MATH 235/W08: Orthogonality & Gram-Schmidt, Assignment 5

Hand in questions 3,9,11,13,14 by 9:30 am on Wednesday March 5, 2008.

Contents

1 Orthogonal Pairs

Determine which pairs of vectors are orthogonal; find the norms of each vector.

a) $u = \begin{pmatrix} i \\ \sqrt{2} \\ 1 \end{pmatrix}, v = \begin{pmatrix} i \\ \sqrt{2} \\ -1 \end{pmatrix}$

b) $u = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, v = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -\sqrt{2} \\ 0 \end{pmatrix}$

2 Representation using Orthogonal Complement

Let $u_1 = (3, 0, 1)^T$, $u_2 = (0, 1, 0)^T$, and $y = (0, 3, 10)^T$. Let $W = \mathrm{span}\{\mathrm{u}_1, \mathrm{u}_2\}$. Represent y as the sum of a vector in W and a vector in the orthogonal complement of W.

3 Basis and Distance ***

Let V be the nullspace of A where $A = \begin{pmatrix} 1 & -1 & -3 & 0 & 2 \\ 0 & 1 & 2 & -1 & 4 \\ 2 & 1 & 0 & 3 & 4 \\ 3 & 2 & 1 & 5 & 6 \end{pmatrix}$ and let

$\mathbf{v} = \begin{pmatrix} 2 & 9 & -3 & 32 & -83 \end{pmatrix}^T$.

(a) Find a basis for V.

(b) Find the distance from $\mathbf{v}$ to V.

4 Orthogonal Basis

Find an orthogonal basis for the span of $X_1 = (1, 1, 0, 1)$, $X_2 = (0, 1, -1, 1)$ and $X_3 = (1, 0, 1, 1)$.

5 Orthogonal Basis for Columns Space

Find an orthogonal basis for the column space of

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

6 Orthogonal Matrix

Find a constant c such that cA is orthogonal, where $A = \begin{pmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{pmatrix}$ has orthogonal columns.

7 Orthogonal Matrix Two

Find a, b, c, d such that $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{2} & -\frac{1}{2} & 0 & b \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & c \\ \frac{1}{2} & -\frac{1}{2} & 0 & d \end{pmatrix}$ is orthogonal.

8 True-False Questions

Provide a proof for each true statement and a counter example (or justification) for each false statement.

(a) There exist 4×4 orthogonal matrices with rank 3.

(b) If A is a $n \times n$ orthogonal matrix and B is $n \times 1$ matrix, then the equation $AX = B$ has a unique solution.

(c) Let Q be an orthogonal matrix. Then

$$Q^2(Q^T)^3Q^{-1}(Q^T)^{-1}(Q^{-1})^T = Q.$$

(d) The following matrix is orthogonal

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

(e) The matrix A from the previous question satisfies $A^2 - A = I$.

9 Angles Between Vectors ***

(a) A vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ has the following properties: $||\mathbf{x}|| = \sqrt{6}$, $\mathbf{x}$ is orthogonal to $(1, -1, 1)$, and the angle between $\mathbf{x}$ and $(1, 1, 0)$ is $\theta = \pi/6$. Find the vector (or vectors) $\mathbf{x}$ which satisfy these properties.

(b) Find a non-zero vector in $\mathbf{C}^2$ which is orthogonal to both $(i, 1)$ and $(1 + i, 2)$, or show that none exists.

10 Orthonormal Bases

Consider the complex vector space $\mathbf{C}^3$ with the standard inner product.

(a) Produce an orthonormal basis for the subspace S spanned by the vectors $\mathbf{v_1} = (1 + i, 1 - 2i, 3i)$ and $\mathbf{v_2} = (2 + 3i, 1, 1 - 2i)$.

(b) Find the orthogonal projection of $\mathbf{v} = (2i, 2 - i, 1)$ along $\mathbf{v_1}$.

11 Basis and Nearest Vectors ***

Let W be the subspace of $\mathbf{R}^4$ spanned by $(1, 0, -1, 2)$ and $(2, 1, 0, -1)$ and take the inner product to be the standard dot product.

(a) Find the vector in W which is closest to the vector $(4, 3, 2, 1)$.

(b) Find a basis for $W^{\perp}$ and the dimension of $W^{\perp}$.

(c) Find the unique $\mathbf{w_1} \in W$ and $\mathbf{w_2} \in W^{\perp}$ such that $(4, 3, 2, 1) = \mathbf{w_1} + \mathbf{w_2}$.

12 Gram-Schmidt Procedure

Let $\mathbf{x_1} = (1, -1, 1, -1)$, $\mathbf{x_2} = (1, 0, 0, 1)$, $\mathbf{x_3} = (1, 1, 3, 3)$ in $\mathbf{R}^4$ and let $S = \mathrm{Span}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$.

(a) If the Gram-Schmidt procedure is applied to the vectors $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ in that order, find the resulting orthonormal basis for S.

(b) Find a basis for $S^{\perp}$.

(c) Find an orthonormal basis for $\mathbf{R}^4$ which extends the orthonormal basis for S found in part (a).

13 Gram-Schmidt Procedure and Inner Products ***

Using the Gram-Schmidt procedure and the standard inner product in $\mathbf{R}^4$, and in $\mathbf{C}^4$:

1. Find an orthonormal basis for the span of the three vectors

$$\{\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}1 & 0 & -1 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 3 & 4 & 0\end{pmatrix}^T\}.$$

2. construct an orthonormal set of vectors from the set:

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} i \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \end{bmatrix}.$$

14 MATLAB: Predator-Prey System ***

(**MATLAB**) **Linear Recurrences: A Predator-Prey System**

Linear recurrence equations are well-suited to studying predator-prey models in the biological sciences. The following example involves studying the dynamics between two inter-related populations–hawks and mice.

Let h_k and m_k denote the number of hawks and mice, respectively, in a certain area in year k. Assume that they are related by the following equations:

$$\begin{aligned} h_{k+1} &= 0.5h_k + 0.01m_k \\ m_{k+1} &= -12.5h_k + 1.25m_k \end{aligned}$$

for $k \geq 0$.

If we write $S_k = \begin{bmatrix} h_k \\ m_k \end{bmatrix}$, then these equations become

$$S_{k+1} = AS_k$$

Find matrix A corresponding to the above.

Find also the diagonalizing matrix P and the diagonal matrix D for A using MATLAB.

Recall that $A = PDP^{-1}$. This means that $A^k = PD^kP^{-1}$ and hence that $S_k = PD^kP^{-1}S_1$, where S_1 is the column vector whose entries are the initial populations of hawks and mice. Let the initial populations consist of 40 hawks and 1500 mice. It follows that the limiting population vector S_L is given by

$$\begin{aligned} S_L &= \lim_{k\to\infty} S_k \\ &= \lim_{k\to\infty} PD^kP^{-1}S_1 \\ &= P\,(\lim_{k\to\infty} D^k)P^{-1}S_1 \end{aligned}$$

Use this formula and MATLAB to calculate the limiting population

$$S_L = \begin{bmatrix} h_L \\ m_L \end{bmatrix}$$

What is the limiting population of hawks, h_L? What is the limiting population of mice, m_L?