0.0025$.