MATH 235/W08 Review of Linear Algebra I
Assignment 1
due by 9:30 am on Wed. Jan. 16/08.

Hand in questions 2,6,10,11,13,15,17.

1. Solve the following system of equations for x and y:

$$2x^2 - 2xy + y^2 = 17$$
$$-x^2 + 2xy + 3y^2 = 20$$
$$x^2 + 4xy + 2y^2 = 7$$

2. Solve $A\mathbf{x} = \mathbf{b}$ and $\mathbf{y}A = \mathbf{c}$ where

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -3 \\ 1 & 0 & -2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}, \mathbf{y} = (y_1, y_2, y_3), \mathbf{c} = (5, -3, -7).$$

3. A system of linear equations in x_1, x_2, x_3 has the augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 1 & 1 & 0 & a \\ 1 & a & b & a \end{array}\right]$,
where a and b are real numbers. Determine values of a and b, if they exist, such that the system has:
(a) no solution;
(b) a unique solution;
(c) infinitely many solutions.

   For the values of a and b that you find in part (c) above, give the general solution to the system of equations.

4. Find A^{-1} for $A = \begin{bmatrix} 1 & i & 2 \\ 2 & 0 & 6 \\ -i & 1 & -i \end{bmatrix}$.

5. Determine if the following set of vectors in $\mathbf{C}^3$ is linearly independent:
$\mathbf{v}_1 = (1, 0, -i), \mathbf{v}_2 = (1 + i, 1, 1 - 2i), \mathbf{v}_3 = (0, i, 2)$.

6. Determine whether the following subsets W are subspaces of V:

   (a) $V = M_{33}, \quad W = \{A \in M_{33} : A \text{ is singular}\}$;

   (b) $B \in V = M_{nn}, \quad W = \{A \in M_{nn} : BAB^T = A$ for a fixed $n \times n$ matrix $B\}$.

7. Let $V = \mathbf{R^4}$. Consider the subspaces:
$U = \{(a_1, a_2, a_3, a_4) \in \mathbf{R^4} : a_1 + 2a_2 + 3a_3 = 0, a_1 + a_2 + a_3 + a_4 = 0\}$,
$W = \{(a_1, a_2, a_3, a_4) \in \mathbf{R^4} : a_1 + a_4 = 0, a_2 + a_3 = 0\}$
(a) Find a basis for U and a basis for W.
(b) Find a basis for $U \cap W$.

8. Let $p_1(x) = 1 + x + \alpha x^2$, $p_2(x) = 1 + \alpha x + x^2$ and $p_3(x) = \alpha + x + x^2$ where $\alpha \in \mathbf{R}$. For what value(s) of α are p_1, p_2, p_3 linearly independent in P_2?

9. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent vectors in a vector space V.
If $\mathbf{x} \notin \text{span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \ \mathbf{v}_n\}$ then prove that $\mathbf{x} + \mathbf{v}_1, \mathbf{x} + \mathbf{v}_2, \ldots, \mathbf{x} + \mathbf{v}_n$ are linearly independent vectors.

10. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator such that

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, T(e_2) = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ -7 \\ 5 \end{pmatrix}, \text{ where } e_1, e_2, e_3 \text{ are the columns of } I_3.$$

Determine if T is one-to-one. Mention an appropriate theorem to justify your answer.

11. Let $T : V \to V$ be a linear operator, and let $\{\mathbf{v_1}, \mathbf{v_2}\}$ be nonzero vectors such that $T(\mathbf{v_1}) = 5\mathbf{v_2}, T(\mathbf{v_2}) = 3\mathbf{v_1}$. Is the set $\{\mathbf{v_1}, \mathbf{v_2}\}$ linearly independent? Why? If not, then find the dependency between $\mathbf{v_1}$ and $\mathbf{v_2}$.

12. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Consider the linear transformation
$T : M_{2\times 3}(\mathbf{R}) \to M_{2\times 3}(\mathbf{R})$ defined by $T(X) = AX - XB$ where $X \in M_{2\times 3}(\mathbf{R})$.

    (a) If $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$, compute $T(X)$.

    (b) Find a basis for Nul(T). What is the dimension of the null space (also known as nullity)?

    (c) Is T one-to-one? Justify.

    (d) Find a basis for range(T). Is T onto? Justify.

(e) Find the matrix A representing T with respect to the basis $E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}$ of $M_{2\times 3}$, where $E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\cdots$, $E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

13. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

(a) Find the largest number of linearly independent vectors among the columns of the matrix A.

(b) Determine the rank of A and the nullity of A.

(c) Give a basis for the column space of A.

(d) Give a basis for the nullspace of A.

(e) Give a basis for the column space of A^T and a basis for the nullspace of A^T. (Hint: Use the steps from row reduction of A.)

(f) State the *Rank Theorem* and use the results above to confirm that it holds for both A and A^T.

14. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}, \qquad \text{a Pascal Matrix.}$$

(a) Apply row elimination to A and find the pivots and the upper triangular U. Factor this "Pascal matrix" into L times U.
(To save arithmetic, you can leave L as the product of elementary matrices. But clearly state what these elementary matrices are and the order of the product.)

(b) How do L and U and the pivots confirm that A is invertible?

(c) How can you change the entry "20" so as to make A singular?

15. Let W be the subspace of $\mathbb{C}^3$ given by

$$W := \text{span} \left\{ \begin{pmatrix} 2 \\ 2i \\ 4 \end{pmatrix} \right\}.$$

(a) Define the *orthogonal complement of* W.

(b) What is the dimension of W and what is the dimension of the orthogonal complement, $W^{\perp}$?

(c) Find a basis for the orthogonal complement $W^\perp$.

(d) Find an orthonormal basis for W; and find an orthonormal basis for $W^\perp$. Denote the latter basis by Γ.

(e) Set $x = \begin{pmatrix} -2 \\ i \\ .5 \end{pmatrix}$.

i. Verify that $x \in W^\perp$.

ii. Find the coordinate vector $[x]_\Gamma$, for the basis Γ found above.

16. Consider $\mathbb{P}_1$, the vector space of all polynomials, with real coefficients, and with degree at most 1. Define the two sets in $\mathbb{P}_1$

$$\Gamma := \{-9+t, -5-t\}, \qquad \beta := \{1-4t, 3-5t\}.$$

(a) Define what is meant by a *basis* for a vector space V.

(b) Show that both Γ and β form a basis for $\mathbb{P}_1$.

(c) Let $p(t) = -14$ and $q(t) = 7t$. Find the coordinates

$$[p(t)]_\Gamma, \qquad [q(t)]_\beta.$$

Explain how the coordinates were found.

(d) Find the change of coordinate matrix from Γ to β.

(e) Find the change of coordinate matrix from β to Γ.

17. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^4$ into $\mathbb{R}^{12}$. (Recall that we define the mapping $T_A(v) = Av$.)

18. Suppose that the vector in the plane $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is given. Define the transformation T on v to be the clockwise rotation in the plane through an angle $\theta = 45$ degrees, i.e. $T(v)$ is the vector in the plane obtained by rotating v clockwise 45 degrees. Similarly, define the transformation S on v to be the counter-clockwise rotation in the plane through an angle $\theta = 60$ degrees, i.e. $S(v)$ is the vector in the plane obtained by rotating v counter-clockwise 60 degrees. (You can assume that both T, S are linear transformations.)

(a) Find the matrix representations A_T, A_S of T and S, respectively.

(b) What is $W(v) = S(T(v))$? Find a simpler description of the product $W = ST$; and find a matrix representation A_W of W.

(c) Confirm that $A_W = A_S A_T$.