MATH 235/W08 Assignment 4B
Eigenvalues, Eigenvectors, Diagonalization

Do NOT hand in this assignment.

1. Let A be an $n \times n$ matrix. Also, let α, β, γ be three distinct eigenvalues of A having corresponding eigenvectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$, respectively. Consider the vector $\mathbf{v} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. Can $\mathbf{v}$ be an eigenvector of A corresponding to an eigenvalue λ (possibly different from α, β, γ? Explain.

2. Recall that the trace of an $n \times n$ matrix $A = [a_{ij}]$, denoted by $\mathrm{tr}(A)$, is the sum of the diagonal elements, that is, $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

   (a) Let C and D be any two $m \times n$ matrices. Prove that $\mathrm{tr}(C^T D) = \mathrm{tr}(DC^T)$.

   (b) Prove that if A and B are similar $n \times n$ matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(A) = \det(B)$.

3. Let M, N be $n \times n$ matrices and suppose that M is invertible. Denote $A = MN$ and $B = NM$. Prove that A is similar to B.

4. Let A and B be invertible $n \times n$ matrices. Prove that AB and BA have the same eigenvalues.

5. Determine if the following pair of matrices are similar to each other:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & 2 \end{bmatrix}.$$