MATH 235/W08: Orthogonality; Least-Squares; & Best Approximation Assignment 6

Hand in questions 1,2,6,8 by 9:30 am on Wednesday March 19, 2008.

Contents

1 Least Squares and the Normal Equations***

Find the least-squares approximation and the error to a solution of $A\mathbf{x} = \mathbf{b}$, by constructing the normal equations for $\hat{\mathbf{x}}$ and then solving for $\hat{\mathbf{x}}$:

(a) $A = \begin{bmatrix} 3 & 5 \\ -2 & -6 \\ -5 & -11 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2i & 1 \\ -1 & 3i \\ -1+2i & -1+3i \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3+i \\ -4+2i \\ 2+i \end{bmatrix}$

Note that the normal equations over $\mathbb{C}$ for $A\mathbf{x} = \mathbf{b}$ are $A^*A\mathbf{x} = A^*\mathbf{b}$ where the operation $*$ denotes complex conjugation plus transposition, that is, $A^* = \overline{A}^T$.

(c)

$$A = \begin{pmatrix} 0 & -2 & -1 & -8 & -6 & -4 \\ 2 & -3 & 8 & 1 & -9 & -2 \\ -3 & 1 & -1 & -1 & -4 & 0 \\ -7 & 3 & -7 & -3 & -1 & 1 \\ 0 & 9 & 8 & 3 & 0 & -4 \\ -2 & -7 & -2 & -2 & 9 & 4 \\ 20 & -51 & -1 & -23 & 37 & 7 \\ 25 & 56 & 17 & 48 & -71 & -18 \\ 17 & -56 & 42 & 7 & -58 & 4 \\ -54 & -107 & -174 & -50 & 60 & 84 \end{pmatrix}, \quad b = \begin{pmatrix} -83 \\ -30 \\ -3 \\ -67 \\ -49 \\ 191 \\ 375 \\ -676 \\ -558 \\ 734 \end{pmatrix}$$

Hint: The file tempoutput.txt at
http://orion.math.uwaterloo.ca/~hwolkowi/henry/teaching/w08/235.w08/miscfiles/tempoutput.txt
and MATLAB may help.

2 Best Polynomial Fit***

You are given the four data points:
$y = 4$ at $t = -2$;
$y = 3$ at $t = -1$;
$y = 1$ at $t = 0$;
$y = 0$ at $t = 2$.

(a) Find the best straight-line fit (least squares) to this data having the form $y = \beta_0 + \beta_1 t$.

(b) Find the best quadratic polynomial fit (least squares) to this data having the form $y = \beta_0 + \beta_1 t + \beta_2 t^2$.

3 Least Squares Solutions and Errors

For the inconsistent linear system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix},$$

find a least-squares solution and the corresponding least-squares error for the solution.

4 Best Quadratic Polynomial Fit

For the data set $\{(x, y) \in \mathbf{R}^2 | (-1, -4), (0, -2), (2, -2), (3, -5)\}$, find the least squares quadratic polynomial of the form $y(x) = a + bx + cx^2$ that best approximates this data.

5 Best Approximation in Continuous Function Space

1. Consider the vector space $C[-1, 1]$ of continuous functions on the interval [-1,1], together with the inner product given by $< f, g >= \int_{-1}^{1} f(t)g(t)dt$, for $f, g \in C[-1, 1]$. Let $\mathbf{P}_2[-1, 1]$ be the subspace of $C[-1, 1]$ having an orthonormal basis given by $B = \left\{ \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3}),\ \sqrt{\frac{3}{2}}t,\ \frac{1}{\sqrt{2}} \right\}$. Find the best quadratic approximation to e^t on $[-1, 1]$.

2. Consider the vector space $C[0, 1]$ of continuous functions on the interval [0,1], together with the inner product given by $< f, g >= \int_{0}^{1} f(t)g(t)dt$, for $f, g \in C[0, 1]$. Let $\mathbf{P}_2[0, 1]$ be the subspace of $C[0, 1]$ having a basis given by $B = \{1, x, x^2\}$. Find the best quadratic approximation to $\sqrt{x}$ on $[0, 1]$.

6 Orthogonality***

1. Suppose that A is a $m \times n$ real matrix. And suppose that $Ax = 0$ and $A^T y = 2y$. Show that x is orthogonal to y.

2. State and justify whether the following three statements are True or False (give an example in *either* case):

   (a) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

   (b) If Q (a $m \times n$ matrix with $m > n$) has orthonormal columns, then $\|Qx\|$ always equals $\|x\|$.

   (c) If Q (a $m \times n$ matrix with $m > n$) has orthonormal columns, then $\|Q^T y\|$ always equals $\|y\|$.

7 MATLAB; Best Line Fit

Over the past four Winter Olympics Games, the Canadian team won the following numbers of medals:

Year	Location	Number of Medals
1994	Lillehammer, Norway	13
1998	Nagano, Japan	15
2002	Salt Lake City, USA	17
2006	Turin, Italy	24

Scaling the years, we can translate this information to the data points $(0, 13)$, $(1, 15)$, $(2, 17)$, $(3, 24)$.

(a) Determine the least squares line $y = \beta_0 + \beta_1 x$ which best fits these data points.

(b) Using your line from (a), predict the total number of medals that Canada will win at the Winter Olympics in Vancouver in 2010.

8 MATLAB; Best Quadratic Fit***

Newton's Laws of Motion imply that an object thrown vertically with a velocity of v meters per second will be at a height of

$$h(t) = vt - \frac{1}{2}gt^2$$

meters after t seconds, where g is the acceleration due to gravity. The values in the table were observed. By fitting a quadratic

$$h(t) = c_0 + c_1 t + c_2 t^2$$

to these data, estimate the values of v and g in this question.

t	0.5	1.0	1.5	2.0
h	23.7	45.1	64.0	80.4

9 BONUS: Eigenvalues and Best Approximation

1. Consider the 3×3 matrix
$$A = \begin{pmatrix} a & b & c \\ 1 & d & e \\ 0 & 1 & f \end{pmatrix}.$$
Determine the entries a, b, c, d, e, f so that:

   The top left 1×1 block is a matrix with eigenvalue 2;

   The top left 2×2 block is a matrix with eigenvalues 3 and -3;

   The matrix A has eigenvalues $0, 1$ and -2.

2. Find the best line $y = c + dt$ by calculus (not MATLAB). Choose c and d to minimize the least squares problem
$$E(c, d) := \int_0^1 (c + dt - t^2)^2 dt.$$

3. (**MATLAB**) First note that: The command **a=ones(n,1)** produces an $n \times 1$ matrix of 1's. The command **r=(1:n)'** produces the vector $\big(1, 2, \ldots, n\big)$, transposed to a column vector. The command $\mathbf{s} = \mathbf{r.^2}$ produces the vector $\big(1^2, 2^2, \ldots, n^2\big)$, because the dot means *a component at a time*.

   This problem looks for the line $y = c + dt$ closest to the parabola $y = t^2$.

   With $n = 10$, choose C and D to give the line $y = C + Dt$ that is closest to t^2 at the points $t = \frac{1}{10}, \frac{2}{10}, \ldots, 1$ (in the vector $r/10$ with r). The inconsistent equations $Ax = b$ are $\begin{pmatrix} a & r/n \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = s/n^2$.