MATH 235 **Assignment 9**

Quadratic Forms, Constrained Optimization and Singular Value Decomposition

Not to be handed in.

1. An $n \times n$ symmetric matrix A with real entries is called *positive definite* if $\mathbf{v}^T A\mathbf{v} > 0$ for all nonzero $\mathbf{v} \in \mathbf{R}^n$.

   (a) Suppose U is an invertible $n \times n$ matrix with real entries. Prove that $A = U^T U$ is positive definite.

   (b) Let $a, b, c \in \mathbf{R}$. Prove that $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if and only if $a > 0$ and $\det A > 0$.

2. (a) If A is symmetric and invertible, show that A^2 is positive definite.

   (b) If $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, show that A^2 is positive definite while A is not positive definite.

   (c) If A is positive definite, show that A^k is positive definite for all positive integers k.

   (d) If A^k is positive definite where k is odd, show that A is positive definite.

3. For each of the quadratic forms below identify the type of conic it represents, and find the equations of the principal axis. Find also the maximum and minimum values of the quadratic form subject to the constraint $||\mathbf{x}|| = 1$ where $\mathbf{x} = (x, y)^T$.

   (a) $9x^2 - 8xy + 3y^2$

   (b) $8x^2 + 6xy$

4. Sketch the conic section given by $2x^2 - 72xy + 23y^2 = -50$ in the xy-plane.

5. Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbf{R}^3$ at which $Q(\mathbf{x})$ is maximized subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$.

6. Find the singular value decomposition (SVD) for each of the following two matrices, A, B. In addition, use the *singular vectors* to find bases for the Four Subspaces. (Then do all the above for A^T and B^T.)

   (a) $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$